Symmetric operations in Algebraic Cobordism

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1 Introduction

Among the oriented generalized cohomology theories on the category of smooth quasiprojective varieties there is the universal one - Ω^* called Algebraic Cobordism. This theory was constructed by M.Levine and F.Morel in [6, 7, 5, 4]. It is an algebro-geometric analog of complex-oriented cobordism in topology. In this article we will describe certain new, so-called, *symmetric* cohomological operations in algebraic cobordism introduced in [17]. We will show that these operations are well-defined on Ω^* - see Theorem 2.24 (in [17] it was shown only that they are well-defined on $(\mathrm{Pre}-\Omega)^*$), and prove various properties. Our operations are interesting from two points of view. First of all, they are trivial on the classes of embeddings, and so provide a natural obstructions for the cobordism class to be represented by the embedding - Proposition 3.2. It is an interesting question to find out what other obstructions exist. Another reason to have symmetric operations is that they permit to work with algebraic cobordism and Chow groups in a bit more subtle way than the Landweber-Novikov operations do, since one does not need to mod out the 2-torsion elements. As a demonstration of this feature we point out the applications to the computation of the algebraic cobordism of a Pfister quadric - Theorem 4.1, and to the questions of the rationality of cycles under function field extensions - Theorem 4.3. We will prove that symmetric operations commute with pull-back morphisms - Proposition 3.4, and behave in a special way with respect to regular push-forwards - Proposition 3.1. We will show also that our operation Φ is a generator for the class of operations with very natural defining properties - Theorem 3.8. The connection between symmetric operations and Landweber-Novikov operations will be described - Proposition 2.13.

In the Appendix we will prove various general results on Algebraic Cobordim used in the paper. Among them, we should mention the formula of Quillen expressing the class of the projective bundle in the Cobordism ring of the base, the formula for the class of the map of degree two, excess intersection formula, and various formulas related to the blow-up morphism.

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2 Operations $\widetilde{\square}$, $\widetilde{C}^2$, Ψ and Φ.

2.1 Preliminaries on Algebraic Cobordism.

Let k be a field of characteristic 0, and X be a smooth quasiprojective variety over k.

In [6],[7] M.Levine and F.Morel have defined the ring of Algebraic cobordism $\Omega^*(X)$. We will briefly recall this definition.

Let $\mathcal{M}^*(X)$ denote the set of isomorphism classes of projective morphisms of pure codimension from a smooth quasiprojective variety Y to X graded by codimension. The disjoint union provides the structure of monoid on $\mathcal{M}^*(X)$. Let $\mathcal{M}^*(X)^+$ be its group completion. The classes $f : Y \to X$ and $g : Z \to X$ are called *elementary cobordant* if there exists a projective morphism h from a smooth quasiprojective variety W to $X \times \mathbb{A}^1$ which is transversal to $X \times \{0\}$ and $X \times \{1\}$ and such that it's restrictions $h^{-1}(X \times \{0\}) \to X$ and $h^{-1}(X \times \{1\}) \to X$ are isomorphic to f and g, respectively. The quotient of $\mathcal{M}^*(X)^+$ by such relations is denoted Pre-$\Omega^*(X)$. In topology this already would be the cobordism, but in algebraic geometry one needs to impose more relations.

If $\mathcal{L}$ is an invertible sheaf on X generated by its global sections, one can define the action $c_1(\mathcal{L})$: Pre-$\Omega^n(X) \to$ Pre-$\Omega^{n+1}(X)$ as follows. For $Y \to X$ we choose the section $s : Y \to E(\mathcal{L}|_Y)$ transversal to the zero-section, and put: $c_1(\mathcal{L}) \cdot [Y \to X] := [Z \to X]$, where $Z \subset Y$ is a smooth subvariety defined by the equation $s = 0$.

Let $F_U(-,-)$ be the *universal formal group law* having the coefficients in the Lazard ring $\mathbb{L}$. The topological realization functor provides the surjection Pre-$\Omega^*(\mathrm{Spec}(k)) \twoheadrightarrow \mathbb{L}$. Let $\tilde{a}_{i,j}$ be any lifting of the coefficients $a_{i,j} \in \mathbb{L}$ of the universal formal group law satisfying $\tilde{a}_{i,j} = \tilde{a}_{j,i}$. Let $\widetilde{\Omega}^*$ be the quotient of Pre-$\Omega^*(X)$ by the relations $c_1(\mathcal{L} \otimes \mathcal{M}) = F_{\tilde{U}}(c_1(\mathcal{L}), c_1(\mathcal{M}))$ for all pairs $\mathcal{L}, \mathcal{M}$ of invertible sheaves generated by the global sections (notice, that the action of $c_1(-)$ is nilpotent). On the ring $\widetilde{\Omega}^*$ there is the natural action of $c_1(\mathcal{N})$ for arbitrary invertible sheaf $\mathcal{N}$, and such an action satisfy the abovementioned rule with respect to tensor product of sheaves. Finally, the algebraic cobordism ring $\Omega^*(X)$ is the quotient of $\widetilde{\Omega}^*$ by the relations generated by $[Y \to X] = c_1(\mathcal{N}_{Y\subset X})[Id_X]$, where Y is a smooth codimension 1 subvariety of X and $\mathcal{N}_{Y\subset X}$ is a normal bundle.

As any oriented generalized cohomology theory, the theory Ω^* has pull-backs for all morphisms between smooth quasiprojective varieties, and push-forwards for projective maps. Moreover, if the morphisms $f : Y \to X$ and $g : Z \to X$ are transversal, and g is projective, then for the cartesian diagram

$$\begin{array}{ccc} Y \times_X Z & \xrightarrow{f'} & Z \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X, \end{array}$$

$f^* g_* = (g')_*(f')^*$ (by [6, Axiom A3]). We will call such transversal squares *proper cartesian*.

Also, Ω^* satisfies the *homotopy invariance* property, and has *projective bundle theorem*.

The topological realization functor defines an isomorphism $\Omega^*(\mathrm{Spec}(k)) \cong \mathbb{L}$, where $\mathbb{L}$ is a Lazard ring, for arbitrary field k of characteristic 0, which gives an action of $\mathbb{L}$ on $\Omega^*(X)$ for arbitrary smooth quasiprojective X. There is a natural ring homomorphism $pr_X : \Omega^*(X) \to \mathrm{CH}^*(X)$ which identifies:

$$\mathrm{CH}^*(X) = \Omega^*(X)/\mathbb{L}_{>0} \cdot \Omega^*(X).$$

Finally, for a regular codimension d subvariety $Z \xrightarrow{j} X$ with open complement $U \xrightarrow{i} X$ one has (by [6, Theorem 2.2]) the exact sequence

$$\Omega^{*-d}(Z) \xrightarrow{j_*} \Omega^*(X) \xrightarrow{i^*} \Omega^*(U) \to 0.$$

2.2 Basic objects

For the closed subscheme Z of X we will denote as $\mathrm{Bl}_{X,Z}$ the respective blow-up variety $\mathrm{Proj}(\oplus_n \mathcal{I}_Z^n)$, and as $\mathcal{O}(1)$ the invertible sheaf on $\mathrm{Bl}_{X,Z}$ corresponding to the graded $\oplus_n \mathcal{I}_Z^n$-module $\oplus_n \mathcal{I}_Z^{n+1}$. If $\mu : \mathrm{Bl}_{X,Z} \to X$ is the blow-up map, then $\mathcal{O}(1) = \mathcal{O}_Y(-E)$, where $E = \mu^{-1}(Z)$ is the exceptional divisor. For a smooth morphism $f : Y \to X$ we will denote as $\square(Y/X)$ the relative square $Y \times_X Y$, as $\widetilde{\square}(Y/X)$ the blow-up variety $\mathrm{Bl}_{\square(Y),\Delta(Y)}$, and as $\widetilde{C}^2(Y/X)$ the quotient of $\widetilde{\square}$ by the natural $\mathbf{Z}/2$-action. Since the locus of fixed points of this action is a smooth (exceptional) divisor on $\widetilde{\square}$, such a quotient is smooth. When $X = \mathrm{Spec(k)}$, we will denote above varieties simply as $\square(Y)$, $\widetilde{\square}(Y)$ and $\widetilde{C}^2(Y)$. Clearly, $\widetilde{C}^2(Y)$ is just $Hilb_2(Y)$ - the variety of subschemes of length 2 on Y.

We get commutative diagram with the left square cartesian:

$$\begin{array}{ccccc} \mathbb{P}_Y(T_{Y/X}) & \xrightarrow{j} & \widetilde{\square}(Y/X) & \xrightarrow{p} & \widetilde{C}^2(Y/X) \\ {\scriptstyle \varepsilon}\downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle D(f)} \\ Y & \xrightarrow[\Delta]{} & \square(Y/X) & \xrightarrow[q]{} & X. \end{array} \qquad (1)$$

Since p is a finite dominant morphism of smooth connected varieties, it is flat. Following M.Rost, let us denote as $\mathcal{L}$ the quotient $p_*(\mathcal{O})/\mathcal{O}$. It is a natural line bundle on $\widetilde{C}^2(Y/X)$ such that $p^*(\mathcal{L}) = \mathcal{O}(1)$ (see Proposition 5.6). We will denote $c_1(\mathcal{L}^{-1}) \in \Omega^1(\widetilde{C}^2(Y))$ as ϱ, and $c_1(\mathcal{O}(-1)) \in \Omega^1(\widetilde{\square}(Y))$ as ρ.

Examples:

1) $\widetilde{\square}(X \coprod X/X) = X \coprod X$ with $\mathcal{O}(1) = \mathcal{O}$;
$\widetilde{C}^2(X \coprod X/X) = X$ with $\mathcal{L} = \mathcal{O}$;

2) $\widetilde{\square}(\mathbb{P}^n) = \mathbb{P}_{\mathrm{Gr}(1,\mathbb{P}^n)}(Tav) \times_{\mathrm{Gr}(1,\mathbb{P}^n)} \mathbb{P}_{\mathrm{Gr}(1,\mathbb{P}^n)}(Tav)$ with $\mathcal{O}(1) = \mathcal{O}(-\Delta)$; $\widetilde{C}^2(\mathbb{P}^n) = \mathbb{P}_{\mathrm{Gr}(1,\mathbb{P}^n)}(S^2(Tav))$ with $\mathcal{L} = \mathcal{O}(-1)$.
Here $\mathrm{Gr}(1,\mathbb{P}^n)$ is the grassmannian of projective lines on $\mathbb{P}^n$, and Tav is a 2-dimensional tautological vector bundle on it.

We have natural maps:

$$\widetilde{\square}(W) \stackrel{A(f)}{\leftarrow} \widetilde{\square}(W/X) \stackrel{B(f)}{\to} X;$$

$$\widetilde{C}^2(W) \stackrel{C(f)}{\leftarrow} \widetilde{C}^2(W/X) \stackrel{D(f)}{\to} X.$$

We will need the following

Lemma 2.1 *Suppose $f : Y \to X$ be a smooth morphism. Then for the natural regular embeddings $A(f) : \widetilde{\square}(Y/X) \to \widetilde{\square}(Y)$ and $C(f) : \widetilde{C}^2(Y/X) \to \widetilde{C}^2(Y)$,*

$$\mathcal{N}_{A(f)} = B(f)^*(T_X) \otimes \mathcal{O}(1), \quad \mathcal{N}_{C(f)} = D(f)^*(T_X) \otimes \mathcal{L},$$

where $B(f)$, $D(f)$ are the natural projections.

Proof: Consider the commutative diagram:

$$\begin{array}{ccccc}
\widetilde{C}^2(Y) & \stackrel{p_Y}{\longleftarrow} & \widetilde{\square}(Y) & \stackrel{\pi_Y}{\longrightarrow} & \square(Y) \\
{\scriptstyle C(f)}\uparrow & & \uparrow{\scriptstyle A(f)} & & \uparrow{\scriptstyle k} \\
\widetilde{C}^2(Y/X) & \underset{p_{Y/X}}{\longleftarrow} & \widetilde{\square}(Y/X) & \underset{\pi_{Y/X}}{\longrightarrow} & \square(Y/X).
\end{array}$$

We know that $\mathcal{N}_k = q^*(T_X)$, and the natural (interchange of factors) $\mathbf{Z}/2$-action on $\mathcal{N}_k$ acts as (-1) on $q^*(T_X)$. Then $\mathcal{N}_{A(f)} = \pi_{Y/X}^* q^*(T_X) \otimes \mathcal{O}(1)$, and the natural $\mathbf{Z}/2$-action on $\mathcal{N}_{A(f)}$ restricted to the complement U of the special fiber is still multiplication by (-1) on $\pi_{Y/X}^* q^*(T_X) \otimes \mathcal{O}(1)|_U = \pi_{Y/X}^* q^*(T_X)|_U$. By Proposition 5.8, $\mathcal{N}_{C(f)} = \mathcal{N}_{A(f)}/(\mathbf{Z}/2) = D(f)^*(T_X) \otimes (\mathcal{O}(1)/(\mathbf{Z}/2))$, where the action on $\mathcal{O}(1)|_U = \mathcal{O}|_U$ is the multiplication by (-1). But $\mathcal{O}(1)/(\mathbf{Z}/2)$ with such action is exactly $\mathcal{L}$ - see Proposition 5.6, cf. [9, Theorem 6.1].

$\square$

2.3 Symmetric push-forwards

In this subsection we assign to each projective morphism $\alpha : Y \to X$ of smooth quasiprojective varieties some push-forward like maps between the cobordism of the respective varieties $\widetilde{\square}$ and $\widetilde{C}^2$, as well as some maps from $\Omega^*(\widetilde{\square}(Y))$ and $\Omega^*(\widetilde{C}^2(Y))$ to $\Omega^*(X)$. Then in Proposition 2.5 we describe the composition formula for such maps.

Let $\alpha : Y \to X$ be projective map between smooth quasiprojective varieties. Decompose it as $f \circ g$, where $g : Y \to W$ is a regular embedding, and $f : W \to X$ is a smooth projective map. In particular, we get regular embeddings $\widetilde{\square}(g) : \widetilde{\square}(Y) \to \widetilde{\square}(W)$ and $\widetilde{C}^2(g) : \widetilde{C}^2(Y) \to \widetilde{C}^2(W)$.

We have natural projective maps

$$\widetilde{\square}(W) \stackrel{a(f)}{\leftarrow} Bl_{\widetilde{\square}(W),\widetilde{\square}(W/X)} \stackrel{b(f)}{\to} \widetilde{\square}(X);$$

$$\widetilde{C}^2(W) \stackrel{c(f)}{\leftarrow} Bl_{\widetilde{C}^2(W),\widetilde{C}^2(W/X)} \stackrel{d(f)}{\to} \widetilde{C}^2(X);$$

$$\widetilde{\square}(W) \stackrel{A(f)}{\leftarrow} \widetilde{\square}(W/X) \stackrel{B(f)}{\to} X;$$

$$\widetilde{C}^2(W) \stackrel{C(f)}{\leftarrow} \widetilde{C}^2(W/X) \stackrel{D(f)}{\to} X.$$

Definition 2.2 *Let us define the maps:*

$$(\widetilde{\square}(\alpha))_* := b(f)_* a(f)^* \widetilde{\square}(g)_* : \Omega_d(\widetilde{\square}(Y)) \to \Omega_d(\widetilde{\square}(X));$$

$$(\widetilde{C}^2(\alpha))_* := d(f)_* c(f)^* \widetilde{C}^2(g)_* : \Omega_d(\widetilde{C}^2(Y)) \to \Omega_d(\widetilde{C}^2(X));$$

$$\Upsilon(\alpha) := B(f)_* A(f)^* \widetilde{\square}(g)_* : \Omega_d(\widetilde{\square}(Y)) \to \Omega_{d-dim(X)}(X);$$

$$\Theta(\alpha) := D(f)_* C(f)^* \widetilde{C}^2(g)_* : \Omega_d(\widetilde{C}^2(Y)) \to \Omega_{d-dim(X)}(X).$$

Proposition 2.3 *The maps* $(\widetilde{\square}(\alpha))_*, (\widetilde{C}^2(\alpha))_*, \Upsilon(\alpha), \Theta(\alpha)$ *do not depend on the choice of the decomposition* $\alpha = f \circ g$.

Proof: We give the proof for $\widetilde{C}^2(\alpha)_*$, the case of $(\widetilde{\square}(\alpha))_*$ is completely analogous.

If $\alpha = f_1 \circ g_1$ and $\alpha = f_2 \circ g_2$ are two such decompositions, we can form the third one, where $W_3 = W_1 \times_X W_2$ and $f_3 = f_1 \times f_2$, $g_3 = g_1 \times g_2$. So, we can reduce our question to the situation where there exists a smooth projective morphism $h : W_2 \to W_1$ such that $f_2 = f_1 \circ h$ and $g_1 = h \circ g_2$.

Consider the commutative diagram:

$$\begin{array}{ccccc}
\widetilde{C}^2(Y) & = & \widetilde{C}^2(Y) & = & \widetilde{C}^2(Y) \\
\downarrow{\scriptstyle \widetilde{C}^2(g_1)} & & \downarrow{\scriptstyle \chi} & (2) & \downarrow{\scriptstyle \widetilde{C}^2(g_2)} \\
\widetilde{C}^2(W_1) & \stackrel{d(h)}{\longleftarrow} & Bl_{\widetilde{C}^2(W_2),\widetilde{C}^2(W_2/W_1)} & \stackrel{c(h)}{\longrightarrow} & \widetilde{C}^2(W_2) \\
\uparrow{\scriptstyle c(f_1)} & (3) & \uparrow{\scriptstyle c} & & \uparrow{\scriptstyle c(f_2)} \\
Bl_{\widetilde{C}^2(W_1),\widetilde{C}^2(W_1/X)} & \stackrel{\delta}{\longleftarrow} & Bl_{\widetilde{C}^2(W_2),(\widetilde{C}^2(W_2/W_1),\widetilde{C}^2(W_2/X))} & \stackrel{\epsilon}{\longrightarrow} & Bl_{\widetilde{C}^2(W_2),\widetilde{C}^2(W_2/X)} \\
\downarrow{\scriptstyle d(f_1)} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d(f_2)} \\
\widetilde{C}^2(X) & = & \widetilde{C}^2(X) & = & \widetilde{C}^2(X)
\end{array}$$

Since $g_1 = h \circ g_2$ is a regular imbedding, the subvarieties $\widetilde{C}^2(Y)$ and $\widetilde{C}^2(W_2/W_1)$ of $\widetilde{C}^2(W_2)$ are disjoint. This makes it possible to define χ and shows that the square (2) is proper cartesian. The square (3) is proper cartesian by Statement 5.15. Then,

$$d(f_1)_* \circ c(f_1)^* \circ \widetilde{C}^2(g_1)_* = d_* \circ \epsilon^* \circ c(f_2)^* \circ \widetilde{C}^2(g_2)_*.$$

Since $\epsilon_*[1_{Bl_{\widetilde{C}^2(W_2),(\widetilde{C}^2(W_2/W_1),\widetilde{C}^2(W_2/X))}}] - [1_{Bl_{\widetilde{C}^2(W_2),\widetilde{C}^2(W_2/X)}}]$ is supported on the proper preimage of $\widetilde{C}^2(W_2/W_1)$ under $c(f_2)$, and $\widetilde{C}^2(Y)$ is disjoint from $\widetilde{C}^2(W_2/W_1)$ in $\widetilde{C}^2(W_2)$, we get from the Statement 5.2 that the latter expression is equal to

$$d(f_2)_* \circ c(f_2)^* \circ \widetilde{C}^2(g_2)_*.$$

For the remaining two maps, we treat only the case of $\Theta(\alpha)$, the other one is similar:

$$
\begin{array}{ccccc}
\widetilde{C}^2(Y) & = & \widetilde{C}^2(Y) & = & \widetilde{C}^2(Y) \\
\downarrow{\scriptstyle \widetilde{C}^2(g_1)} & & \downarrow{\scriptstyle \chi} & (2) & \downarrow{\scriptstyle \widetilde{C}^2(g_2)} \\
\widetilde{C}^2(W_1) & \xleftarrow{d(h)} & Bl_{\widetilde{C}^2(W_2),\widetilde{C}^2(W_2/W_1)} & \xrightarrow{c(h)} & \widetilde{C}^2(W_2) \\
\uparrow{\scriptstyle C(f_1)} & (3) & \uparrow{\scriptstyle C} & & \uparrow{\scriptstyle C(f_2)} \\
\widetilde{C}^2(W_1/X) & \xleftarrow{\delta'} & Bl_{\widetilde{C}^2(W_2/X),\widetilde{C}^2(W_2/W_1)} & \xrightarrow{\epsilon'} & \widetilde{C}^2(W_2/X) \\
\downarrow{\scriptstyle D(f_1)} & & \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle D(f_2)} \\
X & = & X & = & X.
\end{array}
$$

Again, square (2) is proper cartesian by the evident reasons, and (3) is proper cartesian by Statement 5.12. Then

$$D(f_1)_* \circ C(f_1)^* \circ \widetilde{C}^2(g_1)_* = D(f_2)_* \circ C(f_2)^* \circ \widetilde{C}^2(g_2)_*,$$

since $image(\widetilde{C}^2(g_2))$ is disjoint from $\widetilde{C}^2(W_2/W_1)$. □

We have natural maps:

$$X \stackrel{\epsilon}{\leftarrow} \mathbb{P}(T_X) \stackrel{j}{\rightarrow} \widetilde{\square}(X).$$

Let us denote as i the composition $\mathbb{P}(T_X) \stackrel{j}{\rightarrow} \widetilde{\square}(X) \stackrel{p}{\rightarrow} \widetilde{C}^2(X)$.

Consider two composable projective maps between smooth quasiprojective varieties:

$$Z \stackrel{\beta}{\rightarrow} Y \stackrel{\alpha}{\rightarrow} X$$

Definition 2.4 *For* $g(s) \in \Omega^*(Y)[[s]]$ *define:*
$\Upsilon^{g(s)}(\beta) : \Omega^*(\widetilde{\square}(Z)) \rightarrow \Omega^*(Y)$ *as*

$$\Upsilon^{g(s)}(\beta)(z) = \sum_{l\geqslant 0} g_l \cdot \Upsilon(\beta)(\rho^l \cdot z), \text{ and}$$

$\Theta^{g(s)}(\beta) : \Omega^*(\widetilde{C}^2(Z)) \rightarrow \Omega^*(Y)$ *as*

$$\Theta^{g(s)}(\beta)(z) := \sum_{l\geqslant 0} g_l \cdot \Theta(\beta)(\varrho^l \cdot z).$$

For $h(s,u) = \sum_{l,m\geqslant 0} h_{l,m} s^l u^m \in \Omega^*(Y)[[s,u]]$ *define:*
$\Upsilon^{h(s,u)}_{\beta,\alpha} : \Omega^*(\widetilde{\square}(Z)) \rightarrow \Omega^*(\widetilde{\square}(X))$ *as*

$$\Upsilon^{h(s,u)}_{\beta,\alpha}(z) := \sum_{l,m\geqslant 0} \rho^m \cdot j_*\varepsilon^*\alpha_*(h_{l,m} \cdot \Upsilon(\beta)(\rho^l \cdot z)),$$

and $\Theta^{h(s,u)}_{\beta,\alpha} : \Omega^*(\widetilde{C}^2(Z)) \rightarrow \Omega^*(\widetilde{C}^2(X))$ *as*

$$\Theta^{h(s,u)}_{\beta,\alpha}(z) := \sum_{l,m\geqslant 0} \varrho^m \cdot i_*\varepsilon^*\alpha_*(h_{l,m} \cdot \Theta(\beta)(\varrho^l \cdot z)).$$

For a virtual vector bundle $\mathcal{V} = \mathcal{V}_1 - \mathcal{V}_2$ on Y let us denote as $c^{\Omega}(\mathcal{V})(t)$ the expression $\frac{\prod_i (\lambda_i -_{\Omega} t)}{\prod_j (\mu_j -_{\Omega} t)} \in \Omega^*(Y)[[t]][t^{-1}]$, where $\lambda_i, \mu_j \in \Omega^1$ are "roots" of $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively (notice, that λ_i, μ_j are nilpotent).

Let ω be the 1-form on $\mathrm{Spec}(\mathbb{L}[[t]])$ invariant under the universal formal group law (with value dt at $t = 0$).

The following Proposition shows that although the maps $\widetilde{\square}(-)_*$ and $\widetilde{C}^2(-)_*$ do not form a functor, the composition of such push-forwards behaves in some controllable way.

Proposition 2.5 *Let the maps α, β be as above. Then*

(1)
$$\widetilde{\square}(\alpha \circ \beta)_* = \widetilde{\square}(\alpha)_* \circ \widetilde{\square}(\beta)_* + \Upsilon^{h(s,u)}_{\beta,\alpha},$$
$$\widetilde{C}^2(\alpha \circ \beta)_* = \widetilde{C}^2(\alpha)_* \circ \widetilde{C}^2(\beta)_* + \Theta^{h(s,u)}_{\beta,\alpha},$$

where
$$h(s,u) = \operatorname*{Res}_{t=0} \frac{c^{\Omega}(-T_{\alpha})(t) \cdot \omega}{(u -_{\Omega} t)(t -_{\Omega} s)}.$$

(2)
$$\Upsilon(\alpha \circ \beta) = \Upsilon(\alpha) \circ \widetilde{\square}(\beta)_* + \alpha_* \circ \Upsilon^{g(s)}(\beta),$$
$$\Theta(\alpha \circ \beta) = \Theta(\alpha) \circ \widetilde{C}^2(\beta)_* + \alpha_* \circ \Theta^{g(s)}(\beta),$$

where
$$g(s) = \operatorname*{Res}_{t=0} \frac{c^{\Omega}(-T_{\alpha})(t) \cdot \omega}{(t -_{\Omega} s)}.$$

Proof: We can include the morphisms α and β into the diagram:

$$\begin{array}{ccccc}
Z & \xrightarrow{g_2} & W_2 & \xrightarrow{g} & W \\
{\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f} \\
Y & = & Y & \xrightarrow{g_1} & W_1 \\
 & & {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle f_1} \\
 & & X & = & X,
\end{array}$$

where the upper right square is (proper) cartesian, morphisms f_1, f_2, f are smooth projective, and morphisms g_1, g_2, g are regular embeddings. Moreover, we can assume that $\dim(W_1) > 2\dim(Z)$.

(1) We consider the case of $\widetilde{C}^2$, the other one is analogous.

Consider commutative diagram:

$$\begin{array}{ccccc}
\widetilde{C}^2(Z) & & & & \\
{\scriptstyle \widetilde{C}^2(g_2)}\downarrow & & & & \\
\widetilde{C}^2(W_2) & \xleftarrow{c(f_2)} & Bl_{\widetilde{C}^2(W_2),\widetilde{C}^2(W_2/Y)} & \xrightarrow{d(f_2)} & \widetilde{C}^2(Y) \\
{\scriptstyle \widetilde{C}^2(g)}\downarrow & & {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle \widetilde{C}^2(g_1)} \\
\widetilde{C}^2(W) & \xleftarrow{c(f)} & Bl_{\widetilde{C}^2(W),\widetilde{C}^2(W/W_1)} & \xrightarrow{d(f)} & \widetilde{C}^2(W_1) \\
{\scriptstyle c(f_1\circ f)}\uparrow & & {\scriptstyle b}\uparrow & (4) & \uparrow{\scriptstyle c(f_1)} \\
Bl_{\widetilde{C}^2(W),\widetilde{C}^2(W/X)} & \xleftarrow{c} & Bl_{\widetilde{C}^2(W),(\widetilde{C}^2(W/W_1),\widetilde{C}^2(W/X))} & \xrightarrow{d} & Bl_{\widetilde{C}^2(W_1),\widetilde{C}^2(W_1/X)} \\
{\scriptstyle d(f_1\circ f)}\downarrow & & & & \downarrow{\scriptstyle d(f_1)} \\
\widetilde{C}^2(X) & = & \widetilde{C}^2(X) & = & \widetilde{C}^2(X)
\end{array}$$

We have: $\widetilde{C}^2(\alpha\circ\beta)_* = d(f_1\circ f)_* c(f_1\circ f)^* \widetilde{C}^2(g)_* \widetilde{C}^2(g_2)_*$. Observe that any element in the image of $c(f_1\circ f)^* \widetilde{C}^2(g)_* \widetilde{C}^2(g_2)_*$ has support of dimension $\leqslant 2\dim(Z)$. On the other hand, the support of

$$(c_*(1_{Bl_{\widetilde{C}^2(W),(\widetilde{C}^2(W/W_1),\widetilde{C}^2(W/X))}}) - 1_{Bl_{\widetilde{C}^2(W),\widetilde{C}^2(W/X)}})$$

has codimension $\geqslant \dim(W_1)$. Since $\dim(W_1) > 2\dim(Z)$, by Statement 5.2,

$$\widetilde{C}^2(\alpha\circ\beta)_* = d(f_1\circ f)_* c_* c^* c(f_1\circ f)^* \widetilde{C}^2(g)_* \widetilde{C}^2(g_2)_* =$$
$$d(f_1)_* d_* c^* c(f_1\circ f)^* \widetilde{C}^2(g)_* \widetilde{C}^2(g_2)_* = d(f_1)_* d_* b^* c(f)^* \widetilde{C}^2(g)_* \widetilde{C}^2(g_2)_*$$

Since the square (4) is proper cartesian by Statement 5.15, the latter expression is equal to

$$d(f_1)_* c(f_1)^* d(f)_* c(f)^* \widetilde{C}^2(g)_* \widetilde{C}^2(g_2)_*.$$

From Proposition 5.27,

$$c(f)^* \widetilde{C}^2(g)_* = a_* c(f_2)^* - i_* e_* \left(\frac{c_r(\mathcal{M}\otimes\mathcal{O}(1)) - c_r(\mathcal{M})}{c_1(\mathcal{O}(-1))} \cdot \varepsilon^* C(f_2)^* \right),$$

where $i : \mathbb{P}_{\widetilde{C}^2(W/W_1)}(\mathcal{N}) \to Bl_{\widetilde{C}^2(W),\widetilde{C}^2(W/W_1)}$, $e : \mathbb{P}_{\widetilde{C}^2(W_2/Y)}((\widetilde{C}^2(g/g_1))^*\mathcal{N}) \to \mathbb{P}_{\widetilde{C}^2(W/W_1)}(\mathcal{N})$, $\varepsilon : \mathbb{P}_{\widetilde{C}^2(W_2/Y)}((\widetilde{C}^2(g/g_1))^*\mathcal{N}) \to \widetilde{C}^2(W_2/Y)$ are the natural maps, $\mathcal{N} = \mathcal{N}_{\widetilde{C}^2(W/W_1)\subset\widetilde{C}^2(W)} = T_{W_1}\otimes\mathcal{L}$, $r = \dim(W_1) - \dim(Y)$, and $\mathcal{M} = \mathcal{N}_{Y\subset W_1}\otimes\mathcal{L}$.

Observe, that

$$d(f_1)_* c(f_1)^* d(f)_* a_* c(f_2)^* \widetilde{C}^2(g_2)_* = d(f_1)_* c(f_1)^* \widetilde{C}^2(g_1)_* d(f_2)_* c(f_2)^* \widetilde{C}^2(g_2)_* = (\widetilde{C}^2(\alpha))_* \circ (\widetilde{C}^2(\beta))_*.$$

Since the support of any element in the image of $C(f_2)^* \widetilde{C}^2(g_2)_*$ has dimension $\leqslant 2\dim(Z) - \dim(Y)$, and $c_r(\mathcal{M})$ has support of codimension $\geqslant \dim(W_1) - \dim(Y)$, which is greater, by Statement 5.2, we can omit the corresponding term.

Consider commutative diagram

$$\begin{array}{ccccc}
\mathbb{P}_{\widetilde{C}^2(W_2/Y)}(\widetilde{C}^2(g/g_1)^* T_{W_1}\otimes\mathcal{L}) & \xrightarrow{e} & \mathbb{P}_{\widetilde{C}^2(W/W_1)}(T_{W_1}\otimes\mathcal{L}) & \xrightarrow{i} & Bl_{\widetilde{C}^2(W),\widetilde{C}^2(W/W_1)} \\
{\scriptstyle D_2}\downarrow & & {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle d(f)} \\
\mathbb{P}_Y(g_1^* T_{W_1}) & \xrightarrow[e_1]{} & \mathbb{P}_{W_1}(T_{W_1}) & \xrightarrow[i_1]{} & \widetilde{C}^2(W_1)
\end{array}$$

We have: $d(f)_* i_* e_* = (i_1)_*(e_1)_*(D_2)_*$. We also have the diagram:

$$\begin{array}{ccccc} \mathbb{P}_{\widetilde{C}^2(W_2/Y)}(\widetilde{C}^2(g/g_1)^* T_{W_1} \otimes \mathcal{L}) & \xrightarrow{D_2} & \mathbb{P}_Y(g_1^* T_{W_1}) & \xrightarrow{e_1} & \mathbb{P}_{W_1}(T_{W_1}) \\ \varepsilon \downarrow & & \downarrow \eta & & \downarrow \varepsilon_1 \\ \widetilde{C}^2(W_2/Y) & \xrightarrow[D(f_2)]{} & Y & \xrightarrow[g_1]{} & W_1, \end{array}$$

where both squares are proper cartesian (vertical maps are smooth). Then, since $(D_2)^*(\mathcal{O}(1)) = \mathcal{O}(1) \otimes \varepsilon^*(\mathcal{L})$, we get:

$$(D_2)_* \left(\frac{c_r(\mathcal{M} \otimes \mathcal{O}(1))}{c_1(\mathcal{O}(-1))} \cdot \varepsilon^*(-) \right) = (D_2)_* \left(\frac{(D_2)^*(c_r(\mathcal{N}_{Y \subset W_1} \otimes \mathcal{O}(1)))}{((D_2)^* c_1(\mathcal{O}(-1)) -_\Omega \varepsilon^*(\varrho))} \cdot \varepsilon^*(-) \right).$$

Let $\kappa(s,t) = \frac{c^\Omega(\mathcal{N}_{Y \subset W_1})(t)}{(t -_\Omega s)}$. Then

$$\begin{aligned} d(f)_* i_* e_* \left(\frac{c_r(\mathcal{M} \otimes \mathcal{O}(1))}{c_1(\mathcal{O}(-1))} \cdot \varepsilon^* C(f_2)^* \widetilde{C}^2(g_2)_*([v]) \right) &= \\ (i_1)_*(e_1)_*(D_2)_* \Big(\kappa\big(\varepsilon^*(\varrho), c_1(\mathcal{O}(-1))\big) \cdot \varepsilon^* C(f_2)^* \widetilde{C}^2(g_2)_*([v]) \Big) &= \\ (i_1)_* \sum_{l,m} c_1(\mathcal{O}(-1))^m \varepsilon_1^*(g_1)_* \Big(\kappa_{l,m} \cdot D(f_2)_* C(f_2)^* \widetilde{C}^2(g_2)_*([v] \cdot \varrho^l) \Big) &= \Theta^{\kappa(s,t)}_{\beta, g_1}([v]). \end{aligned}$$

Consider the diagram

$$\begin{array}{ccccc} \mathbb{P}_{W_1}(T_{W_1}) & \xleftarrow{c'} & Bl_{\mathbb{P}_{W_1}(T_{W_1}), \mathbb{P}_{W_1}(T_{W_1/X})} & \xrightarrow{d'} & \mathbb{P}_X(T_X) \\ i_1 \downarrow & & \downarrow \tilde{i} & & \downarrow i_X \\ \widetilde{C}^2(W_1) & \xleftarrow[c(f_1)]{} & Bl_{\widetilde{C}^2(W_1), \widetilde{C}^2(W_1/X)} & \xrightarrow[d(f_1)]{} & \widetilde{C}^2(X), \end{array}$$

where the left square is proper cartesian by Statement 5.16. Taking into account that $i_1^*(\varrho) = c_1(\mathcal{O}(-1))$, we get:

$$d(f_1)_* c(f_1)^* \Theta^{\kappa(s,t)}_{\beta, g_1}([v]) = \sum_{l,m} (i_X)_*(d')_*(c')^*(e_1)_* \Big(c_1(\mathcal{O}(-1))^m \cdot \eta^* \Big(\kappa_{l,m} \cdot \Theta(\beta)([v] \cdot \varrho^l) \Big) \Big).$$

From the commutative diagram

$$\begin{array}{ccccc} \mathbb{P}_{W_1}(T_{W_1}) & \xleftarrow{c'} & Bl_{\mathbb{P}_{W_1}(T_{W_1}), \mathbb{P}_{W_1}(T_{W_1/X})} & \xrightarrow{d'} & \mathbb{P}_X(T_X) \\ e_1 \uparrow & & \uparrow \tilde{e} & & \uparrow e_2 \\ \mathbb{P}_Y(g_1^* T_{W_1}) & \xleftarrow[c'']{} & Bl_{\mathbb{P}_Y(g_1^* T_{W_1}), \mathbb{P}_Y(g_1^* T_{W_1/X})} & \xrightarrow[d'']{} & \mathbb{P}_Y(\alpha^* T_X) \end{array}$$

where both square are proper cartesian by Lemma 5.5, we get: $(d')_*(c')^*(e_1)_* = (e_2)_*(d'')_*(c'')^*$. Since $Bl_{\mathbb{P}_Y(g_1^* T_{W_1}), \mathbb{P}_Y(g_1^* T_{W_1/X}))} = \mathbb{P}_{\mathbb{P}_Y(\alpha^* T_X)}(\mathcal{V})$, where $\mathcal{V}$ fits into exact sequence $0 \to (g_1 \circ \varepsilon'')^* T_{W_1/X} \to \mathcal{V} \to \mathcal{O}(-1) \to 0$, and $\mathcal{O}(-1)$ on $\mathbb{P}_{\mathbb{P}_Y(\alpha^* T_X)}(\mathcal{V})$ is the pull-back $(c'')^*(\mathcal{O}(-1))$ - see Statement 5.37, by the Theorem 5.35 (Quillen's formula), we get:

$$\sum_{l,m} (i_X)_*(d')_*(c')^*(e_1)_* \Big(c_1(\mathcal{O}(-1))^m \cdot \eta^* \Big(\kappa_{l,m} \cdot \Theta(\beta)([v] \cdot \varrho^l) \Big) \Big) = \Theta^{-h(s,u)}_{\beta, \alpha}([v]),$$

where

$$h(s,u) = \operatorname*{Res}_{t=0}\left(\omega \cdot c^{\Omega}(-T_{W_1/X})(t)(u -_{\Omega} t)^{-1}\kappa(s,t)\right) = \operatorname*{Res}_{t=0} \frac{\omega \cdot c^{\Omega}(-T_{\alpha})(t)}{(t -_{\Omega} s)(u -_{\Omega} t)}.$$

Here we used the fact that $([-1]_{\Omega})^*(\omega) = -\omega$.

(2) Again, we consider the case of Θ, the other one is analogous.

Consider commutative diagram:

$$\begin{array}{ccccc}
\widetilde{C}^2(Z) & & & & \\
\downarrow{\scriptstyle \widetilde{C}^2(g_2)} & & & & \\
\widetilde{C}^2(W_2) & \xleftarrow{c(f_2)} & Bl_{\widetilde{C}^2(W_2),\widetilde{C}^2(W_2/Y)} & \xrightarrow{d(f_2)} & \widetilde{C}^2(Y) \\
\downarrow{\scriptstyle \widetilde{C}^2(g)} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \widetilde{C}^2(g_1)} \\
\widetilde{C}^2(W) & \xleftarrow{c(f)} & Bl_{\widetilde{C}^2(W),\widetilde{C}^2(W/W_1)} & \xrightarrow{d(f)} & \widetilde{C}^2(W_1) \\
\uparrow{\scriptstyle C(f_1\circ f)} & & \uparrow{\scriptstyle B} & (4) & \uparrow{\scriptstyle C(f_1)} \\
\widetilde{C}^2(W/X) & \xleftarrow{C} & Bl_{\widetilde{C}^2(W/X),\widetilde{C}^2(W/W_1)} & \xrightarrow{D} & \widetilde{C}^2(W_1/X) \\
\downarrow{\scriptstyle D(f_1\circ f)} & & & & \downarrow{\scriptstyle D(f_1)} \\
X & = & X & = & X
\end{array}$$

We have: $\Theta(\alpha \circ \beta) = D(f_1 \circ f)_* C(f_1 \circ f)^* \widetilde{C}^2(g)_* \widetilde{C}^2(g_2)_*$. Observe that any element in the image of $C(f_1 \circ f)^* \widetilde{C}^2(g)_* \widetilde{C}^2(g_2)_*$ has support of dimension $\leqslant 2\dim(Z) - \dim(X)$. On the other hand, the support of

$$(C_*(1_{Bl_{\widetilde{C}^2(W/X),\widetilde{C}^2(W/W_1)}}) - 1_{\widetilde{C}^2(W/X)})$$

has codimension $\geqslant \dim(W_1) - \dim(X)$. Since $\dim(W_1) > 2\dim(Z)$, by Statement 5.2

$$\Theta(\alpha \circ \beta) = D(f_1 \circ f)_* C_* C^* C(f_1 \circ f)^* \widetilde{C}^2(g)_* \widetilde{C}^2(g_2)_* =$$
$$D(f_1)_* D_* C^* C(f_1 \circ f)^* \widetilde{C}^2(g)_* \widetilde{C}^2(g_2)_* = D(f_1)_* D_* B^* c(f)^* \widetilde{C}^2(g)_* \widetilde{C}^2(g_2)_*$$

Since the square (4) is proper cartesian by Statement 5.12, the latter expression is equal to

$$D(f_1)_* C(f_1)^* d(f)_* c(f)^* \widetilde{C}^2(g)_* \widetilde{C}^2(g_2)_*.$$

As we saw above,

$$c(f)^* \widetilde{C}^2(g)_* = a_* c(f_2)^* - i_* e_* \left(\frac{c_r(\mathcal{M} \otimes \mathcal{O}(1))}{c_1(\mathcal{O}(-1))} \cdot \varepsilon^* C(f_2)^* \right).$$

Observe, that

$$D(f_1)_* C(f_1)^* d(f)_* a_* c(f_2)^* \widetilde{C}^2(g_2)_* = D(f_1)_* C(f_1)^* \widetilde{C}^2(g_1)_* d(f_2)_* c(f_2)^* \widetilde{C}^2(g_2)_* = \Theta(\alpha) \circ (\widetilde{C}^2(\beta))_*.$$

Again,

$$d(f)_* i_* e_* \left(\frac{c_r(\mathcal{M} \otimes \mathcal{O}(1))}{c_1(\mathcal{O}(-1))} \cdot \varepsilon^* C(f_2)^* \widetilde{C}^2(g_2)_*([v]) \right) = \Theta^{\kappa(s,t)}_{\beta,g_1}([v]),$$

where $\kappa(s,t) = \frac{c^{\Omega}(N_{Y \subset W_1})(t)}{(t -_{\Omega} s)}$.

Consider the diagram

$$\begin{array}{ccccc} \mathbb{P}_{W_1}(T_{W_1}) & \xleftarrow{C'} & \mathbb{P}_{W_1}(T_{W_1/X}) & \xrightarrow{D'} & X \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle i'} & & \| \\ \widetilde{C}^2(W_1) & \xleftarrow[C(f_1)]{} & \widetilde{C}^2(W_1/X) & \xrightarrow[D(f_1)]{} & X, \end{array}$$

where the left square is proper cartesian by Statement 5.16. Taking into account that $i_1^*(\varrho) = c_1(\mathcal{O}(-1))$, we get:

$$D(f_1)_* C(f_1)^* \Theta^{\kappa(s,t)}_{\beta,g_1}([v]) = \sum_{l,m} (D')_*(C')^*(e_1)_*\Big(c_1(\mathcal{O}(-1))^m \cdot \eta^*\big(\kappa_{l,m} \cdot \Theta(\beta)([v] \cdot \varrho^l)\big)\Big).$$

And by the Theorem 5.35 (Quillen's formula), the latter expression is equal to $\Theta^{-g(s)}_{\beta,\alpha}$, where

$$g(s) = \operatorname*{Res}_{t=0}\Big(\omega \cdot c^{\Omega}(-T_{W_1/X})(t)\kappa(s,t)\Big) = \operatorname*{Res}_{t=0} \frac{\omega \cdot c^{\Omega}(-T_\alpha)(t)}{(t -_{\Omega} s)}.$$

□

2.4 Symmetric operations

Let X be smooth quasiprojective variety, and

$$z = [v : V \to X] - [u : U \to X]$$

be arbitrary representative of some class from $\Omega_r(X)$. Let us define

$$mor(z) : var(z) \to X \quad \text{as} \quad v \coprod u \coprod u : V \coprod U \coprod U \to X.$$

In particular, if $z = [v : V \to X]$ is effective, then $mor(z) : var(z) \to X$ is just $v : V \to X$. Let us also define

$$cl_{\widetilde{C}^2}(z) \in \Omega^0(\widetilde{C}^2(var(z))) \quad \text{as} \quad 1_{\widetilde{C}^2(V)} - 1_{\widetilde{C}^2(U_1)} + 1_{U_1\times U_2} - 1_{V\times U_1},$$

and $cl_{\widetilde{\square}}(z) \in \Omega^0(\widetilde{\square}(var(z)))$ as $p^*(cl_{\widetilde{C}^2}(z))$. In particular, if $z = [v : V \to X]$ is effective, then $cl_{\widetilde{C}^2}(z) = 1_{\widetilde{C}^2(V)}$, and $cl_{\widetilde{\square}}(z) = 1_{\widetilde{\square}(V)}$.

Definition 2.6 *Let $q(t) \in \mathbb{L}[[t]]$. Define the operations:*

$$\widetilde{\square}^{q(t)}(z) := \widetilde{\square}(mor(z))_*(cl_{\widetilde{\square}(z)} \cdot q(\rho)) \in \Omega^*(\widetilde{\square}(X));$$

$$(\widetilde{C}^2)^{q(t)}(z) := \widetilde{C}^2(mor(z))_*(cl_{\widetilde{C}^2(z)} \cdot q(\varrho)) \in \Omega^*(\widetilde{C}^2(X));$$

$$\Psi^{q(t)}(z) := \Upsilon(mor(z))(cl_{\widetilde{\square}(z)} \cdot q(\rho)) \in \Omega^*(X);$$

$$\Phi^{q(t)}(z) := \Theta(mor(z))(cl_{\widetilde{C}^2(z)} \cdot q(\varrho)) \in \Omega^*(X);$$

The aim of the current subsection is to show that $\widetilde{\square}$, $\widetilde{C}^2$, Ψ and Φ are well-defined operations on $\Omega^*(X)$.

We have the pairings:

$$(-,-)_{\widetilde{\square}} : \Omega^*(X) \otimes_{\mathbb{L}} \Omega^*(X) \to \Omega^*(\widetilde{\square}(X))$$

and

$$(-,-)_{\widetilde{C}^2} : \Omega^*(X) \otimes_{\mathbb{L}} \Omega^*(X) \to \Omega^*(\widetilde{C}^2(X)),$$

where $([y],[z])_{\widetilde{\square}} := \pi^*([y] \times [z])$, and $([y],[z])_{\widetilde{C}^2} := p_*([y],[z])_{\widetilde{\square}}$.

Let $y : Y \to X$ and $z : Z \to X$ be projective morphisms between smooth quasiprojective varieties. Let $y \coprod z = f \circ g$ be a decomposition such that $g : Y \coprod Z \to W$ is a regular embedding, and $f : W \to X$ be smooth projective. Then $g_Y \times g_Z : Y \times Z \to \square(W)$ is a regular embedding which does not meet the diagonal, and so can be lifted to $\widetilde{\square}(W)$. We get a class $[\widetilde{Y \times Z}] \in \Omega^*(\widetilde{\square}(W))$.

Proposition 2.7

$$([y],[z])_{\widetilde{\square}} = b(f)_* a(f)^*([\widetilde{Y \times Z}]), \quad ([y],[z])_{\widetilde{C}^2} = d(f)_* c(f)^*(p_*[\widetilde{Y \times Z}]);$$

$$B(f)_* A(f)^*([\widetilde{Y \times Z}]) = D(f)_* C(f)^*(p_*[\widetilde{Y \times Z}]) = [y] \cdot [z].$$

Proof: Consider the diagram

$$\begin{array}{ccccc}
Bl_{\widetilde{\square}(W),\widetilde{\square}(W/X)} & \xrightarrow{\alpha} & Bl_{\square(W),\square(W/X)} & \xrightarrow{\beta} & \widetilde{\square}(X) \\
{\scriptstyle a(f)}\downarrow & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \pi_X}. \\
\widetilde{\square}(W) & \xrightarrow[\pi_W]{} & \square(W) & \xrightarrow[\square(f)]{} & \square(X)
\end{array}$$

Here $\beta \circ \alpha = b(f)$ and $[\widetilde{Y \times Z}] = \pi_W^*([Y \times Z]$. Thus, $b(f)_* a(f)^*([\widetilde{Y \times Z}]) = \beta_* \alpha_* \alpha^* \varepsilon^*([Y \times Z]) = \beta_*(\alpha_*(1_{Bl_{\widetilde{\square}(W),\widetilde{\square}(W/X)}}) \cdot \varepsilon^*([Y \times Z]))$. But the difference $\alpha_*(1_{Bl_{\widetilde{\square}(W),\widetilde{\square}(W/X)}}) - 1_{Bl_{\square(W),\square(W/X)}}$ is supported on the preimage $\varepsilon^{-1}(\Delta(W))$, and by Statement 5.2, $\alpha_*(1_{Bl_{\widetilde{\square}(W),\widetilde{\square}(W/X)}}) \cdot \varepsilon^*([Y \times Z]) = \varepsilon^*([Y \times Z])$, since $Y \times Z$ does not meet the diagonal. Since the right square is proper cartesian by Lemma 5.5, $\beta_* \varepsilon^*([Y \times Z]) = \pi_X^* \square(f)_*([Y \times Z]) = ([y],[z])_{\widetilde{\square}}$.

Consider the diagram

$$\begin{array}{ccccc}
\widetilde{\square}(W) & \xleftarrow{a(f)} & Bl_{\widetilde{\square}(W),\widetilde{\square}(W/X)} & \xrightarrow{b(f)} & \widetilde{\square}(X) \\
{\scriptstyle p_W}\downarrow & & {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle p_X} \\
\widetilde{C}^2(W) & \xleftarrow{c(f)} & Bl_{\widetilde{C}^2(W),\widetilde{C}^2(W/X)} & \xrightarrow{d(f)} & \widetilde{C}^2(X)
\end{array}$$

The left square here is proper cartesian by Statement 5.11. This implies that

$$d(f)_* c(f)^*((p_W)_*[\widetilde{Y \times Z}]) = (p_X)_*(b(f)_* a(f)^*([\widetilde{Y \times Z}]) = (y,z)_{\widetilde{C}^2}.$$

Analogously, consider the diagram

$$\begin{array}{ccccc}
\widetilde{\square}(W/X) & \xrightarrow{\pi_{W/X}} & \square(W/X) & \xrightarrow{\square(f/id)} & X \\
{\scriptstyle A(f)}\downarrow & & \downarrow{\scriptstyle A} & & \downarrow{\scriptstyle \Delta} \\
\widetilde{\square}(W) & \xrightarrow[\pi_W]{} & \square(W) & \xrightarrow[\square(f)]{} & \square(X).
\end{array}$$

Here $\square(f/id) \circ \pi_{W/X} = B(f)$, and the difference $(1_{\square(W/X)} - (\pi_{W/X})_*(1_{\widetilde{\square}(W/X)}))$ is supported on the preimage $A^{-1}(\Delta(W))$. Thus, by Statement 5.2,

$$B(f)_* A(f)^*([\widetilde{Y \times Z}]) = B(f)_* A(f)^* \pi_W^*([Y \times Z]) =$$
$$\square(f/id)_*(\pi_{W/X})_*(\pi_{W/X})^* A^*([Y \times Z]) = \square(f/id)_* A^*([Y \times Z]).$$

And since the right square is proper cartesian, this is equal to $[y] \cdot [z]$. Since the left square in the diagram

$$\begin{array}{ccccc} \widetilde{\square}(W) & \xleftarrow{A(f)} & \widetilde{\square}(W/X) & \xrightarrow{B(f)} & X \\ p_W \downarrow & & p_{W/X} \downarrow & & \| \\ \widetilde{C}^2(W) & \xleftarrow{C(f)} & \widetilde{C}^2(W/X) & \xrightarrow{D(f)} & X \end{array}$$

is proper cartesian by Statement 5.11, we get: $D(f)_* C(f)^*(p_*[\widetilde{Y \times Z}]) = B(f)_* A(f)^*([\widetilde{Y \times Z}]) = [y] \cdot [z]$. $\square$

Proposition 2.8 *Let $[y], [z]$ be formal differences $[y_1] - [y_2]$, $[z_1] - [z_2]$, where $y_i : Y_i \to X$, $z_j : Z_j \to X$ be projective maps between smooth quasiprojective varieties. Then*

$$\widetilde{\square}^{q(t)}([y] + [z]) = \widetilde{\square}^{q(t)}([y]) + \widetilde{\square}^{q(t)}([z]) + q(0) \cdot (([y], [z])_{\widetilde{\square}} + ([z], [y])_{\widetilde{\square}});$$

$$(\widetilde{C}^2)^{q(t)}([y] + [z]) = (\widetilde{C}^2)^{q(t)}([y]) + (\widetilde{C}^2)^{q(t)}([z]) + q(0) \cdot ([y], [z])_{\widetilde{C}^2};$$

$$\Psi^{q(t)}([y] + [z]) = \Psi^{q(t)}([y]) + \Psi^{q(t)}([z]) + 2q(0) \cdot [y] \cdot [z];$$

$$\Phi^{q(t)}([y] + [z]) = \Phi^{q(t)}([y]) + \Phi^{q(t)}([z]) + q(0) \cdot [y] \cdot [z];$$

Proof: Decompose

$$y_1 \coprod z_1 \coprod (y_2)_1 \coprod (y_2)_2 \coprod (z_2)_1 \coprod (z_2)_2 : Y_1 \coprod Z_1 \coprod (Y_2)_1 \coprod (Y_2)_2 \coprod (Z_2)_1 \coprod (Z_2)_2 \to X$$

as $f \circ g$, where g is a regular embedding, and $f : W \to X$ is smooth projective. Now the statement follows from the definition of operations, Proposition 2.7, and the fact that $[\widetilde{Y \times Z}] \cdot \rho = 0$. $\square$

Proposition 2.9 *Let $v_0 : V_0 \to X$ and $v_1 : V_1 \to X$ be elementary cobordant. Then*

$$\widetilde{\square}^{q(t)}([v_0]) = \widetilde{\square}^{q(t)}([v_1]), \quad (\widetilde{C}^2)^{q(t)}([v_0]) = (\widetilde{C}^2)^{q(t)}([v_1]),$$

$$\Psi^{q(t)}([v_0]) = \Psi^{q(t)}([v_1]) \quad \text{and} \quad \Phi^{q(t)}([v_0]) = \Phi^{q(t)}([v_1]).$$

And so, we get a well defined operations:

$$\widetilde{\square}^{q(t)} : (\mathrm{Pre}-\Omega)^*(X) \to \Omega^*(\widetilde{\square}(X));$$

$$(\widetilde{C}^2)^{q(t)} : (\mathrm{Pre}-\Omega)^*(X) \to \Omega^*(\widetilde{C}^2(X));$$

$$\Psi^{q(t)} : (\mathrm{Pre}-\Omega)^*(X) \to \Omega^*(X);$$

$$\Phi^{q(t)} : (\mathrm{Pre}-\Omega)^*(X) \to \Omega^*(X).$$

Proof: We give the proof for the operations $\widetilde{C}^2$ and Φ.

Since $[v_0]$ and $[v_1]$ are elementary cobordant, there exists projective map $t : T \to X \times \mathbb{A}^1$, such that T is smooth quasiprojective, t is transversal to $X \times \{0\}$ and $X \times \{1\}$, and $v_0 = t|_{t^{-1}(X \times \{0\})}$, $v_1 = t|_{t^{-1}(X \times \{1\})}$.

Since T is quasi-projective, we have an embedding $i : T \hookrightarrow \mathbb{P}^n$. Take $W = \mathbb{P}^n \times X$. Then the map $(i, t) : T \to W \times \mathbb{A}^1$ is projective, transversal to $W \times \{0\}$ and $W \times \{1\}$, and its restrictions to the fibers over $\{0\}$ and $\{1\}$ are the maps $g_0 := (i_0, v_0) : V_0 \to W$ and $g_1 := (i_1, v_1) : V_1 \to W$, where i_0 and i_1 are the compositions $V_0 \to T \xrightarrow{i} \mathbb{P}^n$, and $V_1 \to T \xrightarrow{i} \mathbb{P}^n$, respectively. Since the maps g_0 and g_1 are projective and embeddings simultaneously, these are regular embeddings. From the construction, the classes $[g_0]$ and $[g_1]$ are elementary cobordant, and $v_0 = f \circ g_0$, $v_1 = f \circ g_1$, where $f : W \to X$ is the projection.

Lemma 2.10 *Let $\bar{t} : T \to W \times \mathbb{A}^1$ be a regular embedding of smooth quasiprojective varieties, which is transversal to $W \times \{0\}$ and $W \times \{1\}$.*
Let $g_0 : V_0 \to W$, $g_1 : V_1 \to W$ be the restrictions to the respective fibers. Then $[\widetilde{\square}(g_0)]$ is elementary cobordant to $[\widetilde{\square}(g_1)]$ and $[\widetilde{C}^2(g_0)]$ is elementary cobordant to $[\widetilde{C}^2(g_1)]$.

Proof: The morphism $\nu := pr_2 \circ \bar{t} : T \to \mathbb{A}^1$ is smooth in the neigbourhood U of $\{0\}$ and $\{1\}$. Let $T(U) = \bar{t}^{-1}(U)$. Then $\widetilde{\square}(T(U)/U)$ is a smooth subvariety of $\widetilde{\square}(W \times \mathbb{A}^1/\mathbb{A}^1) = \widetilde{\square}(W) \times \mathbb{A}^1$, and $\widetilde{C}^2(T(U)/U)$ is a smooth subvariety of $\widetilde{C}^2(W \times \mathbb{A}^1/\mathbb{A}^1) = \widetilde{C}^2(W) \times \mathbb{A}^1$. Moreover, if $\overline{\widetilde{\square}(T(U)/U)}$ and $\overline{\widetilde{C}^2(T(U)/U)}$ are the closures of the respective varieties, then $pr_2(\overline{\widetilde{\square}(T(U)/U)}\backslash\widetilde{\square}(T(U)/U))$ and $pr_2(\overline{\widetilde{C}^2(T(U)/U)}\backslash\widetilde{C}^2(T(U)/U))$ belong to $\mathbb{A}^1\backslash U$. Thus, for A and B the resolutions of singularities of $\overline{\widetilde{\square}(T(U)/U)}$ and $\overline{\widetilde{C}^2(T(U)/U)}$, the natural maps $a : A \to \widetilde{\square}(W) \times \mathbb{A}^1$ and $b : B \to \widetilde{C}^2(W) \times \mathbb{A}^1$ are transversal to the fibers over $\{0\}$ and $\{1\}$, and the restrictions to those fibers will be $[\widetilde{\square}(g_0)]$ and $[\widetilde{\square}(g_1)]$, for a, and $[\widetilde{C}^2(g_0)]$ and $[\widetilde{C}^2(g_1)]$, for b. $\square$

From Lemma 2.10, $[\widetilde{C}^2(g_0)]$ is elementary cobordant to $[\widetilde{C}^2(g_1)]$, and

$$(\widetilde{C}^2)^{q(t)}([v_0]) = d(f)_*c(f)^*([\widetilde{C}^2(g_0)] \cdot q(\varrho)) = d(f)_*c(f)^*([\widetilde{C}^2(g_1)] \cdot q(\varrho)) = (\widetilde{C}^2)^{q(t)}([v_1]),$$

and analogously,

$$\Phi^{q(t)}([v_0]) = D(f)_*C(f)^*([\widetilde{C}^2(g_0)] \cdot q(\varrho)) = D(f)_*C(f)^*([\widetilde{C}^2(g_1)] \cdot q(\varrho)) = \Phi^{q(t)}([v_1]),$$

Since $(\mathrm{Pre}-\Omega)^*(X)$ is obtained from $\mathcal{M}^*(X)^+$ by moding out the subgroup generated by the elementary cobordism relations, using Proposition 2.8, we get that operations $\widetilde{\square}$, $\widetilde{C}^2$, Ψ and Φ are well-defined on $(\mathrm{Pre}-\Omega)^*(X)$. $\square$

From now on we will work only with the effective classes $[v : V \to X]$. Then our operations look simply as:

$$\Psi^{q(t)}([v]) = \Upsilon(v)(q(\rho)), \text{ and } \Phi^{q(t)}([v]) = \Theta(v)(q(\varrho));$$
$$(\widetilde{\square})^{q(t)}([v]) = (\widetilde{\square}(v))_*(q(\rho)), \text{ and } (\widetilde{C}^2)^{q(t)}([v]) = (\widetilde{C}^2(v))_*(q(\varrho)).$$

Proposition 2.11 *Let $\gamma : Y \to Y'$ be a composition of the regular and open embedding of smooth varieties, and $v' : V' \to Y' \in (\mathrm{Pre}-\Omega)^*(Y')$ is transversal to γ. Let $v \in (\mathrm{Pre}-\Omega)^*(Y)$ be the transversal preimage of v'. Then*

$$\widetilde{\square}^{q(t)}([v]) = (\widetilde{\square}(\gamma))^*(\widetilde{\square}^{q(t)}([v'])), \quad \textit{and} \quad (\widetilde{C}^2)^{q(t)}([v]) = (\widetilde{C}^2(\gamma))^*((\widetilde{C}^2)^{q(t)}([v'])).$$
$$\Psi^{q(t)}([v]) = \gamma^*(\Psi^{q(t)}([v'])), \quad \textit{and} \quad \Phi^{q(t)}([v]) = \gamma^*(\Phi^{q(t)}([v'])).$$

Proof: We consider the case of $\widetilde{C}^2$ and Φ, the over two are similar. Let $v' = f' \circ g'$, where $g' : V' \to W'$ is a regular embedding, and $f' : W' \to Y'$ be a smooth projective morphism. Then f' is transversal to γ, and we get a decomposition $v = g \circ f$, where W is the preimage of W'. Consider the diagram:

$$\begin{array}{ccccccc}
\widetilde{C}^2(V) & \xrightarrow{\widetilde{C}^2(g)} & \widetilde{C}^2(W) & \xleftarrow{c(f)} & Bl_{\widetilde{C}^2(W),\widetilde{C}^2(W/Y)} & \xrightarrow{d(f)} & \widetilde{C}^2(Y) \\
{\scriptstyle \widetilde{C}^2(\gamma|_V)}\downarrow & & \downarrow{\scriptstyle \widetilde{C}^2(\gamma|_W)} & & \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \widetilde{C}^2(\gamma)} \\
\widetilde{C}^2(V') & \xrightarrow[\widetilde{C}^2(g')]{} & \widetilde{C}^2(W') & \xleftarrow[c(f')]{} & Bl_{\widetilde{C}^2(W'),\widetilde{C}^2(W'/Y')} & \xrightarrow[d(f')]{} & \widetilde{C}^2(Y').
\end{array}$$

Since the first and the last square here are proper cartesian, by Statements 5.10 and 5.13, and $\widetilde{C}^2(\gamma|_V)^*(\varrho_{V'}) = \varrho_V$, the statement for $\widetilde{C}^2$ follows.

For Φ consider the diagram:

$$\begin{array}{ccccccc} \widetilde{C}^2(V) & \xrightarrow{\widetilde{C}^2(g)} & \widetilde{C}^2(W) & \xleftarrow{C(f)} & \widetilde{C}^2(W/Y) & \xrightarrow{D(f)} & Y \\ \downarrow{\scriptstyle \widetilde{C}^2(\gamma|_V)} & & \downarrow{\scriptstyle \widetilde{C}^2(\gamma|_W)} & & \downarrow{\scriptstyle \widetilde{C}^2(\gamma|_{(W/Y)})} & & \downarrow{\scriptstyle \gamma} \\ \widetilde{C}^2(V') & \xrightarrow[\widetilde{C}^2(g')]{} & \widetilde{C}^2(W') & \xleftarrow[C(f')]{} & \widetilde{C}^2(W'/Y') & \xrightarrow[D(f')]{} & Y'. \end{array}$$

Again, the statement follows from the fact that the first and the last square are proper cartesian (by Statement 5.14). □

In algebraic cobordism we have the action of the Landweber-Novikov operations. Usually such operations are parametrized by *partitions*, that is, non-ordered sets of natural numbers $\overline{a} = (a_1, \ldots, a_r)$. To such set one can assign a minimal symmetric polynomial $R_{\overline{a}}(\sigma_1, \ldots, \sigma_r)$ containing the monomial $x_1^{a_1} \cdot \ldots \cdot x_r^{a_r}$. Then the respective Landweber-Novikov operation acts as:

$$S^{\overline{a}}_{L.-N.}([v : V \to X]) := v_*(R_{\overline{a}}(c_1, \ldots, c_r)[1_V]),$$

where $c_i = c_i(-T_V + v^*(T_X))$. It gives an operation $\Omega^*(X) \to \Omega^{*+|\overline{a}|}(X)$, where $|\overline{a}| = \sum_i a_i$. For us it will be more convenient to use another parametrization

Definition 2.12 *Let $g \in \Omega^*(X)[\sigma_1, \ldots, \sigma_r]$, be some polynomial. Denote as $S^g_{L.-N.} : \Omega^*(X) \to \Omega^*(X)$ the Landweber-Novikov operation*

$$S^g_{L.-N.}([v : V \to X]) := v_*(g(c_1, c_2, \ldots)[1_V]),$$

where $c_i = c_i(-T_V + v^*(T_X))$. Clearly, $S^g_{L.-N.}$ is just the $\Omega^*(X)$-linear combination of $S^{\overline{a}}_{L.-N.}$'s.

If $h(u) \in \Omega^*(X)[\sigma_1, \ldots, \sigma_r][[u]]$, denote as $S^{h(u)}_{L.-N.} : \Omega^*(X) \to \Omega^*(\widetilde{\square}(X))$ the sum $\sum_l \rho^l \cdot j_* \varepsilon^* S^{h_l}_{L.-N.}$, where $\rho = c_1(\mathcal{O}(-1))$, and the maps are from the diagram (1).

The following statement shows that the operations $\widetilde{\square}$ and Ψ are expressible in terms of the operation (internal and external) square and the Landweber-Novikov operations.

Let $[v : V \to X] \in \Omega^d(X)$ be some representative of the cobordism class. Let us denote as $c^{\Omega}(-T)(t)$ the following expression: $\frac{\prod_{i=1}^{\dim(X)}(\lambda_i -_{\Omega} t)}{\prod_{j=1}^{\dim(V)}(\mu_j -_{\Omega} t)}$, where λ_i, μ_j are formal nilpotent parameters. It clearly depends only on $\dim(X)$ and $\dim(V)$, and can be presented as an element of $\mathbb{L}[[\sigma_1, \ldots]][[t]][t^{-1}]$, where σ_i is the i-th coefficient of the series $\prod_i (1 + \lambda_i) \cdot \prod_j (1 + \mu_j)^{-1}$.

Proposition 2.13

$$\Psi^{q(t)}([v]) = q(0) \cdot [v]^2 + S^g_{L.-N.}([v]), \quad \textit{where } g = -\operatorname*{Res}_{t=0} \frac{q(t) c^{\Omega}(-T)(t) \cdot \omega}{t}.$$

$$\widetilde{\square}^{q(t)}([v]) = q(0) \cdot \pi^*(\square([v])) + S^{h(u)}_{L.-N.}([v]), \quad \textit{where } h(u) = -\operatorname*{Res}_{t=0} \frac{q(t) c^{\Omega}(-T)(t) \cdot \omega}{t(u -_{\Omega} t)}.$$

Proof: Let $v = f \circ g$, where $g : V \to W$ is a regular embedding, and $f : W \to X$ is a smooth projective morphism. We can assume that $\dim(W) > 2\dim(V)$. We have commutative diagram:

$$\begin{array}{ccccccc}
\widetilde{\square}(V) & \xrightarrow{\widetilde{\square}(g)} & \widetilde{\square}(W) & \xleftarrow{A(f)} & \widetilde{\square}(W/X) & \xrightarrow{B(f)} & X \\
\pi_V \downarrow & & \downarrow \pi_W & & \pi_{W/X} \downarrow & & \| \\
\square(V) & \xrightarrow[\square(g)]{} & \square(W) & \xleftarrow[A]{} & \square(W/X) & \xrightarrow[B]{} & X.
\end{array}$$

By Proposition 5.27,

$$\widetilde{\square}(g)_*(1_{\widetilde{\square}(V)}) = \pi_W^* \square(g)_*(1_{\square(V)}) + (j_W)_*(g_1)_* \left(\frac{c_r(\mathcal{N}_g \otimes \mathcal{O}(1)) - c_r(\mathcal{N}_g)}{c_1(\mathcal{O}(-1))} \right),$$

where $r = \dim(\mathcal{N}_g)$, $\mathbb{P}_V(g^*T_W) \stackrel{g_1}{\to} \mathbb{P}_W(T_W) \stackrel{j_W}{\to} \widetilde{\square}(W)$ are the natural maps. Since $r = \dim(W) - \dim(V) > \dim(V)$, the term with $c_r(\mathcal{N}_g)$ can be omitted. Then, since $\widetilde{\square}(g)^*(\rho_W) = \rho_V$,

$$\widetilde{\square}(g)_*(q(\rho)) = \pi_W^* \square(g)_*(1_{\square(V)}) \cdot q(\rho) + (j_W)_*(g_1)_* \left(\frac{q(c_1(\mathcal{O}(-1))) \cdot c_r(\mathcal{N}_g \otimes \mathcal{O}(1))}{c_1(\mathcal{O}(-1))} \right),$$

$$\begin{aligned}
B(f)_* A(f)^*(q(\rho) \cdot \pi_W^* \square(g)_*(1_{\square(V)})) = B(f)_*(q(\rho) \cdot (\pi_{W/X})^* A^* \square(g)_*(1_{\square(V)})) = \\
B_*((\pi_{W/X})_*(q(\rho)) \cdot A^* \square(g)_*(1_{\square(V)})).
\end{aligned}$$

By Theorem 5.25, the element $(1_{\square(W/X)} - (\pi_{W/X})_*(1_{\widetilde{\square}(W/X)}))$, as well as the elements $(\pi_{W/X})_*(\rho^m)$, for $m > 0$, have support of codimension $\geqslant \dim(W) - \dim(X)$. On the other hand, $A^* \square(g)_*(1_{\square(V)})$ has support of dimension $\leqslant 2\dim(V) - \dim(X)$, which is smaller. By Statement 5.2, the latter expression is equal to $q(0) \cdot B_* A^* \square(g)_*(1_{\square(V)})$. and from the proper cartesian square:

$$\begin{array}{ccc}
\square(W) & \xleftarrow{A} & \square(W/X) \\
\square(f) \downarrow & & \downarrow B \\
\square(X) & \xleftarrow[\Delta_X]{} & X
\end{array}$$

this is equal to $q(0) \cdot \Delta_X^* \square(f)_* \square(g)_*(1_{\square(V)}) = q(0) \cdot [v]^2$.

Consider the diagram

$$\begin{array}{ccccc}
\mathbb{P}_V(g^*T_{W/X}) & \xrightarrow{g_2} & \mathbb{P}_W(T_{W/X}) & \xrightarrow{j_{W/X}} & \widetilde{\square}(W/X) \\
A_2 \downarrow & & \downarrow A_1 & & \downarrow A(f) \\
\mathbb{P}_V(g^*T_W) & \xrightarrow[g_1]{} & \mathbb{P}_W(T_W) & \xrightarrow[j_W]{} & \widetilde{\square}(W)
\end{array}$$

with both squares proper cartesian (by Lemma 5.5). We have:

$$\begin{aligned}
B(f)_* A(f)^* (j_W)_*(g_1)_* \left(\frac{q(c_1(\mathcal{O}(-1))) \cdot c_r(\mathcal{N}_g \otimes \mathcal{O}(1))}{c_1(\mathcal{O}(-1))} \right) = \\
(B(f) \circ j_{W/X} \circ g_2)_* \left(\frac{q(c_1(\mathcal{O}(-1))) \cdot c_r(\mathcal{N}_g \otimes \mathcal{O}(1))}{c_1(\mathcal{O}(-1))} \right),
\end{aligned}$$

which, by Theorem 5.35 (Quillen's formula), is equal to $v_*(g')$, where

$$g' = \operatorname*{Res}_{t=0} \frac{-q(t)c^{\Omega}(-T_f)(t)c^{\Omega}(-T_g)(t)\cdot\omega}{t} = -\operatorname*{Res}_{t=0} \frac{q(t)c^{\Omega}(-T_v)(t)\cdot\omega}{t},$$

which is equal to $S^g_{L.-N.}([v])$, with $g = -\operatorname{Res}_{t=0}\frac{q(t)c^{\Omega}(-T)(t)\cdot\omega}{t}$. The first formula is proven.

We have commutative diagram:

$$\begin{array}{ccccccccc}
\widetilde{\square}(V) & \xrightarrow{\widetilde{\square}(g)} & \widetilde{\square}(W) & \xleftarrow{a(f)} & Bl_{\widetilde{\square}(W),\widetilde{\square}(W/X)} & \xrightarrow{b'} & Bl_{\square(W),\square(W/X)} & \xrightarrow{b''} & \widetilde{\square}(X) \\
\pi_V\downarrow & & \downarrow\pi_W & & \pi_1\downarrow & & \downarrow\pi_2 & & \pi_X\downarrow \\
\square(V) & \xrightarrow[\square(g)]{} & \square(W) & = & \square(W) & = & \square(W) & \xrightarrow[\square(f)]{} & \square(X).
\end{array}$$

with the right square proper cartesian (by Lemma 5.5). Again,

$$\widetilde{\square}(g)_*(q(\rho)) = \pi_W^*\square(g)_*(1_{\square(V)})\cdot q(\rho) + (j_W)_*(g_1)_*\left(\frac{q(c_1(\mathcal{O}(-1)))\cdot c_r(\mathcal{N}_g\otimes\mathcal{O}(1))}{c_1(\mathcal{O}(-1))}\right).$$

By the Theorem 5.25, the element $(1_{Bl_{\square(W),\square(W/X)}} - (b')_*(1_{Bl_{\widetilde{\square}(W),\widetilde{\square}(W/X)}}))$, as well as the elements $(b')_*a(f)^*(\rho^m)$, for $m>0$, have support on $(\pi_2)^{-1}(\Delta(W))$. Since $\square(g)_*(1_{\square(V)})$ has support of dimension $2\dim(V) < \operatorname{codim}(W \overset{\Delta}{\subset} \square(W))$, we have:

$$b(f)_*a(f)^*(q(\rho)\cdot\pi_W^*\square(g)_*(1_{\square(V)})) = q(0)\cdot(b'')_*\pi_2^*\square(g)_*(1_{\square(V)}) = q(0)\cdot\pi_X^*(\square([v])).$$

Consider commutative diagram

$$\begin{array}{ccccc}
\mathbb{P}_V(g^*T_W) & \xleftarrow{a_2} & Bl_{\mathbb{P}_V(g^*T_W),\mathbb{P}_V(g^*T_{W/X})} & \xrightarrow{b_2} & \mathbb{P}_V(v^*T_X) \\
g_1\downarrow & (1) & \downarrow\widetilde{g} & & \downarrow g_3 \\
\mathbb{P}_W(T_W) & \xleftarrow{a_1} & Bl_{\mathbb{P}_W(T_W),\mathbb{P}_W(T_{W/X})} & \xrightarrow{b_1} & \mathbb{P}_W(f^*T_X) \\
j_W\downarrow & (3) & \downarrow\widetilde{j} & & \downarrow j_X \\
\widetilde{\square}(W) & \xleftarrow[a(f)]{} & Bl_{\widetilde{\square}(W),\widetilde{\square}(W/X)} & \xrightarrow[b(f)]{} & \widetilde{\square}(X),
\end{array}$$

where square (1) is proper cartesian by evident reasons, and square (3) is proper cartesian by Lemma 5.5 (applied twice). We get:

$$b(f)_*a(f)^*(j_W)_*(g_1)_*\left(\frac{q(c_1(\mathcal{O}(-1)))\cdot c_r(\mathcal{N}_g\otimes\mathcal{O}(1))}{c_1(\mathcal{O}(-1))}\right) =$$
$$(j_X)_*(g_3)_*(b_2)_*(a_2)^*\left(\frac{q(c_1(\mathcal{O}(-1)))\cdot c_r(\mathcal{N}_g\otimes\mathcal{O}(1))}{c_1(\mathcal{O}(-1))}\right).$$

The variety $Bl_{\mathbb{P}_V(g^*T_W),\mathbb{P}_V(g^*T_{W/X})}$ is isomorphic to $\mathbb{P}_{\mathbb{P}_V(v^*T_X)}(\mathcal{X})$, where the bundle $\mathcal{O}(1)$ on the latter is isomorphic to $a_2^*(\mathcal{O}(1))$, and $\mathcal{X}$ fits into the exact sequence $0\to\varepsilon^*g^*T_{W/X}\to\mathcal{X}\to\mathcal{O}(-1)\to 0$, where $\varepsilon:\mathbb{P}_V(v^*T_X)\to V$ is the projection (see Statement 5.37). Thus, by Theorem 5.35, the latter expression is equal to $(j_X)_*(g_3)_*(h'(c_1(\mathcal{O}(-1))))$, where

$$h'(u) = -\operatorname*{Res}_{t=0}\frac{q(t)g^*(c^{\Omega}(-T_f)(t))c^{\Omega}(-T_g)(t)\cdot\omega}{(u -_{\Omega} t)t} = -\operatorname*{Res}_{t=0}\frac{q(t)c^{\Omega}(-T_v)(t)\cdot\omega}{(u -_{\Omega} t)t}.$$

So, our expression is equal to $S^{h(u)}_{L.-N.}([v])$, where

$$h(u) = -\operatorname*{Res}_{t=0} \frac{q(t)c^{\Omega}(-T)(t)\cdot\omega}{(u -_{\Omega} t)t}.$$

The second formula is proven. $\square$

Corollary 2.14 *Operations $\widetilde{\Box}^{q(t)}$ and $\Psi^{q(t)}$ are well-defined on Ω^*.*

The following proposition establishes the relation between the operations $\widetilde{\Box}$ and $\widetilde{C}^2$, and between Ψ and Φ.

Proposition 2.15

$$p_* \circ (\widetilde{\Box})^{h(t)} = (\widetilde{C}^2)^{h(t)\cdot\frac{[2]_\Omega(t)}{t}}$$
$$\Psi^{h(t)} = \Phi^{h(t)\cdot\frac{[2]_\Omega(t)}{t}}.$$

Proof: Let $p_Z : \widetilde{\Box}(Z) \to \widetilde{C}^2(Z)$ be the natural map.
Since $(p_Z)_*(\mathcal{O}_{\widetilde{\Box}(Z)})/\mathcal{O}_{\widetilde{C}^2(Z)} \cong \mathcal{L}$, it follows from Proposition 5.18 that

Statement 2.16

$$(p_Z)_*(1_{\widetilde{\Box}(Z)}) = \frac{[2]_\Omega(t)}{t}(\varrho)\cdot 1_{\widetilde{C}^2(Z)}.$$

Let $v : V \to Y$ be some representative from $(\mathrm{Pre}-\Omega)(Y)$, and $v = f \circ g$, where $g : V \to W$ is regular embedding, and $f : W \to X$ is smooth projective morphism. Consider the diagram:

$$\begin{array}{ccccccc}
\widetilde{\Box}(V) & \xrightarrow{\widetilde{\Box}(g)} & \widetilde{\Box}(W) & \xleftarrow{a(f)} & Bl_{\widetilde{\Box}(W),\widetilde{\Box}(W/Y)} & \xrightarrow{b(f)} & \widetilde{\Box}(Y) \\
{\scriptstyle p_V}\downarrow & & {\scriptstyle p_W}\downarrow & & {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_Y} \\
\widetilde{C}^2(V) & \xrightarrow{\widetilde{C}^2(g)} & \widetilde{C}^2(W) & \xleftarrow{c(f)} & Bl_{\widetilde{C}^2(W),\widetilde{C}^2(W/Y)} & \xrightarrow{d(f)} & \widetilde{C}^2(Y)
\end{array}$$

Since the middle square is proper cartesian by Statement 5.11, we get:

$$\begin{aligned}
(p_Y)_*(\widetilde{\Box}^{h(t)}([v])) =& (p_Y)_* b(f)_* a(f)^* \widetilde{\Box}(g)_*(h(\rho)) = \\
d(f)_* c(f)^* \widetilde{C}^2(g)_*(p_V)_*(h(\rho)) =& d(f)_* c(f)^* \widetilde{C}^2(g)_*(h(\varrho)\cdot(p_V)_*(1_{\widetilde{\Box}(V)})),
\end{aligned}$$

since $p_V^*(\varrho) = \rho$. And the latter expression is equal to $(\widetilde{C}^2)^{h(t)\cdot\frac{[2]_\Omega(t)}{t}}([v])$ by the Statement 2.16.
For operations Ψ and Φ, analogously, consider the diagram:

$$\begin{array}{ccccccc}
\widetilde{\Box}(V) & \xrightarrow{\widetilde{\Box}(g)} & \widetilde{\Box}(W) & \xleftarrow{A(f)} & \widetilde{\Box}(W/Y) & \xrightarrow{B(f)} & Y \\
{\scriptstyle p_V}\downarrow & & {\scriptstyle p_W}\downarrow & & {\scriptstyle p_{W/Y}}\downarrow & & \| \\
\widetilde{C}^2(V) & \xrightarrow{\widetilde{C}^2(g)} & \widetilde{C}^2(W) & \xleftarrow{C(f)} & \widetilde{C}^2(W/Y) & \xrightarrow{D(f)} & Y.
\end{array}$$

Again, the middle square is proper cartesian by Statement 5.11, and

$$\begin{aligned}
&\Psi^{h(t)}([v]) = \Upsilon(v)(h(\rho)) = B(f)_* A(f)^* \widetilde{\Box}(g)_*(h(\rho)) = \\
&D(f)_* C(f)^* \widetilde{C}^2(g)_* \left(h(\varrho)\cdot\frac{[2]_\Omega(\varrho)}{\varrho}\right) = \Theta(v)\left(h(\rho)\cdot\frac{[2]_\Omega(\rho)}{\rho}\right) = \Phi^{h(t)\cdot\frac{[2]_\Omega(t)}{t}}([v]).
\end{aligned}$$

$\square$

Corollary 2.17 (1) *Let X be smooth quasi-projective variety such that $\Omega^*(X)$ has no 2-torsion. Then operations $\Phi^{q(t)}$ are well-defined on $\Omega^*(X)$.*

(2) *Let X be smooth quasi-projective variety such that $\Omega^*(\widetilde{C}^2(X))$ has no 2-torsion. Then operations $(\widetilde{C}^2)^{q(t)}$ are well-defined on $\Omega^*(X)$.*

Proof: Let $q(t)$ be divisible by t^n. Use decreasing induction on n. For large n, operations $\Phi^{q(t)}$ and $(\widetilde{C}^2)^{q(t)}$ will be zero by dimensional considerations, and so, well-defined. Now, by Proposition 2.16, $2 \cdot \Phi^{q(t)} - \Psi^{q(t)} = \Phi^{r(t)}$, and similarly, $2 \cdot (\widetilde{C}^2)^{q(t)} - p_* \widetilde{\square}^{q(t)} = (\widetilde{C}^2)^{r(t)}$, where $r(t)$ is divisible by t^{n+1}. By induction hypothesis, $2 \cdot \Phi^{q(t)}$ and $2 \cdot (\widetilde{C}^2)^{q(t)}$ are well-defined. Since there is no 2-torsion, operations $\Phi^{q(t)}$ and $(\widetilde{C}^2)^{q(t)}$ are also well-defined. □

We say that the variety X/k is *of type P*, if there exists a sequence $\mathrm{Spec}(k) = P_0 \leftarrow P_1 \leftarrow \ldots \leftarrow P_r = X$, with maps $\pi_{i,j} : P_i \to P_j$ such that $P_{i+1} = \mathbb{P}_{P_i}(\mathcal{V}_i)$, where $\mathcal{V}_i = \oplus_{l=1}^{L(i)} \mathcal{L}_{i,l}$, where $\mathcal{L}_{i,l} = \otimes_{j<i} \pi_{i,j}^*(\mathcal{O}(n_{i,l,j})_j)$.

The proof of the following statement is due to A.Kuznetsov.

Proposition 2.18 *If X is of type P, then X and $\widetilde{C}^2(X)$ are cellular varieties.*

Proof: The fact that X itself is cellular is trivial, since X is a consecutive projective bundle.
The proof of the $\widetilde{C}^2(X)$ part is based on the following result of Brosnan, Bialynicki-Birula, Hesselink and Iversen:

Theorem 2.19 ([2, Theorem 3.3]) *Let X be smooth projective with the action of $\mathbb{G}_m$ having isolated k-rational fixed points. Then X is cellular.*

We have:

Lemma 2.20 *Let X be of type P. Then on X there is action of $\mathbb{G}_m$ which has only isolated k-rational fixed points, and for each such point x, the induced action on $\mathbb{P}(T_{X,x})$ has the same property.*

Proof: Let us prove by the induction on i that on the system $\mathcal{V}_j, j < i$ we have a $\mathbb{G}_m$-action, such that it is linear on $\mathcal{V}_j$, commutes with the maps $\mathcal{V}_j \to P_j$, and with the maps $(\mathcal{V}_j \backslash s_0(P_j)) \to P_{j+1}$, and it has only isolated k-rational fixed points on P_i and on $\mathbb{P}(T_{P_i,x})$ for x-fixed (the latter means that all eigenvalues on $T_{P_i,x}$ are different).
Base of induction ($i = 0$) is trivial.
Induction step ($i \Rightarrow i+1$): Suppose we have such action for $\mathcal{V}_j, j < i$. Let us denote it as ρ_i. It induces the action on $\mathcal{O}(n_{i,l,j})_j$ for all $j < i$, and hence some action on $\mathcal{V}_i$ which we still denote as ρ_i. Let $x_k, 1 \leqslant k \leqslant M$ be all fixed points of ρ_i on P_i, and $\mathbb{P}(t_{k,n}), 1 \leqslant n \leqslant Q(k)$ be all fixed points of ρ_i on $\mathbb{P}(T_{P_i,x_k})$.
Let ρ_i acts on $\mathcal{L}_{i,l}|_{x_k}$ as $\lambda \mapsto \cdot\lambda^{s(l,k)}$, and on $t_{k,n}$ as $\lambda \mapsto \cdot\lambda^{r(k,n)}$. Let $N \in \mathbb{N}$ be such that $N/3 > max(|s(l,k)|, |r(k,n)|)$.
Define new action ρ_{i+1} on $\mathcal{L}_{i,l}$ as $\rho_i \cdot \lambda^{Nl}$. Then, for fixed k, the eigenvalues of ρ_{i+1} on $\mathcal{L}_{i,l}|_{x_k}, 1 \leqslant l \leqslant L(i)$ will all be different, and thus ρ_{i+1} will have only isolated k-rational fixed points $y_{k,l} := \mathbb{P}(\mathcal{L}_{i,l}|_{x_k}), 1 \leqslant l \leqslant L(i); 1 \leqslant k \leqslant M$ on P_{i+1}.
We have short exact sequence: $0 \to T_{fiber} \to T_{P_{i+1},y_{k,l}} \to T_{P_i,x_k} \to 0$, where the eigenvalues of the ρ_{i+1} on T_{P_i,x_k} are $\lambda^{r(k,n)}$, and on T_{fiber} such eigenvalues are $\lambda^{s(l',k)-s(l,k)+N(l'-l)}$, where $l' \neq l$. Since

$N > |s(l',k)| + |s(l,k)| + |r(k,n)|$, all these eigenvalues are pairwise different. The induction step is proven. □

Now, we just observe that $\mathbb{G}_m$-action on X gives an action on $\widetilde{C}^2(X)$ whose fixed points are either the non ordered pair (x,y) of fixed points on X, or the pair $(x,\mathbb{P}(t))$, where x is a fixed point on x, and $\mathbb{P}(t)$ is a fixed point on $\mathbb{P}(T_{X,x})$. Since we have only finitely many such pairs, and they are k-rational, by Theorem 2.19, $\widetilde{C}^2(X)$ is cellular. □

Corollary 2.21 *Let X be smooth projective of type P. Then $\Omega^*(X)$ and $\Omega^*(\widetilde{C}^2(X))$ is a free (finitely generated) $\mathbb{L}$-module. In particular, it has no $\mathbb{L}$-torsion, and the operations $\Phi^{q(t)}$, $(\widetilde{C}^2)^{q(t)}$ are well defined on $\Omega^*(X)$.*

Proof: The fact that Ω^* of a cellular variety is a free $\mathbb{L}$-module follows from the [10, Theorem 6.5], or [19, Corollary 2.9]. The rest follows from the Corollary 2.17. □

Lemma 2.22 *Let V be smooth quasi-projective variety, $u : U \to V$ be smooth divisor, and $\mathcal{O}(U) = \mathcal{L}_1 \otimes (\mathcal{L}_2)^{\pm 1}$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are generated by global sections. Let $[u']$ be $c_1(\mathcal{L}_1) \pm_{\Omega} c_1(\mathcal{L}_2) \in (\mathrm{Pre}-\Omega)^1(V)$. Then*

(1) $\Phi^{q(t)}([u]) = \Phi^{q(t)}([u'])$;

(2) $(\widetilde{C}^2)^{q(t)}([u]) = (\widetilde{C}^2)^{q(t)}([u'])$.

Proof: Let $\mathcal{L}$ be some ample line bundle on V. The triple $(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L})$ gives a map $\beta : V \to P := \mathbb{P}^N \times \mathbb{P}^M \times \mathbb{P}^L$, which is a composition of a regular and open embedding such that $\mathcal{O}(U) = \beta^*(\mathcal{O}(1)_1 \otimes (\mathcal{O}(1)_2)^{\pm 1})$. This gives a proper cartesian diagram:

$$\begin{array}{ccc} \mathbb{P}_V(\mathcal{O}(U) \oplus \mathcal{O}) & \xrightarrow{\gamma} & \mathbb{P}_P(\mathcal{O}(1)_1 \otimes (\mathcal{O}(1)_2)^{\pm 1} \oplus \mathcal{O}) \\ {\scriptstyle \pi_V}\downarrow & & \downarrow{\scriptstyle \pi_P} \\ V & \xrightarrow[\beta]{} & P. \end{array}$$

Let $[q']$ be $c_1(\mathcal{O}(1)_1) \pm_{\Omega} c_1(\mathcal{O}(1)_2)$. In general, $[q']$, as well as $[u']$, is a difference of the classes of two projective maps. We have a section $s : V \to \mathbb{P}_V(\mathcal{O}(U) \oplus \mathcal{O})$ such that s is transversal to the 0-section $s_0 := \mathbb{P}(\mathcal{O}(U))$, $s^*([s_0(V)]) = [U]$, and which does not meet ∞-section $s_\infty := \mathbb{P}(\mathcal{O})$.

Consider $[s_\infty(V)] +_{\Omega} \pi_V^*[u'] \in (\mathrm{Pre}-\Omega)^1(\mathbb{P}_V(\mathcal{O}(U) \oplus \mathcal{O}))$. This is a transversal preimage under γ of $[s_\infty(P)] +_{\Omega} \pi_P^*[q']$. And $[s_0(V)]$ is a transversal preimage under γ of $[s_0(P)]$. By Proposition 2.11,

$$\Phi^{q(t)}([s_\infty(V)] +_{\Omega} \pi_V^*[u']) = \gamma^* \Phi^{q(t)}([s_\infty(P)] +_{\Omega} \pi_P^*[q']), \text{ and}$$

$$\Phi^{q(t)}([s_0(V)]) = \gamma^* \Phi^{q(t)}([s_0(P)]). \text{ Analogously,}$$

$$(\widetilde{C}^2)^{q(t)}([s_\infty(V)] +_{\Omega} \pi_V^*[u']) = \widetilde{C}^2(\gamma)^*(\widetilde{C}^2)^{q(t)}([s_\infty(P)] +_{\Omega} \pi_P^*[q']), \text{ and}$$

$$(\widetilde{C}^2)^{q(t)}([s_0(V)]) = \widetilde{C}^2(\gamma)^*(\widetilde{C}^2)^{q(t)}([s_0(P)]).$$

But in $\Omega^*(\mathbb{P}_P(\mathcal{O}(1)_1 \otimes (\mathcal{O}(1)_2)^{\pm 1} \oplus \mathcal{O}))$, $[s_0(P)] = [s_\infty(P)] +_{\Omega} \pi_P^*[q']$. Since variety $\mathbb{P}_P(\mathcal{O}(1)_1 \otimes (\mathcal{O}(1)_2)^{\pm 1} \oplus \mathcal{O})$ is of type P, by Corollary 2.21,

$$\Phi^{q(t)}([s_0(P)]) = \Phi^{q(t)}([s_\infty(P)] +_{\Omega} \pi_P^*[q']), \text{ and}$$

$$(\widetilde{C}^2)^{q(t)}([s_0(P)]) = (\widetilde{C}^2)^{q(t)}([s_\infty(P)] +_\Omega \pi_P^*[q']).$$

Consequently, $\Phi^{q(t)}([s_0(V)]) = \Phi^{q(t)}([s_\infty(V)] +_\Omega \pi_P^*[u'])$, and $(\widetilde{C}^2)^{q(t)}([s_0(V)]) = (\widetilde{C}^2)^{q(t)}([s_\infty(V)] +_\Omega \pi_P^*[u'])$.

Since $s : V \to \mathbb{P}_V(\mathcal{O}(U) \oplus \mathcal{O})$ is transversal to $[s_\infty(V)] +_\Omega \pi_P^*[u']$ and to $[s_0(V)]$, and the transversal preimages under s are $[u']$ and $[u]$, respectively, we have: $\Phi^{q(t)}([u]) = \Phi^{q(t)}([u'])$, and $(\widetilde{C}^2)^{q(t)}([u]) = (\widetilde{C}^2)^{q(t)}([u'])$. □

Lemma 2.23 *Under the conditions of Lemma 2.22, let $v : V \to X$ be a projective map of smooth quasi-projective varieties. Then*

(1) $\Phi^{q(t)}([v \circ u]) = \Phi^{q(t)}([v \circ u'])$;

(2) $(\widetilde{C}^2)^{q(t)}([v \circ u]) = (\widetilde{C}^2)^{q(t)}([v \circ u'])$.

Proof: Follows immediately from Lemma 2.22 and Proposition 2.5. □

Now, we can prove:

Theorem 2.24 *Operations $(\widetilde{C}^2)^{q(t)}$ and $\Phi^{q(t)}$ are well-defined on $\Omega^*(X)$.*

Proof: One could observe that $\Omega^*(X)$ is obtained from $(\mathrm{Pre}-\Omega)^*(X)$ by moding out the following relations. Let $v : V \to X$ be a projective map from a smooth variety, and $u : U \to V$ be smooth divisor on V. Let $\mathcal{O}(U) = \mathcal{L}_1 \otimes (\mathcal{L}_2)^{\pm 1}$, where $\mathcal{L}_1, \mathcal{L}_2$ are generated by the global sections, and $[u']$ is $c_1(\mathcal{L}_1) \pm_\Omega c_1(\mathcal{L}_2) \in (\mathrm{Pre}-\Omega)^1(V)$. Then, in $\Omega^*(X)$, $[v \circ u] = [v \circ u']$. It follows from the Lemma 2.23 that the value of $(\widetilde{C}^2)^{q(t)}$ and $\Phi^{q(t)}$ on such elements coincide. Hence, these operations are well-defined on Ω^*. □

3 Some properties of Ψ and Φ

3.1 Pull-backs and regular push-forwards

Proposition 3.1 *Let $X \xrightarrow{j} Y$ be a regular imbedding of smooth varieties. Then for arbitrary $[v : V \to X]$, and for arbitrary $q(t) \in \mathbb{L}[[t]]$, we have:*

$$\Phi^{q(t)}(j_*([v])) = j_*(\Phi^{q(t)\cdot c^\Omega(\mathcal{N}_j)(t)}([v])).$$

Proof: We can find such smooth projective morphism $f_Y : W_Y \to Y$ that v can be decomposed as $V \xrightarrow{g_X} W_X \xrightarrow{f_X} X$, where g_X is an embedding, and f_X is the pull-back of f_Y under j. Consider commutative diagram:

$$\begin{array}{ccc}
\widetilde{C}^2(V) & = & \widetilde{C}^2(V) \\
{\scriptstyle \widetilde{C}^2(g_X)}\downarrow & & \downarrow{\scriptstyle \widetilde{C}^2(g_Y)} \\
\widetilde{C}^2(W_X) & \xrightarrow{\widetilde{C}^2(j_W)} & \widetilde{C}^2(W_Y) \\
{\scriptstyle C(f_X)}\uparrow & & \uparrow{\scriptstyle C(f_Y)} \\
\widetilde{C}^2(W_X/X) & \xrightarrow{\widetilde{C}^2(j_W/j)} & \widetilde{C}^2(W_Y/Y) \\
{\scriptstyle D(f_X)}\downarrow & & \downarrow{\scriptstyle D(f_Y)} \\
X & \xrightarrow[j]{} & Y
\end{array}$$

Notice, that the middle square is cartesian.

By Lemma 2.1, $\mathcal{N}_{C(f_X)} = D(f_X)^*(T_X) \otimes \mathcal{L}$, $\mathcal{N}_{C(f_Y)} = D(f_X)^*(T_Y) \otimes \mathcal{L}$, and thus, $\widetilde{C}^2(j_W/j)^*\mathcal{N}_{C(f_Y)}/\mathcal{N}_{C(f_X)} = D(f_X)^*(\mathcal{N}_j) \otimes \mathcal{L}$. Let $d = codim(X \subset Y)$. Then, by the Excess Intersection Formula (Theorem 5.19),

$$C(f_Y)^* \circ \widetilde{C}^2(j_W)_*(a) = \widetilde{C}^2(j_W/j)_*(C(f_X)^*(a) \cdot c_d(D(f_X)^*(\mathcal{N}_j) \otimes \mathcal{L})),$$

where $c_d(D(f_X)^*(\mathcal{N}_j) \otimes \mathcal{L}) = c^{\Omega}(\mathcal{N}_j)(\varrho)$. Hence,

$$\begin{aligned} \Phi^{q(t)}(j_*([v])) = (D(f_Y))_*(q(\varrho) \cdot C(f_Y)^* \circ (g_Y)_*[1_{\widetilde{C}^2(V)}]) = \\ (D(f_Y))_*(q(\varrho) \cdot C(f_Y)^* \circ \widetilde{C}^2(j_W)_* \circ (g_X)_*[1_{\widetilde{C}^2(V)}]) = \\ (D(f_Y))_* \circ \widetilde{C}^2(j_W/j)_*(c^{\Omega}(\mathcal{N}_j)(\varrho) \cdot q(\varrho) \cdot C(f_X)^* \circ (g_X)_*[1_{\widetilde{C}^2(V)}]) = \\ j_* \circ D(f_X)_*(c^{\Omega}(\mathcal{N}_j)(\varrho) \cdot q(\varrho) \cdot C(f_X)^* \circ (g_X)_*[1_{\widetilde{C}^2(V)}])) = j_*(\Phi^{q(t) \cdot c^{\Omega}(\mathcal{N}_j)(t)}([v])). \end{aligned}$$

□

The following statement shows that the nontriviality of the symmetric operations provide an obstruction for the cobordism class to be represented by the smooth subvariety.

Proposition 3.2 *Operations $\Phi^{q(t)}$, $\Psi^{q(t)}$ are trivial on the classes of embeddings*

Proof: If $v : V \to X$ is a regular embedding, we can take $W = X$, and observe that the varieties $\widetilde{\square}(X/X)$ and $\widetilde{C}^2(X/X)$ are empty. □

Question 3.3 *What other obstructions exist?*

Proposition 3.4 *Let $h : X \to Y$ be a morphism of smooth quasiprojective varieties, and $q(t) \in \mathbb{L}[[t]]$. Then*

$$\Phi^{q(t)} \circ h^* = h^* \circ \Phi^{q(t)}.$$

Proof: Any morphism of smooth quasiprojective varieties can be decomposed into the composition of the open embedding, regular imbedding and a smooth projective morphism.

1) The case of an open embedding is clear from the construction.

2) Let $\pi : X \to Y$ be smooth projective morphism, and $[v : V \to Y]$ be the morphism of smooth projective varieties representing some class in $\Omega^d(Y)$. Decompose $v : V \to Y$ into the composition $V \stackrel{g}{\hookrightarrow} W \stackrel{f}{\to} Y$, where g is a regular imbedding and f is smooth projective morphism. Let $W_X := W \times_Y X$, and $V_X := V \times_Y X$. Then, for arbitrary $q(t) \in \mathbb{L}[[t]]$, $\Phi^{q(t)}([v]) = D(f)_* \circ C(f)^*\widetilde{C}^2(g)_*(q(\varrho))$.

Consider commutative diagram:

$$\begin{array}{ccccc} \widetilde{C}^2(V) & \xleftarrow{d(\pi_V)} & Bl_{\widetilde{C}^2(V_X),\widetilde{C}^2(V_X/V)} & \xrightarrow{c(\pi_V)} & \widetilde{C}^2(V_X) \\ {\scriptstyle \widetilde{C}^2(g)}\downarrow & (1) & {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \widetilde{C}^2(g_X)} \\ \widetilde{C}^2(W) & \xleftarrow{d(\pi_W)} & Bl_{\widetilde{C}^2(W_X),\widetilde{C}^2(W_X/W)} & \xrightarrow{c(\pi_W)} & \widetilde{C}^2(W_X) \\ {\scriptstyle C(f)}\uparrow & & {\scriptstyle \beta}\uparrow & (4) & \uparrow{\scriptstyle C(f_X)} \\ \widetilde{C}^2(W/Y) & \xleftarrow{d(\pi_{W/Y})} & \widetilde{C}^2(W_X/X) & = & \widetilde{C}^2(W_X/X) \\ {\scriptstyle D(f)}\downarrow & (5) & {\scriptstyle D(f_X)}\downarrow & & \downarrow{\scriptstyle D(f_X)} \\ Y & \xleftarrow{\pi} & X & = & X \end{array}$$

The squares (1),(5) and (4) in this diagram are proper cartesian, by Statements 5.13, 5.14 and Lemma 5.5, respectively (notice, that $\widetilde{C}^2(W_X/W)$ and $\widetilde{C}^2(W_X/X)$ do not intersect). Also, $d(\pi_{W/Y})^* \circ C(f)^*(\varrho_W) = C(f_X)^*(\varrho_{W_X})$. We get:

$$\pi^*(\Phi^{q(t)}([v])) = \pi^* \circ D(f)_* \circ C(f)^*(q(\varrho_W) \cdot \widetilde{C}^2(g)_*[1_{\widetilde{C}^2(V)}]) =$$
$$D(f_X)_*(q(\varrho_{W_X/X}) \cdot \beta^* \circ \alpha_* \circ d(\pi_V)^*[1_{\widetilde{C}^2(V)}]) =$$
$$D(f_X)_* \circ C(f_X)^*(q(\varrho_{W_X}) \cdot \widetilde{C}^2(g_X)_* \circ c(\pi_W)_*[1_{Bl_{\widetilde{C}^2(V_X),\widetilde{C}^2(V_X/V)}}]).$$

By [5, Proposition 3.2] (Proposition 5.25),

$$\widetilde{C}^2(g_X)_*(c(\pi_V)_*[1_{Bl_{\widetilde{C}^2(V_X),\widetilde{C}^2(V_X/V)}}] - [1_{\widetilde{C}^2(V_X)}])$$

is supported on $\widetilde{C}^2(W_X/W)$. But, subvarieties $\widetilde{C}^2(W_X/W)$ and $\widetilde{C}^2(W_X/X)$ do not intersect in $\widetilde{C}^2(W_X)$. Thus,

$$C(f_X)^* \circ \widetilde{C}^2(g_X)_*(c(\pi_V)_*[1_{Bl_{\widetilde{C}^2(V_X),\widetilde{C}^2(V_X/V)}}] - [1_{\widetilde{C}^2(V_X)}]) = 0,$$

and our expression is equal to

$$D(f_X)_* \circ C(f_X)^*(q(\varrho_{W_X}) \cdot \widetilde{C}^2(g_X)_*[1_{\widetilde{C}^2(V_X)}]) = \Phi^{q(t)}(\pi^*[v]).$$

3) It remains to consider the case of a regular embedding: $j : X \to Y$.

Lemma 3.5 *Let B be smooth quasiprojective variety, and $\pi : \mathbb{A}_B(\mathcal{N}) \to B$ be an affine bundle for certain vector bundle $\mathcal{N}$ with zero section $i : B \to \mathbb{A}_B(\mathcal{N})$. Then i^* commutes with the operation $\Phi^{q(t)}$.*

Proof: By the homotopy invariance, the map $\pi^* : \Omega^*(B) \to \Omega^*(\mathbb{A}_B(\mathcal{N}))$ is an isomorphism. Since $\pi \circ i = id$, and π is a composition of an open embedding and a smooth projective morphism, we have: $i^*\Phi^{q(t)}(\pi^*([v])) = i^*\pi^*\Phi^{q(t)}([v]) = \Phi^{q(t)}([v]) = \Phi^{q(t)}(i^*\pi^*([v]))$. $\square$

Lemma 3.6 *Let $f : B \to A$ be regular embedding of smooth quasiprojective varieties, $\overline{f} = f \times id : B \times \mathbb{P}^1 \to A \times \mathbb{P}^1$ and $i = id \times \{1\} : A \to A \times \mathbb{P}^1$. Then for arbitrary $[v] \in \Omega^*(A)$,*

$$\Phi^{q(t)}(\overline{f}^* i_*([v])) = \overline{f}^* \Phi^{q(t)}(i_*([v])).$$

Proof: Consider the standard deformation to the normal cone construction.

Let $\widehat{A} := Bl_{A\times\mathbb{P}^1, B\times\{0\}}$, $\widehat{B} := Bl_{B\times\mathbb{P}^1, B\times\{0\}} = B \times \mathbb{P}^1$, $D := \mathbb{P}_B(\mathcal{N}_B \oplus \mathcal{O})$, and $C := Bl_{A\times\{0\}, B\times\{0\}} = Bl_{A,B}$. Notice, that C does not meet $\widehat{B}$. We have natural proper cartesian squares:

$$\begin{array}{ccccc} B & \xrightarrow{k_B} & \widehat{B} & \xleftarrow{i_B} & B \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \widehat{f}} & & \downarrow{\scriptstyle f} \\ D & \xrightarrow[i_D]{} & \widehat{A} & \xleftarrow[i_A]{} & A, \end{array}$$

where the left objects live over $\{0\}$, right ones over $\{1\}$, and j is given by the embedding of $\mathbb{P}_B(\mathcal{O})$ into $\mathbb{P}_B(\mathcal{N}_f \oplus \mathcal{O})$. Let $\widehat{\pi} : \widehat{A} \to A$ be the natural projection, and $u := \widehat{\pi}^*(v)$. Then $v = (i_A)^*(u)$. Let $\eta : \widehat{A} \to \mathbb{P}^1$ be the projection.

Since $\eta^*(\mathcal{O}(1)) = \mathcal{O}(D) \otimes \mathcal{O}(C)$, the difference $\delta := (c_1(\eta^*\mathcal{O}(1)) - c_1(\mathcal{O}(D)))$ is supported on C.

Since $\Phi^{q(t)}$ commutes with the pull-backs for open embeddings, the support of $\Phi^{q(t)}(x)$ belongs to the support of x. Since C does not meet $\widehat{B}$, $\widehat{f}^*(\delta \cdot z) = 0$ and $\widehat{f}^* \Phi^{q(t)}(\delta \cdot z) = 0$. Thus,

$$\Phi^{q(t)} \widehat{f}^*((i_A)_*(v)) = \Phi^{q(t)} \widehat{f}^*(u \cdot c_1(\eta^*(\mathcal{O}(1)))) = \Phi^{q(t)} \widehat{f}^*(u \cdot c_1(\mathcal{O}(D))) = \Phi^{q(t)}(\widehat{f}^*(i_D)_*(i_D)^*(u)) = \Phi^{q(t)}((k_B)_* j^*(i_D)^*(u))$$

By Proposition 3.1 and Lemma 3.5, this is equal to

$$(k_B)_* \Phi^{q(t)(\text{-}_\Omega t)}(j^*(i_D)^*(u)) = (k_B)_* j^* \Phi^{q(t)(c_1(\mathcal{O}(D)) \text{-}_\Omega t)}((i_D)^*(u)) = \widehat{f}^*(i_D)_* \Phi^{q(t)(c_1(D) \text{-}_\Omega t)}((i_D)^*(u)) = \widehat{f}^* \Phi^{q(t)}((i_D)_*(i_D)^*(u)) = \widehat{f}^* \Phi^{q(t)}(u \cdot c_1(\mathcal{O}(D))) = \widehat{f}^* \Phi^{q(t)}(u \cdot c_1(\eta^*(\mathcal{O}(1)))) = \widehat{f}^* \Phi^{q(t)}((i_A)_*(i_A)^*(u)) = \widehat{f}^* \Phi^{q(t)}((i_A)_*(v)).$$

Let $\nu := (\widehat{\pi}, \eta) : \widehat{A} \to A \times \mathbb{P}^1$. Since the maps i and ν are transversal, $\nu^* \circ i_* = (i_A)_*$.

$$\Phi^{q(t)}(\overline{f}^* i_*(v)) = \Phi^{q(t)}(\widehat{f}^* \nu^* i_*(v)) = \Phi^{q(t)}(\widehat{f}^*(i_A)_*(v)) = \widehat{f}^* \Phi^{q(t)}((i_A)_*(v)) = \widehat{f}^* \Phi^{q(t)}(\nu^* i_*(v)),$$

And the latter expression is equal to $\widehat{f}^* \nu^* \Phi^{q(t)}(i_*(v)) = \overline{f}^* \Phi^{q(t)}(i_*(v))$ by Proposition 2.11, since ν is transversal to any map of the form $i \circ v$. □

We immediately get:

Lemma 3.7 *Let $f : B \to A$ be regular embedding, and $q(t) \in \mathbb{L}[[t]]$ be divisible by t. Then $f^* \Phi^{q(t)}(v) = \Phi^{q(t)} f^*(v)$.*

Proof: Let $\overline{A} = A \times \mathbb{P}^1$, $\overline{B} = B \times \mathbb{P}^1$ with the projections $\overline{\pi}_A : \overline{A} \to A$, $\overline{\pi}_B : \overline{B} \to B$, embeddings $e_A = id \times \{1\} : A \to \overline{A}$, $e_B = id \times \{1\} : B \to \overline{B}$ and the map $\overline{f} = f \times id : \overline{B} \to \overline{A}$.

We have proper cartesian square:

$$\begin{array}{ccc} \overline{B} & \xleftarrow{e_B} & B \\ \downarrow{\scriptstyle \overline{f}} & & \downarrow{\scriptstyle f} \\ \overline{A} & \xleftarrow[e_A]{} & A. \end{array}$$

From Lemma 3.6 we have: for any $p(t) \in \mathbb{L}[[t]]$,

$$\Phi^{p(t)}(\overline{f}^*(e_A)_*(v)) = \overline{f}^* \Phi^{p(t)}((e_A)_*(v)).$$

But $\Phi^{p(t)}(\overline{f}^*(e_A)_*(v)) = \Phi^{p(t)}((e_B)_* f^*(v)) = (e_B)_* \Phi^{p(t)(c_1(\mathcal{O}(1)) \text{-}_\Omega t)}(f^*(v)) = (e_B)_* \Phi^{p(t)(\text{-}_\Omega t)}(f^*(v))$, and $\overline{f}^* \Phi^{p(t)}((e_A)_*(v)) = \overline{f}^*(e_A)_* \Phi^{p(t)(c_1(\mathcal{O}(1)) \text{-}_\Omega t)}(v) = (e_B)_* f^* \Phi^{p(t)(c_1(\mathcal{O}(1)) \text{-}_\Omega t)}(v) = (e_B)_* f^* \Phi^{p(t)(\text{-}_\Omega t)}(v)$. Since $(e_B)_*$ is injective, we get: $\Phi^{p(t)(\text{-}_\Omega t)}(f^*(v)) = f^* \Phi^{p(t)(\text{-}_\Omega t)}(v)$. The Lemma follows. □

It remains to treat the case of Φ^1.

Here we consider some modification of the deformation to the normal cone construction.

Consider two projective lines $\mathbb{P}^1_x$, $\mathbb{P}^1_y$ on projective plane $\mathbb{P}^2$, whose intersection is the point $\{0\}$. Let $\{1\}$ be some other point on $\mathbb{P}^1_x$.

Let $\widehat{\widehat{A}} := Bl_{A \times \mathbb{P}^2, B \times \mathbb{P}^1_y}$, $\widehat{A} = Bl_{A \times \mathbb{P}^1_x, B \times \{0\}}$, $\widehat{\widehat{B}} = Bl_{B \times \mathbb{P}^2, B \times \mathbb{P}^1_y} = B \times \mathbb{P}^2$, $\widehat{B}_x = Bl_{B \times \mathbb{P}^1_x, B \times \{0\}} = B \times \mathbb{P}^1_x$, $\widehat{D} = \mathbb{P}_{B \times \mathbb{P}^1_y}(\mathcal{N}_f \oplus \mathcal{O}(1))$, $\widehat{C} = Bl_{A \times \mathbb{P}^1_y, B \times \mathbb{P}^1_y} = Bl_{A,B} \times \mathbb{P}^1_y$, and $\widehat{B}_y = \widehat{\widehat{B}} \cap \widehat{D} = B \times \mathbb{P}^1_y$. Notice that $\widehat{C}$ does not meet $\widehat{\widehat{B}}$.

We have natural commutative diagram

$$\begin{array}{ccccccc} \widehat{B}_y & \xrightarrow{\widehat{k}_B} & \widehat{\widehat{B}} & \xleftarrow{\widehat{i}_B} & \widehat{B}_x & \xleftarrow{i_B} & B \\ \widehat{j}\downarrow & & \downarrow\widehat{\widehat{f}} & & \downarrow\widehat{f} & & \downarrow f \\ \widehat{D} & \xrightarrow[\widehat{i}_D]{} & \widehat{\widehat{A}} & \xleftarrow[\widehat{i}_A]{} & \widehat{A} & \xleftarrow[i_A]{} & A, \end{array}$$

with all squares proper cartesian, where the very right objects live over $\{1\}$.

Denote as $\mu : \widehat{\widehat{A}} \to \mathbb{P}^2$ the projection. Then $\mu^*(\mathcal{O}(1)) = \mathcal{O}(\widehat{A}) = \mathcal{O}(\widehat{D}) \otimes \mathcal{O}(\widehat{C})$. This implies that the element $\widehat{\delta} := (c_1(\mu^*(\mathcal{O}(1))) - c_1(\mathcal{O}(\widehat{D})))$ has support on $\widehat{C}$. Since $\Phi^{q(t)}$ commutes with pull-backs for open embeddings, the elements of the form $\Phi^{q(t)}(\widehat{\delta} \cdot z)$ have also support on $\widehat{C}$. Thus, $\widehat{\widehat{f}}^*(z \cdot \widehat{\delta}) = 0$ and $\widehat{\widehat{f}}^* \Phi^{q(t)}(z \cdot \widehat{\delta}) = 0$.

Finally, we have natural projections $\widehat{\widehat{\pi}} : \widehat{\widehat{A}} \to A$ and $\widehat{\pi} : \widehat{A} \to A$. Let v be arbitrary element of $\Omega^*(A)$. Let us denote: $w := \widehat{\widehat{\pi}}^*(v)$, $u := \widehat{\pi}^*(v)$. Clearly, $u = \widehat{i}_A^*(w)$, and $i_A^*(u) = v$.

Using Proposition 3.1, Lemma 3.5 and proper cartesian diagrams above, we get:

$$\begin{aligned} (\widehat{i}_B)_* \Phi^{(c_1(\mu^*(\mathcal{O}(1))) -_{\Omega} t)}(\widehat{f}^*(u)) = \Phi^1((\widehat{i}_B)_* \widehat{f}^*(u)) = \Phi^1(\widehat{\widehat{f}}^* (\widehat{i}_A)_*(u)) = \Phi^1(\widehat{\widehat{f}}^* (\widehat{i}_A)_* (\widehat{i}_A)^*(w)) = \\ \Phi^1(\widehat{\widehat{f}}^* (w \cdot c_1(\mu^*(\mathcal{O}(1))))) = \Phi^1(\widehat{\widehat{f}}^* (w \cdot c_1(\mathcal{O}(\widehat{D})))) = \Phi^1(\widehat{\widehat{f}}^* (\widehat{i}_D)_*(\widehat{i}_D)^*(w)) = \Phi^1((\widehat{k}_B)_* \widehat{j}^* (\widehat{i}_D)^*(w)) = \\ (\widehat{k}_B)_* \widehat{j}^* \Phi^{(c_1(\mathcal{O}(\widehat{D})) -_{\Omega} t)}((\widehat{i}_D)^*(w)) = \widehat{\widehat{f}}^* (\widehat{i}_D)_* \Phi^{(c_1(\mathcal{O}(\widehat{D})) -_{\Omega} t)}((\widehat{i}_D)^*(w)) = \widehat{\widehat{f}}^* \Phi^1((\widehat{i}_D)_*(\widehat{i}_D)^*(w)) = \\ \widehat{\widehat{f}}^* \Phi^1(w \cdot c_1(\mathcal{O}(\widehat{D}))) = \widehat{\widehat{f}}^* \Phi^1(w \cdot c_1(\mu^*(\mathcal{O}(1)))) = \widehat{\widehat{f}}^* \Phi^1((\widehat{i}_A)_*(\widehat{i}_A)^*(w)) = \widehat{\widehat{f}}^* \Phi^1((\widehat{i}_A)_*(u)) = \\ \widehat{\widehat{f}}^* (\widehat{i}_A)_* \Phi^{(c_1(\mu^*(\mathcal{O}(1))) -_{\Omega} t)}(u) = (\widehat{i}_B)_* \widehat{f}^* \Phi^{(c_1(\mu^*(\mathcal{O}(1))) -_{\Omega} t)}(u). \end{aligned}$$

From Lemma 3.7 and the fact that $(\widehat{i}_B)_*$ is injective, we get:

$$\Phi^{c_1(\mu^*(\mathcal{O}(1)))}(\widehat{f}^*(u)) = \widehat{f}^* \Phi^{c_1(\mu^*(\mathcal{O}(1)))}(u).$$

But then (again, using Lemma 3.5):

$$\begin{aligned} (i_B)_* \Phi^1(f^*(v)) = (i_B)_* \Phi^1(f^*(i_A)^*(u)) = (i_B)_* \Phi^1((i_B)^* \widehat{f}^*(u)) = (i_B)_*(i_B)^* \Phi^1(\widehat{f}^*(u)) = \\ \Phi^{c_1(\mu^*(\mathcal{O}(1)))}(\widehat{f}^*(u)) = \widehat{f}^* \Phi^{c_1(\mu^*(\mathcal{O}(1)))}((u)) = \widehat{f}^*(i_A)_*(i_A)^* \Phi^1(u) = (i_B)_* f^*(i_A)^* \Phi^1(u) = \\ (i_B)_* f^* \Phi^1((i_A)^*(u)) = (i_B)_* f^* \Phi^1(v). \end{aligned}$$

Since $(i_B)_*$ is injective, we get:

$$\Phi^1(f^*(v)) = f^* \Phi^1(v).$$

Proposition 3.4 is proven. □

3.2 The generating property of Φ

It follows from Proposition 3.4, Proposition 3.1, Proposition 3.2, and Proposition 2.8, that $\Phi^{q(t)}$ gives an operation $R^{q(t)} : \Omega^*(X) \to \Omega^*(X)$ for smooth quasiprojective varieties over k, which is $\Omega^*(X)$-linear on $q(t)$, and satisfies the following properties (with $r_0 = 1$):

(1) $R^{q(t)}$ commutes with the pull-backs;

(2) For the regular embedding $j : X \to Y$,

$$R^{q(t)}(j_*([v])) = j_*(R^{q(t)\cdot c^{\Omega}(N_j)(t)}([v]));$$

(3) $R^{q(t)}(1) = 0$.

(4) There exists $r_0 \in \mathbb{L}$ such that

$$R^{q(t)}(a+b) = R^{q(t)}(a) + R^{q(t)}(b) + q(0)\cdot r_0 \cdot a \cdot b.$$

In fact, all such operations are expressible in terms of Φ.

Theorem 3.8 *There is one-to-one correspondence between the operations $R^{q(t)}$ satisfying conditions* (1)–(4) *above, and the set of power series $r_R(t) \in \mathbb{L}[[t]]$ given by:*

$$R^{q(t)} = \Phi^{r_R(t)\cdot q(t)}.$$

Proof: Let us show that any operation $R^{q(t)}$ satisfying above conditions can be presented in the form $\Phi^{r_R(t)\cdot q(t)}$, for certain $r_R(t)$.

Considering $R^{q(t)} - \Phi^{r_0\cdot q(t)}$, we can assume that $r_0 = 0$, and the operation $R^{q(t)}$ is linear.

Lemma 3.9 *Operation $R^{q(t)}$ satisfying* $(1)-(4)$ *is uniquely determined by $R^{q(t)}|_{\mathbb{L}}$.*

Proof: We need to show that for arbitrary smooth quasiprojective X, and for arbitrary $[v : V \to X]$, $R^{q(t)}([v])$ is determined by the action on the Lazard ring. By the result of Hironaka ([3]) and [6, Theorem 2.2], we can assume that X is an open subvariety of a smooth projective variety $\overline{X}$, and v is the restriction of some cobordism class from $\overline{X}$. From the condition (1), it is enough to consider the case X-projective. Let $U = image(v)$. Use the induction on $\dim(U)$ (for all varieties X simultaneously). For $\dim(U) < 0$, U is empty, and so, the value is 0. Let now U be nonempty.

By the result of Hironaka, there exists the consecutive blow up $\mu : \widetilde{X} \to X$ with smooth centers having dimensions smaller than U, such that the proper preimage of U under μ is a regular subvariety $\widetilde{U}$ of $\widetilde{X}$. Let $a \in \mathbb{L}$ be the fiber of v over $k(U)$. Then $[v] - \mu_*[\widetilde{u}]\cdot a \in \Omega^*(X)$ has support of dimension less than the dimension of U. So, the value of $R^{q(t)}$ on this element is determined. Thus, we need only to show that the value on $\mu_*[\widetilde{u}]\cdot a$ is determined. Since $\mu_*(1) \in \Omega^0(X)$ is invertible by [6, Lemma 1.6], by the projection formula, $\mu^* : \Omega^*(X) \to \Omega^*(\widetilde{X})$ is injective. From property (1), it is sufficient to check that the value on $\mu^*\mu_*[\widetilde{u}]\cdot a$ is determined. Let $i : B \to A$ be regular embedding, and

$$\begin{array}{ccc} \mathbb{P}_B(\mathcal{N}_i) & \xrightarrow{j} & Bl_{A,B} \\ \varepsilon\downarrow & & \downarrow\pi \\ B & \xrightarrow[i]{} & A \end{array}$$

the blow-up diagram. Then it follows from the proof of Corollary 5.26 that for any $x \in \Omega^*(Bl_{A,B})$, $\pi^*\pi_*(x) - x = j_*(y)$, for some $y \in \Omega^*(\mathbb{P}_B(\mathcal{N}_i))$. And y can be expressed as a polynomial in $c_1(\mathcal{O}(-1))$ with coefficients in $\varepsilon^*\Omega^*(B)$. So, if $\dim(B) < \dim(U)$, by inductive assumptions, the value on $(\pi^*\pi_*(x) - x)$ is determined. Since the dimension of our centers is smaller than that of U, we get that the value on the element $(\mu^*\mu_*[\widetilde{u}] - [\widetilde{u}])\cdot a$ is determined. Thus, we need only to show that the value of $R^{q(t)}$ on $[\widetilde{u}]\cdot a$ is determined. But due to the condition (2), this value is equal to $R^{q(t)\cdot c^{\Omega}(\mathcal{N}_{\widetilde{u}})(t)}(f^*(a))$, where $f : \widetilde{U} \to \mathrm{Spec}(k)$ is the projection. Due to the condition (1) this value is determined by the values on a. $\square$

Let us extend the operation $R^{q(t)}$ to $\Omega^*(X)\otimes_{\mathbf{Z}}\mathbb{Q}$ by linearity.

Over the ring $\mathbb{L}\otimes_{\mathbf{Z}}\mathbb{Q}$, the universal formal group law $x+_{\Omega}y$ is equivalent to the additive one. That is, there exists a power series $log(x)\in\mathbb{L}\otimes_{\mathbf{Z}}\mathbb{Q}[[x]]$ with formal inverse $exp(z)$, such that $x+_{\Omega}y=exp(log(x)+log(y))$. Then $dlog(t)=\omega$ - the unique invariant differential form on $\mathrm{Spec}(\mathbb{L}[[t]])$ satisfying $\omega(0)=dt$. By the result of Mischenko, $log(x)=\sum_{i\geqslant 0}\frac{[\mathbb{P}^i]}{i+1}x^{i+1}$. Let us denote: $p_i:=\frac{[\mathbb{P}^i]}{i+1}$. By the result of Milnor, $\mathbf{L}\otimes_{\mathbf{Z}}\mathbb{Q}=\mathbb{Q}[p_1,p_2,\ldots]$.

Proposition 3.10 *Let z be the class of the smooth divisor on X, and $[v]\in\Omega^*(X)$ be arbitrary element. Let $R^{q(t)}$ be linear operation, satisfying $(1)-(3)$. Then we have the following identity:*

$$R^{q(t)}(v\cdot log(z))=R^{q(t)\cdot(-_{\Omega}t)}([v])\cdot log(z).$$

Proof: Let $q(t)$ be fixed, $u:=log(c_1(\mathcal{O}(1)))\in\Omega^1(\mathbb{P}^\infty)$, and $T(u):=R^{q(t)}([v]\cdot u)\in\Omega^*(X\times\mathbb{P}^\infty)=\Omega^*(X)[[u]]$.

Applying condition (1) to the Segre embedding $X\times\mathbb{P}^\infty\times\mathbb{P}^\infty\to X\times\mathbb{P}^\infty$, we get:

$$T(u_1)+T(u_2)=R^{q(t)}\Big([v]\cdot\Big(log(c_1(\mathcal{O}(1)_1))+log(c_1(\mathcal{O}(1)_2))\Big)\Big)=$$
$$R^{q(t)}\Big([v]\cdot log\Big(c_1(\mathcal{O}(1)_1)\underset{\Omega}{+}c_1(\mathcal{O}(1)_2)\Big)\Big)=R^{q(t)}\Big([v]\cdot log\Big(c_1(\mathcal{O}(1)_1\otimes\mathcal{O}(1)_2)\Big)\Big)=T(u_1+u_2).$$

Thus, the function $T(u)$ is linear, and so, $R^{q(t)}([v]\cdot u)=\lambda\cdot u$, for certain $\lambda\in\Omega^*(X)$.

Recounting that $[v]\cdot log(x)=[v]\cdot x+[v]\cdot p_1x^2+\ldots$, and using conditions (1) and (2), we get the equality:

$$R^{q(t)\cdot(x-_{\Omega}t)x}([v])+R^{q(t)\cdot(x-_{\Omega}t)^2x^2}([v]\cdot p_1)+\ldots=\lambda\cdot(x+p_1x^2+\ldots).$$

Comparing coefficients at x, we get: $\lambda=R^{q(t)\cdot(-_{\Omega}t)}([v])$. So, we proved the formula for the case $x=c_1(\mathcal{O}(1))$ on $X\times\mathbb{P}^\infty$. In reality, this equality is equivalent to the certain set of linear relations among $R^{q(t)t^r}([v]\cdot p_i)$ for all r and i - these are just coefficients at x^m for various m in the relation above. Now, the equality for arbitrary z (the class of a smooth divisor) follows from the obtained relations, by adding them with coefficients z^m with given z. □

In the remaining part of the proof of Theorem 3.8 let s will denote $(-_{\Omega}t)$, and our operation will be $R^{f(s)}$ (just different choice of the uniformizer). The difference will be that the condition (2) will now look as (we just need the special case of a divisor):

$(2')$ For the smooth divisor $j:D\to X$,

$$R^{f(s)}(j_*([v]))=j_*(R^{f(s)(s+_{\Omega}c_1(N_j))}([v]))$$

Lemma 3.11 *Let $R^{f(s)}$ be the additive operation satisfying conditions $(1)-(3)$. Then the restriction $R^{f(s)}|_{\mathbb{L}}$ is determined by the set $R^1(p_i)$ and $R^s(p_i)$, $i\geqslant 1$.*

Proof: Since $\mathbb{L}\otimes_{\mathbf{Z}}\mathbb{Q}=\mathbb{Q}[p_1,p_2,\ldots]$ has $\mathbb{Q}$-basis consisting of monomials $p_I:=\prod_{i\in I}p_i$, where I is the collection of natural numbers (with multiplicity), we need to show that $R^{s^m}(p_I)$, for all $m\geqslant 0$ is determined by $R^1(p_i)$ and $R^s(p_i)$.

Let us denote $\alpha_i:=\frac{\deg(c_i(-T_{\mathbb{P}^i}))}{(i+1)}=\frac{\binom{-(i+1)}{i}}{(i+1)}$, and $\alpha_I:=\prod_{i\in I}\alpha_i$.

Sublemma 3.12 *Let $R^{f(s)}$ be the additive operation satisfying conditions $(1)-(3)$. Then for arbitrary p_I and arbitrary $m\geqslant 0$, the difference*

$$R^{s^{m+2j}}(p_I\cdot p_j)-\alpha_jR^{s^m}(p_I)$$

is a linear combination with $\mathbb{L}_{>0}$-coefficients of $R^{s^n}(p_I\cdot p_l)$ with $n>m+2l$ and $l<j$.

Proof: For the homogeneous element $a \in \mathbb{L}$, and $m \geqslant 0$ let us call the *degree* of the term $R^{s^m}(a)$ the number $2\dim(a) - m$.

From Proposition 3.10 and properties (1) and (2′) we have the identities (for $m + r \geqslant 0$):

$$\sum_{i\geqslant 0} R^{s^{m+r}\cdot(s+_{\Omega}x)^{i+1}x^{i+1}}([v]\cdot p_i) = R^{s^{m+r+1}}([v])\cdot\sum_{i\geqslant 0} p_i\cdot x^{i+1}.$$

Take $[v] = p_I$, $0 \leqslant r \leqslant j-1$ and consider the coefficient at x^{r+2}. This coefficient will be the $\mathbb{L}$-linear combination of terms $R^{s^n}(p_I \cdot p_l)$ of degree $\leqslant$ degree of $R^{s^m}(p_I)$, where $l \leqslant r+1$. Moreover, the only term with $l = r+1$ will be $R^{s^{m+2(r+1)}}(p_I \cdot p_{r+1})$, and the terms of degree = degree of $R^{s^m}(p_I)$ will look as:

$$\sum_{i=[r/2]}^{r+1} R^{s^{m+2i}}(p_I\cdot p_i)\binom{i+1}{2i-r}.$$

Adding these equations for $0 \leqslant r \leqslant j-1$ with appropriate coefficients, we get that there exists $\lambda \in \mathbf{Z}$ such that $(R^{s^{m+2j}}(p_I\cdot p_j) - \lambda\cdot R^{s^m}(p_I))$ is a $\mathbb{L}_{>0}$-linear combination of terms $R^{s^n}(p_I\cdot p_l)$ of smaller degree with $l < j$. It remains to find λ.

Clearly, λ does not depend on m. If I is empty, then $R^{s^m}(p_I) = R^{s^m}(1) = 0$, by the condition (3), and the statement follows. Otherwise, take $m = 2\dim(p_I) - 1$. Observe that the same identities will be valid for any other additive operation satisfying $(1)-(3)$. In particular, for $\Phi^{s\cdot f(s)}$ and $\Psi^{s\cdot f(s)}$. Then the degree of $\Psi^{s\cdot s^m}(p_I)$ is zero, so the terms of smaller degree will disappear (this time (for Φ and Ψ), the degree is equal to the dimension), and we get the equality: $\Psi^{s^{2\dim(p_I\cdot p_j)}}(p_I\cdot p_j) - \lambda\cdot\Psi^{s^{2\dim(p_I)}}(p_I) = 0$. Since $\Psi^{s^{2\dim(X)}}([X]) = \deg(c_{\dim(X)}(-T_X))$, $\lambda = \frac{\alpha_{I\cup j}}{\alpha_I} = \alpha_j$. □

Sublemma 3.13 *Let $R^{f(s)}$ be the additive operation satisfying conditions $(1)-(3)$. Then for arbitrary homogeneous $a \in \mathbb{L}$, $R^{s^m}(a) = 0$, for arbitrary $m \geqslant 2\dim(a)$.*

Proof: If we would assume operation $R^{f(s)}$ graded (with s of degree -1, and R^1 of the same degree as operation square), then (at least, for $m > 2\dim(a)$) the statement would follow just from dimensional consideration (since $\mathbb{L}_{<0} = 0$). But, the point is, this triviality is, anyway, encoded in the conditions $(1)-(3)$. Really, using the induction on $\dim(p_I)$, it immediately follows from the Sublemma 3.12 that for $m \geqslant 2\dim(a)$, $R^{s^m}(p_I)$ is a $\mathbb{L}$-linear combination of $R^{s^k}(1)$ for certain nonnegative k, which is zero. Since p_I form a $\mathbb{Q}$-basis of $\mathbb{L}\otimes_{\mathbf{Z}}\mathbb{Q}$, the statement follows. □

Let us show now that for arbitrary $n \geqslant 0$, and for arbitrary I, $R^{s^n}(p_I)$ is expressed in terms of $R^1(p_i)$ and $R^s(p_i)$. Use induction on the degree of $R^{s^n}(p_I)$. It follows from the Sublemma 3.13, that for the degree $\leqslant 0$, $R^{s^n}(p_I)$ is zero. To make induction step, apply Sublemma 3.12, which shows that, modulo terms of smaller degree, $R^{s^{m+2j}}(p_I\cdot p_j)$ is expressed through $R^{s^m}(p_I)$, and vise-versa. Thus, all terms $R^{s^r}(p_J)$ of the given degree are expressed, modulo smaller terms, through any given one. Since the terms $R^1(p_i)$, $R^s(p_i)$, for $i \geqslant 1$ cover all positive degrees, we are done.

Lemma 3.11 is proven. □

Let us now inductively define $r_i \in \mathbb{L}\otimes_{\mathbf{Z}}\mathbb{Q}$ as

$$r_{2k+1} := \frac{R^t(p_{k+1}) - \Phi^{r_1t^2+\ldots+r_{2k}t^{2k+1}}(p_{k+1})}{\Phi^{t^{2(k+1)}}(p_{k+1})};$$

$$r_{2k+2} := \frac{R^1(p_{k+1}) - \Phi^{r_1t+\ldots+r_{2k+1}t^{2k+1}}(p_{k+1})}{\Phi^{t^{2(k+1)}}(p_{k+1})}.$$

Notice, that the denominator here is just $\frac{1}{2(k+2)}\binom{-(k+2)}{k+1} \in \mathbb{Q}$. We obtain the power series $r_R(t) := r_1t + r_2t^2 + \ldots \in \mathbb{L} \otimes_{\mathbf{Z}} \mathbb{Q}[[t]]$.

Then the operation $S^{q(t)} := \Phi^{r_R(t)\cdot q(t)}$ satisfy: $S^1(p_i) = R^1(p_i)$, and $S^t(p_i) = R^t(p_i)$. By Lemmas 3.9 and 3.11, operations $R^{q(t)}$ and $S^{q(t)}$ coincide.

It remains to check that r_i belongs to $\mathbb{L}$, and not just to $\mathbb{L} \otimes_{\mathbf{Z}} \mathbb{Q}$. Use induction on i. Suppose, for $j < i$, $r_j \in \mathbb{L}$. Since $R^{q(t)}|_{\mathbb{L}} : \mathbb{L} \to \mathbb{L}$, we have: $\Phi^{t^k\cdot(r_it^i + r_{i+1}t^{i+1}+\ldots)}(\mathbb{L})$ takes values in $\mathbb{L}$. In particular, for arbitrary $n \geqslant i/2$, $\Phi^{r_it^{2n}}([\mathbb{P}^n])$ belongs to $\mathbb{L}$. But this element is equal to $r_i \cdot \frac{1}{2}\binom{-(n+1)}{n}$. Since for any given prime p, $\frac{1}{2}\binom{-(p^l+1)}{p^l}$ is not divisible by p, and we can choose l such that $p^l \geqslant i/2$, we get that $r_i \in \mathbb{L}$.

Thus, we have proved that any operation $R^{q(t)} : \Omega^*(X) \to \Omega^*(X)$ satisfying $(1)-(4)$ can be expressed in the form $\Phi^{r_R(t)\cdot q(t)}$, for certain $r_R(t) = r_0 + r_1t + \ldots \in \mathbb{L}[[t]]$ (we easily return to the general (non-additive) case).

It remains to notice, that different power series $r_R(t)$ give different operations. Really, let $r_R(t) \neq 0$, and r_lt^l - the first nonzero term. Then, for any $n \geqslant l/2$, $\Phi^{r_R(t)t^{2n-l}}([\mathbb{P}^n]) = r_l \cdot \binom{-(n+1)}{n} \neq 0$. Thus, the operation is nonzero.

Theorem 3.8 is proven.

□

Example: As we saw above, for the operation Ψ,

$$r_\Psi(t) = \frac{[2]_\Omega(t)}{t} = 2 + a_{1,1}t + 2a_{2,1}t^2 + (2a_{3,1} + a_{2,2})t^3 + \ldots.$$

3.3 Chow traces

Denote as $\psi^{q(t)} : \Omega^*(X) \to \mathrm{CH}^*(X)$ and $\phi^{q(t)} : \Omega^*(X) \to \mathrm{CH}^*(X)$ the Chow traces of operations Ψ and Φ, respectively. For $[v] \in \Omega^*(X)$ let us denote as $pr([v]) \in \mathrm{CH}^*(X)$ its Chow trace.

The following statement follows immediately from Propositions 2.13 and 2.15. Notice the discrepancy in notations between [17] and the current article: $\Phi^{codim([v])+r}$ of [17] is now $\Phi^{(\text{-}_\Omega\, t)^r}$.

Proposition 3.14 ([17, Propositions 3.8,3.9])

(1) $\psi^{q(t)}([v]) = q(0)\cdot(pr([v]))^2 - pr \circ S^g_{L.-N.}$, *where* $g = \sum_l (-1)^l q_l \cdot \sigma_{l+d}$, *and* $d = codim([v])$.

(2) $\psi^{q(t)} = 2\cdot\phi^{q(t)}$.

For a smooth projective variety X, M.Rost defines invariant $\eta_2(X) \in \mathbf{Z}$ as the degree of zero-cycle $c_1(\mathcal{L})^{2\dim(X)} \cdot [\widetilde{C}^2(X)]$. This is nothing else, but $\Phi^{t^{2\dim(X)}}([X]) \in \Omega_0(\mathrm{Spec}(k)) = \mathbf{Z}$. M.Rost showed (see [9, Theorem 6.1]) that this number can also be expressed as $\frac{-deg(c_{\dim(X)}(-T_X))}{2}$. One can see it from Proposition 3.14.

Let $[v : V \to X] \in \Omega^*(X)$, and $[u : U \to \mathrm{Spec(k)}] \in \mathbb{L}_{>0}$. The following Proposition shows that if you have at your disposal the element $[u]\cdot[v]$, then you can get, may be, not $[v]$ itself, but, at least, up to the multiple $\eta_2(U)$, Chow-traces of certain Landweber-Novikov operations from $[v]$. One can get analogous result just with the help of the Landweber-Novikov operations, but then the multiple will be $2\eta_2(U)$, which is crusial, if one studies 2-torsion effects.

Proposition 3.15 *Let* $r = (\mathrm{codim}(v) - 2\dim(u))$. *Then*

$$\phi^{q(t)}([v]\cdot[u]) = \eta_2(U)\cdot pr \circ S^h_{L.-N.}([v]),$$

where $h = \sum_{i\geqslant\max(r;0)} (-1)^{i-r} q_{i-r}\cdot\sigma_i$.

Proof: We have the composition $V \times U \stackrel{\beta}{\to} V \stackrel{v}{\to} X$. $\Phi^{q(t)}([v]\cdot[u]) = \Theta(v\circ\beta)(q(\varrho))$. From Proposition 2.5,

$$\Theta(v\circ\beta) = \Theta(v)\circ \widetilde{C}^2(\beta)_* + v_*(\Theta^{g(s)}(\beta)),$$

where $g(s) = \mathrm{Res}_{t=0}\frac{c^{\Omega}(-T_v)(t)\cdot\omega}{(t-_{\Omega}s)}$. Consequently,

$$\phi^{q(t)}([v]\cdot[u]) = pr\circ\Theta(v)\circ\widetilde{C}^2(\beta)_*(q(\varrho)) + v_*(pr\circ\Theta^{f(s)}(\beta)(q(\varrho))),$$

where $f(s) = \mathrm{Res}_{t=0}\frac{c^{\mathrm{CH}}(-T_v)(t)\cdot\omega^{\mathrm{CH}}}{(t-s)} = \sum_{i=0}^{d}(-1)^{d-i}c_i(-T_v)s^{d-i}$, where $d = \mathrm{codim}(v)$. Let us first study the second summand. Let

$$\widetilde{C}^2(V\times U) \stackrel{C(\beta)}{\longleftarrow} \widetilde{C}^2(V\times U/V) = V\times\widetilde{C}^2(U) \stackrel{D(\beta)}{\to} V$$

be the standard maps. Then

$$v_*(pr\circ\Theta^{f(s)}(\beta)(q(\varrho))) = v_*(pr\circ\Theta(\beta)(f(\varrho)\cdot q(\varrho))) =$$
$$v_*(pr\circ D(\beta)_*C(\beta)^*(f(\varrho)\cdot q(\varrho))) = v_*(D(\beta)_*^{\mathrm{CH}}((f\cdot q)(\varrho_{CH}))) =$$
$$v_*((f\cdot q)_{2\dim(U)}\cdot\eta_2(U)\cdot[V]) = v_*(\eta_2(U)\sum_{i=\max(r;0)}^{d}(-1)^{d-i}q_{2\dim(U)-d+i}\cdot c_i(-T_v)[V]) =$$
$$\eta_2(U)\cdot\sum_{i=\max(r;0)}^{d}(-1)^{i-r}pr(q_{i-r}\cdot S^i_{L.-N.}([v])),$$

where $S^i_{L.-N.}$ is the Landweber-Novikov operation, corresponding to the characteristic number c_i, and we assume $q_j = 0$, for $j < 0$.

Let us denote as $\widetilde{C}^2(-)^{CH}_*$, $\Theta(-)^{CH}$, etc. ...- the maps analogous to $\widetilde{C}^2(-)_*$, $\Theta(-)$, ..., but defined for Chow groups.

To study the first term of our equation, we will need the following lemma.

Lemma 3.16

$$\widetilde{\square}(\beta)^{CH}_*(\rho^m_{CH}) = \begin{cases} 0, & \textit{if } m\leqslant 2\dim(U);\\ -2\eta_2(U)\cdot\rho_{CH}^{m-2\dim(U)}, & \textit{otherwise.}\end{cases}$$

$$\widetilde{C}^2(\beta)^{CH}_*(\varrho^m_{CH}) = \begin{cases} 0, & \textit{if } m\leqslant 2\dim(U);\\ -2\eta_2(U)\cdot\varrho_{CH}^{m-2\dim(U)}, & \textit{otherwise.}\end{cases}$$

Proof: We have natural maps

$$\widetilde{\square}(V\times U) \stackrel{a(\beta)}{\longleftarrow} Bl_{\widetilde{\square}(V\times U),V\times\widetilde{\square}(U)} \stackrel{b_1}{\to} \widetilde{\square}(V)\times\square(U) \stackrel{b_2}{\to} \widetilde{\square}(V),$$

where b_1 is blow-down map with the center $Z = \mathbb{P}_V(T_V)\times U \to \widetilde{\square}(V)\times\square(U)$, and $b_2\circ b_1 = b(\beta)$. Moreover, $a(\beta)^*(\mathcal{O}(-\rho)) = \mathcal{O}(b_1^{-1}(Z))$, and
$b_1^{-1}(Z) = \mathbb{P}_Z(\mathcal{X})$, where $\mathcal{X} = \mathcal{O}(-1)\oplus T_U$.

Since $\dim(U) > 0$, clearly, $\widetilde{\square}(\beta)^{CH}_*(1) = 0$ by dimensional reasons. For $m > 0$, $\widetilde{\square}(\beta)^{CH}_*(\rho^m_{CH}) = j_*\eta_*\varepsilon_*((-1)^{m-1}c_1(\mathcal{O}(1))^{m-1}_{CH}[\mathbb{P}_Z(\mathcal{X})])$, where
$\varepsilon : \mathbb{P}_Z(\mathcal{X}) \to Z$, $\eta : Z = \mathbb{P}_V(T_V)\times U \to \mathbb{P}_V(V)$ are the projections, and $j : \mathbb{P}_V(T_V) \to \widetilde{\square}(V)$ is the standard embedding. So, we get:

$$j_*\eta_*((-1)^{m-1}c^{CH}_{m-1-\dim(U)}(-\mathcal{X})[Z]) = deg(c_{\dim(U)}(-T_U)[U])\cdot j_*((-1)^{m-1}c_1^{CH}(\mathcal{O}(1))^{m-1-2\dim(U)}),$$

and the latter expression is zero for $m \leqslant 2\dim(U)$, and equal to $-2\eta_2(U)\cdot \rho_{CH}^{m-2\dim(U)}$, for $m > 2\dim(U)$. The first formula is proven.

Let now V be such that $\mathrm{CH}^*(\widetilde{C}^2(V))$ has no 2-torsion. Consider commutative diagram:

$$\begin{array}{ccccc} \widetilde{\square}(V\times U) & \xleftarrow{a(\beta)} & Bl_{\widetilde{\square}(V\times U),V\times\widetilde{\square}(U)} & \xrightarrow{b(\beta)} & \widetilde{\square}(V) \\ {\scriptstyle p_{V\times U}}\downarrow & & \downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle p_V} \\ \widetilde{C}^2(V\times U) & \xleftarrow{c(\beta)} & Bl_{\widetilde{C}^2(V\times U),V\times\widetilde{C}^2(U)} & \xrightarrow{d(\beta)} & \widetilde{C}^2(V), \end{array}$$

with the left square proper cartesian (by Statement 5.11). Then

$$\widetilde{C}^2(\beta)_*^{CH}(2\cdot\varrho_{CH}^m) = \widetilde{C}^2(\beta)_*^{CH}((p_{V\times U})_*(\rho_{CH}^m)) = (p_V)_*\widetilde{\square}(\beta)_*^{CH}(\rho_{CH}^m) =$$
$$-2\eta_2(U)\cdot(p_V)_*(\rho_{CH}^{m-2\dim(U)}) = -4\eta_2(U)\cdot\varrho_{CH}^{m-2\dim(U)}$$

Since $\mathrm{CH}^*(\widetilde{C}^2(V))$ has no 2-torsion, we are done.

Let now V be arbitrary smooth quasi-projective variety. Let $i: V \to W$ be the composition of open and regular embedding into some variety, $\widetilde{C}^2$ of which has no 2-torsion in Chow groups (say, $W = \mathbb{P}^N$). We have commutative diagram:

$$\begin{array}{ccccc} \widetilde{C}^2(V\times U) & \xleftarrow{c(\beta)} & Bl_{\widetilde{C}^2(V\times U),V\times\widetilde{C}^2(U)} & \xrightarrow{d(\beta)} & \widetilde{C}^2(V) \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i_3} \\ \widetilde{C}^2(W\times U) & \xleftarrow{c(\beta')} & Bl_{\widetilde{C}^2(W\times U),W\times\widetilde{C}^2(U)} & \xrightarrow{d(\beta')} & \widetilde{C}^2(W), \end{array}$$

where the right square is proper cartesian (by Statement 5.13). Then,

$$\widetilde{C}^2(\beta)_*^{CH}(\varrho_{CH}^m) = \widetilde{C}^2(\beta)_*^{CH}((i_1)^*(\varrho_{CH}^m)) = (i_3)^*\widetilde{C}^2(\beta')_*^{CH}(\varrho_{CH}^m) =$$
$$(i_3)^*(-2\eta_2(U)\cdot\varrho_{CH}^{m-2\dim(U)}) = -2\eta_2(U)\cdot\varrho_{CH}^{m-2\dim(U)}$$

The second formula is proven. $\square$

Consider the standard maps: $\mathbb{P}_V(T_V) \xrightarrow{j} \widetilde{\square}(V) \xrightarrow{p} \widetilde{C}^2(V)$. Then

$$[j] = c_1(\mathcal{O}(-1)) = \rho \in \Omega^1(\widetilde{\square}(V)), \text{ and } [p\circ j] = [2]_\Omega(\varrho) \in \Omega^1(\widetilde{C}^2(V)).$$

Since $p^*(\varrho) = \rho$, we have $2\varrho_{CH}^{m-2\dim(U)} = p_*(\rho_{CH}^{m-2\dim(U)})$, and

$$\Theta(v)^{CH}(2\varrho_{CH}^{m-2\dim(U)}) = \Upsilon(v)^{CH}(\rho_{CH}^{m-2\dim(U)}) = \psi^{t^{m-2\dim(U)}}([v]) = (-1)^{m+1}pr\circ S_{L.-N.}^{m+d-2\dim(U)}([v]),$$

by Proposition 3.14

$$\Theta(v)^{CH}\circ\widetilde{C}^2(\beta)_*^{CH}(q(\varrho_{CH})) = -\eta_2(U)\cdot\Theta(v)^{CH}(\sum_{j>2\dim(U)} 2q_j\cdot\varrho_{CH}^{j-2\dim(U)}) =$$
$$\eta_2(U)\cdot\sum_{j>2\dim(U)}(-1)^j pr(q_j\cdot S_{L.-N.}^{j+d-2\dim(U)}([v])) = \eta_2(U)\cdot\sum_{i>d}(-1)^{i-r}pr(q_{i-r}\cdot S_{L.-N.}^i([v])).$$

Putting things together, we get:

$$\phi^{q(t)}([v]\cdot[u]) = \eta_2(U)\cdot\sum_{l\geqslant\max(r;0)}(-1)^{l-r}pr(q_{l-r}\cdot S_{L.-N.}^l([v])).$$

$\square$

3.4 Few words about $\widetilde{\square}$

Here I just wanted to mention that the operation $\widetilde{\square}$ can be used to construct Steenrod operations ($mod\, 2$).

Theorem 3.17 ([17, Theorem 2.2]) *We have the following commutative diagram:*

$$\begin{array}{ccc} \Omega^*(X) & \xrightarrow{j^*\circ(\widetilde{\square})^1} & \Omega^*(\mathbb{P}_X(T_X)) \\ {\scriptstyle pr/2}\downarrow & & \downarrow{\scriptstyle pr/2} \\ \mathrm{CH}^*(X)/2 & \xrightarrow[S]{} & \mathrm{CH}^*(\mathbb{P}_X(T_X))/2, \end{array}$$

where $S(z) = \oplus_{l=0}^{d} \rho^{d-l} S^l(z)$, $d = codim(z)$, and S^l is the Steenrod operation (see [21] and [1]).

4 Some applications

4.1 Algebraic cobordism of a Pfister quadric

The Pfister quadric is a rare example of noncellular variety for which the ring of algebraic cobordism is computed - see [19]. This was possible since the extension of scalars map is injective in this case. So, one needs only to find out which elements over algebraic closure are defined over the base field. But the proof of injectivity from [19, Proposition 4.4] uses the original computation by M.Rost of the Chow groups of a Pfister quadric (in the case of $MGL^{2*,*}$ there is an independent computation, using motivic homotopy theory - see [19, Theorem 7.2]). In the current subsection we would like to give another proof of this fact, which is based on symmetric operations, does not use the computations of M.Rost, and, in turn, gives a new way to compute the Chow groups.

Let $\alpha = \{a_1, \dots, a_n\} \in K_n^M(k)/2$ be a pure symbol. Let Q_α be corresponding Pfister quadric.

Theorem 4.1 ([19, Proposition 4.4]) *Let $E/F/k$ be any extension of fields. Then the map*

$$\Omega^*(Q_\alpha|_F) \to \Omega^*(Q_\alpha|_E)$$

is injective.

Proof: It was shown by M.Rost (see [15]) that the Chow motive of Q_α decomposes into simpler, so-called, Rost motives: $M^{CH}(Q_\alpha) = \oplus_{i=0}^{2^{n-1}-1} M_\alpha(i)[2i]$, where M_α is indecomposable, and $M_\alpha|_{\overline{k}} = \mathbf{Z} \oplus \mathbf{Z}(2^{n-1}-1)[2^n-2]$. It follows from [19, Corollary 2.8] that the same kind of decomposition exists for the cobordism motive: $M^\Omega(Q_\alpha) = \oplus_{i=0}^{2^{n-1}-1} M_\alpha^\Omega(i)[2i]$.

Clearly, it is sufficient to prove injectivity for $\Omega^*(M_\alpha^\Omega)$. Use induction on n.

($n=0$) $M_\alpha^\Omega = M^\Omega(\mathrm{Spec}(k\sqrt{a_1}))$, and $\Omega^*(M_\alpha)$ is either $\mathbb{L}$, or $\mathbb{L}\oplus\mathbb{L}$, depending on: if $\sqrt{a_1} \in k$, or not.

($n=1$) $M_\alpha^\Omega = M^\Omega(C_{\{a_1,a_2\}})$, where $C_{\{a_1,a_2\}} =: C$ is a conic. By [5, Theorem 2.2], we have exact sequence:

$$\oplus_{p\in(C|_L)^{(1)}} \Omega^*(\mathrm{Spec}(L(p))) \longrightarrow \Omega^*(C|_L) \longrightarrow \Omega^*(L(C)) \longrightarrow 0$$

Since $\Omega_0(C|_L) = \mathrm{CH}_0(C|_L) = \mathbf{Z}$ with generator - the class of the point p_0 of degree either 1, or 2, we get commutative diagram

$$\begin{array}{ccccccc} \mathbb{L} & \xrightarrow{j} & \Omega^*(C|_L) & \xrightarrow{e^*} & \mathbb{L} & \longrightarrow & 0 \\ & & {\scriptstyle \pi_*}\downarrow & & & & \\ & & \mathbb{L} & & & & , \end{array}$$

with exact row and $\pi_* \circ j$ be either multiplication by 1, or by 2 (π here is the projection $C|_L \to \mathrm{Spec}(L)$). Thus, the map $(e^*, \pi_*) : \Omega^*(C|_L) \to \mathbb{L} \oplus \mathbb{L}$ is injective. Hence, the extension of scalars map is injective too.

($1 \leqslant n \Rightarrow n+1$) Let $\alpha = \{a_1, \dots, a_{n+1}\}$. Consider $\beta = \{a_1, \dots, a_n\}$. The motive M^{Ω}_{β} is a direct summand in $M^{\Omega}(P)$, where P is $2^{n-1} - 1$-dimensional (norm-)quadric. Moreover, the projection $M^{\Omega}(P) \to M^{\Omega}(\mathrm{Spec}(k)) = \mathbf{L}$ can be decomposed as: $M^{\Omega}(P) \xrightarrow{f} M^{\Omega}_{\beta} \xrightarrow{g} \mathbf{L}$. Consider the diagram:

$$\begin{array}{ccccc} \Omega^*(M^{\Omega}_{\alpha}|_F) & \xrightarrow{g^*} & \Omega^*(M^{\Omega}_{\alpha} \otimes M^{\Omega}_{\beta}|_F) & \xrightarrow{f^*} & \Omega^*(M^{\Omega}_{\alpha} \otimes M^{\Omega}(P)|_F) \\ \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i_3} \\ \Omega^*(M^{\Omega}_{\alpha}|_E) & \xrightarrow[g^*]{} & \Omega^*(M^{\Omega}_{\alpha} \otimes M^{\Omega}_{\beta}|_E) & \xrightarrow[f^*]{} & \Omega^*(M^{\Omega}_{\alpha} \otimes M^{\Omega}(P)|_E) \end{array}$$

Since $M^{CH}_{\alpha} \otimes M^{CH}_{\beta} = M^{CH}_{\beta} \oplus M^{CH}_{\beta}(2^n - 1)[2^{n+1} - 2]$, the same is true for cobordism-motives due to [19, Corollary 2.8]. By inductive hypothesis, i_2 is injective, and $\mathrm{Ker}(i_1) \subset \mathrm{Ker}(g^*) \subset \mathrm{Ker}((g \circ f)^*)$. Let $x \in \Omega_r(M^{\Omega}_{\alpha})$ belongs to $\mathrm{Ker}(i_1)$. Then $0 = \pi_*\pi^*(x) = x \cdot [P]$, where $\pi : P \to \mathrm{Spec}(k)$. If $r = 0$, then $\Omega_0 = \mathrm{CH}_0$ and the absence of the kernel follows from the theorem of Swan. If $r > 0$, then $2\dim(P) \geqslant \mathrm{codim}(x)$, and we can apply Proposition 3.15:

$$0 = \phi^{t^{2^{n-1}-1}}(x \cdot [P]) = (-1)^{\mathrm{codim}(x)} \eta_2(P) \cdot pr(x).$$

Notice, that $\dim(P) > 0$. Since $\eta_2(P)$ is odd, this implies: $pr(x) = 0$. Consequently, $\mathrm{Ker}(i_1) \subset \mathbb{L}_{>0} \cdot \Omega^*(M^{\Omega}_{\alpha}|_F)$. Hence,

$$\mathrm{CH}^*(M_{\alpha}|_F) = \Omega^*(M^{\Omega}_{\alpha}|_F)/\mathbb{L}_{>0} \cdot \Omega^*(M^{\Omega}_{\alpha}|_F) = image(i_1)/\mathbb{L}_{>0} \cdot image(i_1),$$

and taking $E = \overline{F}$, we get the Rost's description of the Chow groups of M_{α} from [19, Theorem 3.4]. Now we can proceed as in the proof of [19, Proposition 4.4]. More precisely, it immediately follows from [19, Proposition 4.3] that there is an ($\mathbb{L}$-module) section $image(i_1) \to \Omega^*(M^{\Omega}_{\alpha}|_F)$. Or, in other words, $\mathrm{Ker}(i_1)$ is a direct summand of $\Omega^*(M^{\Omega}_{\alpha}|_F)$. Consequently, $\mathrm{Ker}(i_1)/\mathbb{L}_{>0} \cdot \mathrm{Ker}(i_1) = 0$, which implies: i_1 is injective for $E = \overline{F}$. Since, if both F and E are algebraically closed, then i_1 is an isomorphism, we get that i_1 is monomorphic for arbitrary extension E/F. □

Let $e^* : \Omega^*(M^{\Omega}_{\alpha}) \to \mathbb{L}^*$ be the restriction to the generic point, and $\pi_* : \Omega_*(M^{\Omega}_{\alpha}) \to \mathbb{L}_*$ be the projection to the point. One gets:

Theorem 4.2 ([19, Theorem 3.4,Proposition 4.4]) *The map*

$$(e^*, \pi_*) : \Omega^*(M^{\Omega}_{\alpha}) \to \mathbb{L} \oplus \mathbb{L}$$

is injective, and its image is equal to $\mathbb{L} \cdot (1, [Q_{2^{n-1}-1}]) \oplus I(2, n-2) \cdot (0, 1)$*, where* $I(2, n-2)$ *- the ideal in* $\mathbb{L}$ *generated by the classes of quadrics* $[Q_r]$ *of dimensions from* 0 *to* $2^{n-2} - 1$.

4.2 Rationality of cycles

Let Y be smooth quasiprojective variety over the field k. In many situations one needs to compute the image of the restriction of scalars map: $ac : \mathrm{CH}^m(Y) \to \mathrm{CH}^m(Y|_{\overline{k}})$. If $\overline{y} \in image(ac)$ we say that $\overline{y}$ is *k-rational.* It appears, that sometimes this condition may be checked over some bigger field $k(Q)/k$, where the situation could be simpler. For example, it is a standard application of the Rost degree formula, that if p is a prime number, and Q is a ν_n-variety of dimension $p^{n-1} - 1$, then for arbitrary Y of dimension

$< \dim(Q)$, Y has a zero cycle of degree prime to p if and only if $Y|_{k(Q)}$ does. For quadrics this gives: if $\dim(Q) \geqslant 2^r - 1 > \dim(Y)$, then the existence of points of odd degree on Y and $Y|_{k(Q)}$ is equivalent. Moreover this condition on the dimension is the optimal one. Operation Φ permits to address similar questions about the cycles of arbitrary dimension. We have the following:

Theorem 4.3 ([18, Corollary 3.5]) *Let Y be a smooth quasiprojective variety over a field of characteristic 0, and Q be a smooth projective quadric. Suppose $\overline{y} \in \mathrm{CH}^m(Y|_{\overline{k}})/2$, and $m < [\dim(Q) + 1/2]$. Then $\overline{y}$ is k-rational if and only if $\overline{y}|_{\overline{k(Q)}}$ is $k(Q)$-rational.*

There is a generalization of the above result, which roughly speaking says that even if the condition $m < [\dim(Q) + 1/2]$ is not satisfied, but $\overline{y}|_{\overline{k(Q)}}$ is $k(Q)$-rational, you still get some rational cycles over k: certain Steenrod operations applied to $\overline{y}$.

This result is proven with the help of Proposition 3.15. If one does not mind moding out 2-torsion in Chow groups, then similar result can be obtained just with the help of the usual Landweber-Novikov operations, with Proposition 3.15 substituted by the multiplicative properties of such operations. But to get the "clean" statement as above, the use of Φ is essencial.

5 Appendix

The aim of this appendix is to provide the proof of certain facts we use in the main part of the paper which were not contained in the original version of [5]. Since then some of these facts, probably, appeared somewhere, but I decided to provide independent proofs here to make the article more self-contained.

5.1 On the definition of Algebraic Cobordism

Here I would like to comment on the definiton of Algebraic Cobordism given in Subsection 2.1. This definition is a bit different from the standard one appearing in [5]. But it corresponds to the construction given on p. 727 of [6].

Let me show that both definitions are equivalent. I would like to thank the referee for providing these arguments.

Let us denote the object obtained by constructed from Subsection 2.1 as $\overline{\Omega}_*$. Then $\overline{\Omega}_*$ is provided with the sequence of surjections

$$\mathcal{M}_*(X) \to \mathrm{Pre}-\Omega_*(X) \to \tilde{\Omega}_*(X) \to \overline{\Omega}_*(X).$$

On $\mathrm{Pre}-\Omega_*(X)$ one has well-defined first Chern class operators $\tilde{c}_1(\mathcal{L})$ for globally generated $\mathcal{L}$. On $\tilde{\Omega}(X)$ one imposes a formal group law for the $\tilde{c}_1(\mathcal{L})$. Namely, following the arguments from the proof of [8, Proposition 2.5.15], one obtains, that for the power series $F_{\tilde{U}}$ constructed in Subsection 2.1, the pair $(\tilde{\Omega}_*, F_{\tilde{U}})$ gives a commutative formal group. This enables the extension of the operations $\tilde{c}_1(\mathcal{L})$ to all $\mathcal{L}$, and gives a ring homomorphism $\mathbb{L} \to \tilde{\Omega}_*(\mathrm{Spec}(k))$. Finally, on $\tilde{\Omega}_*(X)$ one imposes the relation $\tilde{c}_1(\mathcal{O}_Y(D))(1_Y) = [i : D \to Y]$ if $i : D \to Y$ is the inclusion of a smooth Cartier divisor on a smooth Y. In addition, the external product on $\mathcal{M}_*$ descends to give an external product on $\overline{\Omega}_*$.

Now, it is a formal exercise to check that $\overline{\Omega}_*$ is an "oriented Borel-Moore $\mathbb{L}$-functor of geometric type on Sch_k" in the sense of [8, Sections 2.1,2.2].

Since all the relations for $\overline{\Omega}_*$ are valid in Ω_*, the canonical map $\mathcal{M}_*(X) \to \Omega_*(X)$ (which is shown by M.Levine-F.Morel to be a surjection - see [8, Lemma 2.5.11]) factors through $\overline{\Omega}_*(X) \to \Omega_*(X)$. Since Algebraic Cobordism is the universal oriented Borel-Moore $\mathbb{L}$-functor of geometric type, we get canonical map $\Omega_* \to \overline{\Omega}_*$. It is easy to check that the composition $\overline{\Omega}_* \to \Omega_* \to \overline{\Omega}_*$ is the identity. Hence, $\overline{\Omega}_* \to \Omega_*$ is an isomorphism.

5.2 The dimension of support

Definition 5.1 *Let $[v] \in \Omega^*(X)$. We say that $[v]$ has support of codimension $\geqslant m$, if there exists closed subscheme $Z \subset X$ of codimension m with open compliment $i : U \to X$, such that $i^*([v]) = 0$.*

It is a consequence of [8, Theorem 1.2.14] that for any $[v]$ with codimension of support $\geqslant m$ there exist $z_i : Z_i \to X$ and $\lambda_i \in \mathbb{L}$ such that $[z_i] \in \Omega^{\geqslant m}(X)$, and

$$[v] = \sum_i \lambda_i \cdot [z_i].$$

We get the following:

Statement 5.2 *Let $[v], [w] \in \Omega^*(X)$ have support of codimension $\geqslant m$ and $\geqslant n$, respectively. Then:*

(1) *$[v] \cdot [w]$ has support of codimension $\geqslant (m + n)$. In particular, if $m + n > \dim(X)$, then $[v] \cdot [w] = 0$;*

(2) *If the intersection of the supports of $[v]$ and $[w]$ (in some realization) is empty, then $[v] \cdot [w] = 0$.*

Proof: (1) Follows from the fact that the product is associative, and thus $\mathbb{L}$-linear.

(2) Let Z_1, Z_2 be the supports for $[v]$ and $[w]$, respectively, and U_1, U_2 be the open compliments. Now, we just need to observe that the map $v : V \to X$ factors through $U_2 \to X$, and thus, $v^*([w]) = 0$. □

5.3 Transversality

Let me remind that the pair of morphisms $Y \xrightarrow{f} X \xleftarrow{g} Z$ of smooth varieties is called *transversal*, if for the cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ {\scriptstyle g'}\uparrow & & \uparrow{\scriptstyle g} \\ U & \xrightarrow[f']{} & Z \end{array}$$

the map $(g')^*T_Y \oplus (f')^*T_Z \to (f \circ g')^*T_X$ is surjective. Then U is smooth, and the sequence

$$0 \to T_U \to (g')^*T_Y \oplus (f')^*T_Z \to (f \circ g')^*T_X \to 0$$

is exact.

The following straightforward Lemma helps to check transversality

Lemma 5.3 *Let $Y \xrightarrow{f} X \xleftarrow{g} Z$ be the pair of morphisms. Suppose on X there is a pair of filtrations $X^y_m \subset \ldots \subset X^y_0 = X = X^z_0 \supset \ldots \supset X^z_n$. Let $Y_{i,\mu}$ and $Z_{j,\nu}$ be the irreducible components of $f^{-1}(X^y_i \backslash X^y_{i+1})$, and $g^{-1}(X^z_j \backslash X^z_{j+1})$, respectively. Suppose*

1) *$f : Y_{i,\mu} \to X^y_i \backslash X^y_{i+1}$ and $g : Z_{j,\nu} \to X^z_j \backslash X^z_{j+1}$ are smooth.*

2) *X^y_i and X^z_j are transversal on X.*

Then f and g are transversal.

The following observation is evident:

Observation 5.4 *Let in the diagram*

$$\begin{array}{ccccc} A & \longleftarrow & C & \longleftarrow & E \\ \uparrow & & \uparrow & & \uparrow \\ B & \longleftarrow & D & \longleftarrow & F \end{array}$$

both small squares are proper cartesian. Then the large square (containing A, B, E and F) is proper cartesian too.

Below we will establish the proper cartesian property for various types of diagrams. We have the following statements.

Lemma 5.5 *Let $B \stackrel{f}{\to} A \stackrel{g}{\leftarrow} C$ be the pair of maps of smooth quasiprojective varieties, and $D := B \times_A C$. Suppose $f : B \to A$ and $f' : D \to C$ are regular embeddings (in particular, the sequence*

$$0 \to T_D \to (g')^* T_B \oplus (f')^* T_C \to (g' \circ f)^* T_A$$

is exact). Then the right square in the following diagram is always proper cartesian, and the left one is proper cartesian provided f and g are transversal.

$$\begin{array}{ccccc} A & \xleftarrow{\pi_A} & Bl_{A,B} & \xleftarrow{j_A} & \mathbb{P}_B(\mathcal{N}_B) \\ g\uparrow & & g_{Bl}\uparrow & & \uparrow g_{\mathbb{P}} \\ C & \xleftarrow[\pi_C]{} & Bl_{C,D} & \xleftarrow[j_C]{} & \mathbb{P}_D(\mathcal{N}_D). \end{array}$$

Proof: The statement about the right square is clear.

Transversality of g and π_A can be checked separately on B and $A \backslash B$. Finally, the fact that the left square is cartesian follows from the fact that $g^* \mathcal{I}_B = \mathcal{I}_D$, where $\mathcal{I}_B$, $\mathcal{I}_D$ are sheaves of ideals of B and D, respectively. □

Proposition 5.6 *Let A be smooth quasiprojective variety equipped with $(\mathbf{Z}/2)$-action, such that the locus of fixed points is smooth divisor R_A. Let $\overline{A} = A/(\mathbf{Z}/2)$. Then $\overline{A}$ is smooth, $p : A \to \overline{A}$ is flat of degree 2, and for $\mathcal{U}_A := p_*(\mathcal{O})/\mathcal{O}$, $p^*(\mathcal{U}_A) = \mathcal{O}(-R_A)$. Moreover, $\mathcal{O}(-R_A)$ has natural $(\mathbf{Z}/2)$-action, which is equal to $(-id)$ outside R_A (where $\mathcal{O}(-R_A)$ is naturally isomorphic to $\mathcal{O}$), and such that $\mathcal{O}(-R_A)/(\mathbf{Z}/2) = \mathcal{U}_A$.*

Proof: From the results of D.Mumford we know that the quotient exists. Outside R_A the action will be free, and so $p|_{A \backslash R_A}$ is étale of degree 2. If $x \in R_A$, then in $\mathcal{O}_{A,x}$ the equation t_1 defining R_A can be chosen as one of the local parameters. The action splits $\mathcal{O}_{A,x}$ into $(+1)$ and (-1) eigenspaces $V_0 \oplus V_1$. If $f \in V_1$, then $f \in \mathcal{I}_{R_A}$. That is, f is divisible by t_1. Hence, $\cdot t_1 : V_0 \to V_1$ is an isomorphism. This shows that p is flat of degree 2. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{A,x}$, and $\mathfrak{n}$ be the maximal ideal of V_0. Since $\mathfrak{m}^{\mathfrak{l}}$ is preserved under the action, and characteristic is not 2, $\mathfrak{m}/\mathfrak{m}^2 = (V_0 \cap \mathfrak{m})/(V_0 \cap \mathfrak{m}^2) \oplus (V_1 \cap \mathfrak{m})/(V_1 \cap \mathfrak{m}^2) = V_0^1 \oplus k(x) \cdot t_1$. Then it is not difficult to see that V_0^1 and t_1^2 generate $\mathfrak{n}/\mathfrak{n}^2$. Hence, V_0 is a smooth local ring, and $\overline{A}$ is smooth.

Since p is flat of degree 2, $p_*(\mathcal{O}) =: \mathcal{V}_A$ is a two-dimensional vector bundle on $\overline{A}$. On $\overline{A}$ we have a natural map $\mathcal{O} \to \mathcal{V}_A$ which identifies the image with the $(+1)$-eigensubbundle, and $\mathcal{V}_A = \mathcal{O} \oplus \mathcal{U}_A$, where the latter is a (-1)-eigensubbundle.

Then $A = \mathrm{Spec}_{\overline{A}}(\mathcal{V}_A)$, and the multiplicative structure on $\mathcal{V}_A$ is given by the map $\mathcal{U}_A^{\otimes 2} \stackrel{\mu_A}{\to} \mathcal{O}$. We have natural map

$$p^*(\mathcal{U}_A) = (\mathcal{O}_{\overline{A}} \oplus \mathcal{U}_A) \otimes_{\mathcal{O}_{\overline{A}}} \mathcal{U}_A = \mathcal{U}_A \oplus \mathcal{U}_A^{\otimes 2} \stackrel{id \oplus \mu_A}{\to} \mathcal{U}_A \oplus \mathcal{O}_{\overline{A}}$$

of $\mathcal{O}_A = (\mathcal{O}_{\overline{A}} \oplus \mathcal{U}_A)$-modules. The zeroes of this map is exactly the fixed point divisor R_A. Thus, $p^*(\mathcal{U}_A) = \mathcal{I}_{R_A} = \mathcal{O}(-R_A)$.

To prove the last statement, define the action on $\mathcal{O}(-R_A) = p^*(\mathcal{U}_A) = \mathcal{U}_A \oplus \mathcal{U}_A^{\otimes 2}$ as (id) on the first summand, and $(-id)$ on the second. Then, clearly, $\mathcal{O}(-R_A)/(\mathbf{Z}/2) = \mathcal{U}_A$, and the action otside R_A (under the natural identification $id \oplus \mu : \mathcal{I}_{R_A} \to \mathcal{O}_A$) will be $(-id)$. □

Proposition 5.7 *Let A, B be smooth quasiprojective varieties equipped with $(\mathbf{Z}/2)$-action, such that the locuses of fixed points are smooth divisors R_A and R_B. Suppose $f : B \to A$ is a $(\mathbf{Z}/2)$-equivariant map which is transversal to R_A, and $R_B = f^{-1}(R_A)$. Then for $\overline{A} := A/(\mathbf{Z}/2)$, $\overline{B} := B/(\mathbf{Z}/2)$ the natural diagram*

$$\begin{array}{ccc} A & \xleftarrow{f} & B \\ {\scriptstyle p_A}\downarrow & & \downarrow{\scriptstyle p_B} \\ \overline{A} & \xleftarrow[\overline{f}]{} & \overline{B} \end{array}$$

is proper cartesian.

Proof: By Proposition 5.6, we know that $\overline{A}$ and $\overline{B}$ are smooth, and $A = \mathrm{Spec}_{\overline{A}}(\mathcal{V}_A)$, $B = \mathrm{Spec}_{\overline{B}}(\mathcal{V}_B)$, for $\mathcal{V}_A := (p_A)_*(\mathcal{O})$, $\mathcal{V}_B := (p_B)_*(\mathcal{O})$. Then $A \times_{\overline{A}} \overline{B} = \mathrm{Spec}_{\overline{B}}(\overline{f}^*(\mathcal{V}_A))$. Morphism f gives a map $\eta : \overline{f}^*(\mathcal{V}_A) \to \mathcal{V}_B$, which decomposes as $id \oplus \varepsilon$, where $\varepsilon : \overline{f}^*(\mathcal{U}_A) \to \mathcal{U}_B$ is certain map. We would want to show that it is an isomorphism. Really, multiplicative structure in $\mathcal{V}_A$ and $\mathcal{V}_B$ is given by certain maps $\mu_A : \mathcal{U}_A^{\otimes 2} \to \mathcal{O}$ and $\mu_B : \mathcal{U}_B^{\otimes 2} \to \mathcal{O}$, which should fit in the commutative diagram

$$\begin{array}{ccc} \overline{f}^*(\mathcal{U}_A)^{\otimes 2} & \xrightarrow{\overline{f}^*\mu_A} & \mathcal{O} \\ {\scriptstyle \varepsilon^{\otimes 2}}\downarrow & & \| \\ \mathcal{U}_B^{\otimes 2} & \xrightarrow[\mu_B]{} & \mathcal{O}. \end{array}$$

But the zeroes of μ_A and μ_B are exactly the images $\overline{R_A}$, $\overline{R_B}$ of the fixed points divisors. These will be again smooth divisors on $\overline{A}$ and $\overline{B}$. Since $p_B^{-1}\overline{f}^{-1}(\overline{R_A}) = f^{-1}p_A^{-1}(\overline{R_A}) = f^{-1}(R_A) = R_B$, we get: $\overline{f}^{-1}(\overline{R_A}) = \overline{R_B}$. If t_1^A is a local parameter defining R_A at $x = f(y)$, where $y \in R_B$, then $f^*(t_1^A) = t_1^B$ is a local parameter defining R_B at y. Then $\overline{f}^*(d(t_1^A)^2) = d(t_1^B)^2$. This implies that $\overline{f}_* : T_{y,\overline{B}}/T_{y,\overline{R_B}} \to T_{x,\overline{A}}/T_{x,\overline{R_A}}$ is an isomorphism, and $\overline{f}$ is transversal to $\overline{R_A}$. Since $\overline{f}^{-1}(\overline{R_A})$ coinside with the divisor of zeroes of $\overline{f}^*\mu_A$ and with $\overline{R_B}$ - the divisor of zeroes of μ_B, ε is an isomorphism. Hence, $B = A \times_{\overline{A}} \overline{B}$. The transversality of the maps p_A and $\overline{f}$ can be checked separately on $\overline{R_A}$, where it was proven above, and on $\overline{A} \backslash \overline{R_A}$, where p_A is etale. □

Proposition 5.8 *In the situation of Proposition 5.7, suppose that f is a regular embedding. Then*

$$\mathcal{N}_f = p_B^* \mathcal{N}_{\overline{f}}, \quad \textit{and} \quad \mathcal{N}_{\overline{f}} = \mathcal{N}_f/(\mathbf{Z}/2).$$

Proof: Since the square from Proposition 5.7 is proper cartesian and p_B is flat, we get an isomorphism $\mathcal{N}_f \cong p_B^* \mathcal{N}_{\overline{f}}$. Then $\mathcal{N}_f/(\mathbf{Z}/2) = \mathcal{N}_{\overline{f}} \otimes_{\mathcal{O}_{\overline{B}}} (\mathcal{O}_B/(\mathbf{Z}/2)) = \mathcal{N}_{\overline{f}}$. □

Remark: Notice, that $T_{\overline{A}} = T_A/(\mathbf{Z}/2)$, but T_A is not isomorphic to $p_A^* T_{\overline{A}}$.
example: $A = \mathbb{P}^1$ with $\sigma(z) = -z$. Then $\overline{A} = \mathbb{P}^1$, $T_{\overline{A}} = \mathcal{O}(2)$, $T_A = \mathcal{O}(2)$, but $\pi_A^* T_{\overline{A}} = \mathcal{O}(4)$.

The following proposition shows that to construct a proper cartesian square for $\widetilde{C}^2$'s one just need to construct one for $\widetilde{\square}$'s.

Proposition 5.9 *Let A, B, C, D are smooth quasiprojective varieties equipped with the $(\mathbf{Z}/2)$-action, with the locuses of fixed point the smooth divisors R_A, R_B, R_C, R_D, and smooth quotient-varieties $\overline{A}, \overline{B}, \overline{C}, \overline{D}$. Suppose, they fit into the proper cartesian diagram:*

$$\begin{array}{ccc} A & \xleftarrow{\ f\ } & B \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle g'} \\ C & \xleftarrow[\ f'\ ]{} & D \end{array}$$

of $(\mathbf{Z}/2)$-equivariant maps, and:

(a) either f is transversal to R_A, f' is transversal to R_C, $f^{-1}(R_A) = R_B$, $(f')^{-1}(R_C) = R_D$;

(b) or g is transversal to R_A, g' is transversal to R_B, $g^{-1}(R_A) = R_C$, $(g')^{-1}(R_B) = R_D$.

Then the quotients also fit into the proper cartesian diagram:

$$\begin{array}{ccc} \overline{A} & \xleftarrow{\ \overline{f}\ } & \overline{B} \\ {\scriptstyle \overline{g}}\uparrow & & \uparrow{\scriptstyle \overline{g}'} \\ \overline{C} & \xleftarrow[\ \overline{f}'\ ]{} & \overline{D} \end{array}$$

Proof: From the symmetry we can assume that we are in the situation (b). Thus we can apply Proposition 5.7 to morphisms g, and g'. We have: $C = \overline{C} \times_{\overline{A}} A$, and $D = B \times_{\overline{B}} \overline{D}$. We have a natural map $\nu : \overline{D} \to \overline{B} \times_{\overline{A}} \overline{C}$. Then $p_B^*(\nu) : D = B \times_{\overline{B}} \overline{D} \to B \times_{\overline{B}} \overline{B} \times_{\overline{A}} \overline{C}$. We want to show that this is an isomorphism. Really, we have $B \times_{\overline{B}} \overline{B} \times_{\overline{A}} \overline{C} = B \times_{\overline{A}} \overline{C} = B \times_A A \times_{\overline{A}} \overline{C} = B \times_A C = D$, and it is not difficult to follow, what is happening with the map $p_B^*(\nu)$ under these transformations: it is just given by the only natural maps to the factors. So, in the end we get an identity, and $p_B^*(\nu)$ is an isomorphism. Since p_B is flat, ν is an isomorphism. Thus, our square is indeed cartesian. The transversality of $\overline{f}$ and $\overline{g}$ can be checked separately on $\overline{R_A}$ and $\overline{A} \backslash \overline{R_A}$. $\square$

Now we can prove the proper cartesian property for various squares we are using in the text.

Statement 5.10 *Let $X \xrightarrow{g} Y \xleftarrow{h} Z$ be transversal regular embeddings of smooth quasiprojective varieties, and $U = X \times_Y Z$. Then the following natural squares are proper cartesian:*

$$\begin{array}{ccc} \widetilde{\Box}(Y) & \xleftarrow{\ \widetilde{\Box}(h)\ } & \widetilde{\Box}(Z) \\ {\scriptstyle \widetilde{\Box}(g)}\uparrow & & \uparrow{\scriptstyle \widetilde{\Box}(g')} \\ \widetilde{\Box}(X) & \xleftarrow[\ \widetilde{\Box}(h')\ ]{} & \widetilde{\Box}(U) \end{array} \quad \textit{and} \quad \begin{array}{ccc} \widetilde{C}^2(Y) & \xleftarrow{\ \widetilde{C}^2(h)\ } & \widetilde{C}^2(Z) \\ {\scriptstyle \widetilde{C}^2(g)}\uparrow & & \uparrow{\scriptstyle \widetilde{C}^2(g')} \\ \widetilde{C}^2(X) & \xleftarrow[\ \widetilde{C}^2(h')\ ]{} & \widetilde{C}^2(U) \end{array}$$

Proof: Transversality of $\widetilde{\Box}(g)$ and $\widetilde{\Box}(h)$ can be checked separately on $\mathbb{P}_Y(T_Y)$, where it follows from the transversality of $\mathbb{P}_X(T_X)$ and $\mathbb{P}_Z(T_Z)$, and on $\widetilde{\Box}(Y) \backslash \mathbb{P}_Y(T_Y)$, where it follows from the transversality of $\Box(g)$ and $\Box(h)$. The easy inspection shows that the intersection of $\widetilde{\Box}(X)$ and $\widetilde{\Box}(Z)$ is $\widetilde{\Box}(U)$, thus, the left square is proper cartesian. For the right square one needs to apply Proposition 5.9, and observe that $\widetilde{\Box}(g)$ and $\widetilde{\Box}(g')$ are transversal to the fixed point locuses $\mathbb{P}_Y(T_Y)$ and $\mathbb{P}_Z(T_Z)$, and the respective preimages are $\mathbb{P}_X(T_X)$ and $\mathbb{P}_U(T_U)$, by Lemma 5.5. $\square$

Statement 5.11 *Let $Z \xrightarrow{f} Y \xrightarrow{g} X$ be the smooth morphisms of smooth quasiprojective varieties. Then the following natural squares are proper cartesian:*

$$\begin{array}{ccccc} \widetilde{\square}(Z/Y) & \xrightarrow{A(f/g)} & \widetilde{\square}(Z/X) & \xleftarrow{a(f/g)} & Bl_{\widetilde{\square}(Z/X),\widetilde{\square}(Z/Y)} \\ {\scriptstyle p_{Z/Y}}\downarrow & & {\scriptstyle p_{Z/X}}\downarrow & & \downarrow{\scriptstyle \tilde{p}} \\ \widetilde{C}^2(Z/Y) & \xrightarrow[C(f/g)]{} & \widetilde{C}^2(Z/X) & \xleftarrow[c(f/g)]{} & Bl_{\widetilde{C}^2(Z/X),\widetilde{C}^2(Z/Y)} \end{array}$$

In particular, $Bl_{\widetilde{C}^2(Z/X),\widetilde{C}^2(Z/Y)} = Bl_{\widetilde{\square}(Z/X),\widetilde{\square}(Z/Y)}/(\mathbf{Z}/2)$.

Proof: The left square is proper cartesian by Proposition 5.7, since $A(f/g) : \widetilde{\square}(Z/Y) \to \widetilde{\square}(Z/X)$ is transversal to the fixed point set $R_{\widetilde{\square}(Z/X)} = \mathbb{P}_Z(T_{Z/X})$, and $A(f/g)^{-1}(\mathbb{P}_Z(T_{Z/X})) = \mathbb{P}_Z(T_{Z/Y})$. For the right one, it remains to apply Lemma 5.5. Since $\tilde{p}$ is flat of degree 2, $(\mathbf{Z}/2)$-equivariant with trivial action on $Bl_{\widetilde{C}^2(Z/X),\widetilde{C}^2(Z/Y)}$, we get the last equality (from the proof of the Statement 5.12 one can see directly that the locus of fixed points on $Bl_{\widetilde{\square}(Z/X),\widetilde{\square}(Z/Y)}$ is a smooth divisor). $\square$

Statement 5.12 *Let $Z \xrightarrow{h} Y \xrightarrow{f} X$ be smooth morphisms of smooth quasiprojective varieties. Then the following natural squares are proper cartesian:*

$$\begin{array}{ccc} \widetilde{\square}(Y) & \xleftarrow{b(h)} & Bl_{\widetilde{\square}(Z),\widetilde{\square}(Z/Y)} \\ {\scriptstyle A(f)}\uparrow & & \uparrow{\scriptstyle A} \\ \widetilde{\square}(Y/X) & \xleftarrow[b]{} & Bl_{\widetilde{\square}(Z/X),\widetilde{\square}(Z/Y)} \end{array} \quad \text{and} \quad \begin{array}{ccc} \widetilde{C}^2(Y) & \xleftarrow{d(h)} & Bl_{\widetilde{C}^2(Z),\widetilde{C}^2(Z/Y)} \\ {\scriptstyle C(f)}\uparrow & & \uparrow{\scriptstyle C} \\ \widetilde{C}^2(Y/X) & \xleftarrow[d]{} & Bl_{\widetilde{C}^2(Z/X),\widetilde{C}^2(Z/Y)} \end{array}.$$

Proof: Let us start with the first square. We have:

$$Bl_{\widetilde{\square}(Z),\widetilde{\square}(Z/Y)} = Bl_{\square(Z),(\Delta(Z),\square(Z/Y))} =$$
$$Bl_{Bl_{\square(Z),\square(Z/Y)},\mathbb{P}_Z(h^*T_Y)} = Bl_{(\widetilde{\square}(Y)\times_{\square(Y)}\square(Z)),\mathbb{P}_Z(h^*T_Y)},$$

where the embedding $\mathbb{P}_Z(h^*T_Y) \to (\widetilde{\square}(Y)\times_{\square(Y)}\square(Z))$ is given on two factors by $\mathbb{P}_Z(h^*T_Y) \to \mathbb{P}_Y(T_Y) \to \widetilde{\square}(Y)$, and $\mathbb{P}_Z(h^*T_Y) \to Z \xrightarrow{\Delta} \square(Z)$, respectively.

$\mathbb{P}_Z(h^*T_Y)$ is transversal to $(\widetilde{\square}(Y/X)\times_{\square(Y/X)}\square(Z/X))$, and the intersection is $\mathbb{P}_Z(h^*T_{Y/X})$. Applying Lemma 5.5 and Observation 5.4, we get that

$$Bl_{(\widetilde{\square}(Y)\times_{\square(Y)}\square(Z)),\mathbb{P}_Z(h^*T_Y)} \times_{\widetilde{\square}(Y)} \widetilde{\square}(Y/X) =$$
$$Bl_{(\widetilde{\square}(Y)\times_{\square(Y)}\square(Z)),\mathbb{P}_Z(h^*T_Y)} \times_{(\widetilde{\square}(Y)\times_{\square(Y)}\square(Z))} (\widetilde{\square}(Y) \times_{\square(Y)} \square(Z)) \times_{\widetilde{\square}(Y)} \widetilde{\square}(Y/X) =$$
$$Bl_{(\widetilde{\square}(Y)\times_{\square(Y)}\square(Z)),\mathbb{P}_Z(h^*T_Y)} \times_{(\widetilde{\square}(Y)\times_{\square(Y)}\square(Z))} (\widetilde{\square}(Y/X) \times_{\square(Y/X)} \square(Z/X)) =$$
$$Bl_{(\widetilde{\square}(Y/X)\times_{\square(Y/X)}\square(Z/X)),\mathbb{P}_Z(h^*T_{Y/X})} = Bl_{\widetilde{\square}(Z/X),\widetilde{\square}(Z/Y)},$$

and the respective (our first) square is proper cartesian.

To show that the second square is proper cartesian, by Proposition 5.9 and the last claim of Statement 5.11, it is sufficient to check that $A(f)$ is transversal to the locus of fixed points on $\widetilde{\square}(Y)$, A is transversal to the locus of fixed points on $Bl_{\widetilde{\square}(Z),\widetilde{\square}(Z/Y)}$, and the respective preimages are the locuses of fixed points on $\widetilde{\square}(Y/X)$ and $Bl_{\widetilde{\square}(Z/X),\widetilde{\square}(Z/Y)}$. But such locuses are $\mathbb{P}_Y(T_Y)$, the special divisor on $Bl_{(\widetilde{\Box}(Y)\times_{\Box(Y)}\Box(Z)),\mathbb{P}_Z(h^*T_Y)}$, $\mathbb{P}_Y(T_{Y/X})$, and the special divisor on $Bl_{(\widetilde{\Box}(Y/X)\times_{\Box(Y/X)}\Box(Z/X)),\mathbb{P}_Z(h^*T_{Y/X})}$, respectively. Everything follows from Lemma 5.5. $\Box$

Statement 5.13 *Let $X \xrightarrow{g} Y \xleftarrow{h} Z$ be morphisms of smooth quasiprojective varieties, where g is a regular embedding, h is a smooth morphism, and $U = X \times_Y Z$. Then the following natural squares are proper cartesian:*

$$\begin{array}{ccc} \widetilde{\Box}(Y) & \xleftarrow{b(h)} & Bl_{\widetilde{\Box}(Z),\widetilde{\Box}(Z/Y)} \\ {\scriptstyle \widetilde{\Box}(g)}\uparrow & & \uparrow{\scriptstyle \widetilde{\widetilde{\Box}(g')}} \\ \widetilde{\Box}(X) & \xleftarrow[b(h')]{} & Bl_{\widetilde{\Box}(U),\widetilde{\Box}(U/X)} \end{array} \quad \text{and} \quad \begin{array}{ccc} \widetilde{C}^2(Y) & \xleftarrow{d(h)} & Bl_{\widetilde{C}^2(Z),\widetilde{C}^2(Z/Y)} \\ {\scriptstyle \widetilde{C}^2(g)}\uparrow & & \uparrow{\scriptstyle \widetilde{\widetilde{C}^2(g')}} \\ \widetilde{C}^2(X) & \xleftarrow[d(h')]{} & Bl_{\widetilde{C}^2(U),\widetilde{C}^2(U/X)} \end{array}.$$

Proof: The proof is the same as for the Statement 5.12. One should just observe, that the subvariety $\mathbb{P}_Z(h^*T_Y)$ of $(\widetilde{\Box}(Y)\times_{\Box(Y)}\Box(Z))$ is transversal to $(\widetilde{\Box}(X)\times_{\Box(X)}\Box(U))$, and the intersection is $\mathbb{P}_U((h')^*T_X)$. The facts that $\widetilde{\Box}(g)$ and $\widetilde{\widetilde{\Box}(g')}$ are transversal to the locuses of fixed points again follows from Lemma 5.5. $\Box$

Statement 5.14 *Let $X \xrightarrow{g} Y \xleftarrow{h} Z$ be morphisms of smooth quasiprojective varieties, where g is a regular embedding, h is a smooth morphism, and $U = X \times_Y Z$. Then the following natural squares are proper cartesian:*

$$\begin{array}{ccc} Y & \xleftarrow{B(h)} & \widetilde{\Box}(Z/Y) \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle \widetilde{\Box}(g'/g)} \\ X & \xleftarrow[B(h')]{} & \widetilde{\Box}(U/X) \end{array} \quad \text{and} \quad \begin{array}{ccc} Y & \xleftarrow{D(h)} & \widetilde{C}^2(Z/Y) \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle \widetilde{C}^2(g'/g)} \\ X & \xleftarrow[D(h')]{} & \widetilde{C}^2(U/X) \end{array}$$

Proof: The morphisms $B(h)$ and $D(h)$ are smooth. Thus, it remains only to observe that the preimage of X is $\widetilde{\Box}(U/X)$ and $\widetilde{C}^2(U/X)$, respectively. $\Box$

Statement 5.15 *Let $Z \xrightarrow{h} Y \xrightarrow{f} X$ be smooth morphisms of smooth quasiprojective varieties. Then the following natural squares are proper cartesian:*

$$\begin{array}{ccc} \widetilde{\Box}(Y) & \xleftarrow{b(h)} & Bl_{\widetilde{\Box}(Z),\widetilde{\Box}(Z/Y)} \\ {\scriptstyle a(f)}\uparrow & & \uparrow{\scriptstyle a} \\ Bl_{\widetilde{\Box}(Y),\widetilde{\Box}(Y/X)} & \xleftarrow[b]{} & Bl_{\widetilde{\Box}(Z),(\widetilde{\Box}(Z/Y),\widetilde{\Box}(Z/X))} \end{array} \quad \text{and} \quad \begin{array}{ccc} \widetilde{C}^2(Y) & \xleftarrow{d(h)} & Bl_{\widetilde{C}^2(Z),\widetilde{C}^2(Z/Y)} \\ {\scriptstyle c(f)}\uparrow & & \uparrow{\scriptstyle c} \\ Bl_{\widetilde{C}^2(Y),\widetilde{C}^2(Y/X)} & \xleftarrow[d]{} & Bl_{\widetilde{C}^2(Z),(\widetilde{C}^2(Z/Y),\widetilde{C}^2(Z/X))} \end{array}.$$

Proof: This follows directly from Proposition 5.12 and Lemma 5.5. Take $A = \widetilde{\Box}(Y)$, $B = \widetilde{\Box}(Y/X)$, $C = Bl_{\widetilde{\Box}(Z),\widetilde{\Box}(Z/Y)}$ for the first square, and $A = \widetilde{C}^2(Y)$, $B = \widetilde{C}^2(Y/X)$, $C = Bl_{\widetilde{C}^2(Z),\widetilde{C}^2(Z/Y)}$ for the second. $\Box$

Finally, we need two more squares.

Statement 5.16 *Let* $Y \xrightarrow{f} X$ *be the smooth morphism of smooth quasiprojective varieties. Then the following natural squares are proper cartesian:*

$$\begin{array}{ccccc} \mathbb{P}_Y(T_{Y/X}) & \xrightarrow{E(f)} & \mathbb{P}_Y(T_Y) & \xleftarrow{e(f)} & Bl_{\mathbb{P}_Y(T_Y),\mathbb{P}_Y(T_{Y/X})} \\ {\scriptstyle i_{Y/X}}\downarrow & & {\scriptstyle i_Y}\downarrow & & \downarrow{\scriptstyle \tilde{i}} \\ \widetilde{C}^2(Y/X) & \xrightarrow[C(f)]{} & \widetilde{C}^2(Y) & \xleftarrow[c(f)]{} & Bl_{\widetilde{C}^2(Y),\widetilde{C}^2(Y/X)} \end{array}.$$

Proof: Our left square is the large square of the diagram

$$\begin{array}{ccccc} \mathbb{P}_Y(T_Y) & \xrightarrow{j_Y} & \widetilde{\square}(Y) & \xrightarrow{p_Y} & \widetilde{C}^2(Y) \\ {\scriptstyle E(f)}\uparrow & & {\scriptstyle A(f)}\uparrow & & \uparrow{\scriptstyle C(f)}, \\ \mathbb{P}_Y(T_{Y/X}) & \xrightarrow[j_{Y/X}]{} & \widetilde{\square}(Y/X) & \xrightarrow[p_{Y/X}]{} & \widetilde{C}^2(Y/X) \end{array}$$

where the left small square is proper cartesian by Lemma 5.5, and the right one is proper cartesian by Statement 5.11. Now one needs to apply the Observation 5.4. Our right square is proper cartesian by Lemma 5.5. □

5.4 Morphisms of degree 2

Let $\pi : Y \to Z$ be a finite morphism of degree 2 between smooth varieties. The morphism π is automatically flat. The aim of this subsection is to compute $\pi_*([1_Y]) \in \Omega^0(Z)$. A similar computation was earlier independently performed by M.Rost and A.Smirnov.

The direct image $\pi_*(\mathcal{O}_Y) =: \mathcal{V}$ is a 2-dimensional vector bundle on Z which has a structure of the sheaf of commutative $\mathcal{O}_Z$-algebras. It has natural subbundle $\mathcal{O}_Z$ generated by the unit section, and the quotient is an invertible bundle which we denote $\mathcal{U}$.

Consider the symmetric $\mathcal{O}_Z$-algebra $S^*(\mathcal{V})$. Then $\mathrm{Spec}(S^*(\mathcal{V}))$ is just the total space of the bundle $\mathcal{V}^\wedge$. We have a projection $\varepsilon' : \mathrm{Spec}(S^*(\mathcal{V})) \to Z$, and there is natural closed embedding $j' : Y \to \mathrm{Spec}(S^*(\mathcal{V}))$ such that $\pi = \varepsilon' \circ j'$. This embedding is given by the surjective $\mathcal{O}_Z$-algebra homomorphism $S^*(\mathcal{V}) \to \mathcal{V}$ which sends the zero-degree component to the subbundle $\mathcal{O}_Z$ of V, and the first-degree component identically to $\mathcal{V}$. The image of j' is disjoint from the zero section of ε'. Indeed, the intersection of the two closed subsets is given by the ideal which contains the augmentation ideal of $S^*(\mathcal{V})$ as well as the zero-degree component, since the image of the latter in the algebra V is contained in the image of the component of degree 1. Thus, the map $\pi : Y \to Z$ factors as $Y \xrightarrow{j} \mathrm{Proj}(S^*(\mathcal{V})) = \mathbb{P}(\mathcal{V}^\wedge) \xrightarrow{\varepsilon} X$.

Let us compute $j_*([1_Y]) \in \Omega^1(\mathbb{P}(\mathcal{V}^\wedge))$.

Proposition 5.17 $j_*([1_Y]) = c_1(\mathcal{U}^{-2}(2))[1_{\mathbb{P}(\mathcal{V}^\wedge)}]$.

Proof: The image of j is given by the homogeneous sheaf of ideals of $S^*(\mathcal{V})$ which is generated by the relations $image(id - 1 \otimes_s \mu) : S^2(\mathcal{V}) \to S^2(\mathcal{V})$, where $\mu : S^2(\mathcal{V}) \to \mathcal{V}$ is a multiplication map in $\mathcal{V}$, 1 is considered as an element of $S^1(\mathcal{V}) = \mathcal{V}$, and $\otimes_s$ is a product in $S^*(\mathcal{V})$.

Notice that $(id - 1 \otimes_s \mu)(\mathcal{O}_Z \otimes_s \mathcal{V}) = 0$. Thus, our ideal is generated by $(id - 1 \otimes_s \mu)(S^2(\mathcal{V})/(\mathcal{O}_Z \otimes_s \mathcal{V})) = (id - 1 \otimes_s \mu)(S^2(\mathcal{U}))$. So, our Y is just zeroes of some linear map $\mathcal{U}^2 \to \mathcal{O}(2)$. Or, which is the same, of some linear map $\mathcal{O}_{\mathbb{P}(\mathcal{V}^\wedge)} \to \mathcal{U}^{-2}(2)$. Since Y is smooth, by [5, Definition 2.2(Sect)], $j_*([1_Y]) = c_1(\mathcal{U}^{-2}(2))[1_{\mathbb{P}(\mathcal{V}^\wedge)}]$. □

Theorem 5.18 *$\pi_*([1_Y]) = \frac{[2]_\Omega(t)}{t}(c_1(\mathcal{U}^{-1}))[1_Z]$, where $[2]_\Omega(t)$ is the multiplication by 2 in the sense of the universal formal group law F_Ω.*

Proof: Since the characteristic of the base-field is not equal to 2, the sequence $0 \to \mathcal{O}_Z \to \mathcal{V} \to \mathcal{U} \to 0$ is split by the $trace/2 : \mathcal{V} \to \mathcal{O}_Z$ map, and so is the sequence $0 \to \mathcal{U}^{-1} \to \mathcal{V}^\wedge \to \mathcal{O}_Z \to 0$. Then we have a closed embedding $i : \mathbb{P}(\mathcal{O}_Z) \hookrightarrow \mathbb{P}(\mathcal{V}^\wedge)$, and $i_*([1_{\mathbb{P}(\mathcal{O}_Z)}]) = c_1(\mathcal{U}^{-1}(1))[1_{\mathbb{P}(\mathcal{V}^\wedge)}]$. Notice that $\mathbb{P}(\mathcal{O}_Z)$ can be naturally identified with Z and, under this identification, $\varepsilon \circ i : Z \to Z$ is the identity. At the same time, $\mathcal{U}^{-1}(1)|_{\mathbb{P}(\mathcal{O}_Z)}$ is identified with $\mathcal{U}^{-1}$ on Z. Then

$$\pi_*([1_Y]) = \varepsilon_* \circ j_*([1_Y]) = \varepsilon_* c_1(\mathcal{U}^{-2}(2))[1_{\mathbb{P}(\mathcal{V}^\wedge)}] =$$
$$\varepsilon_* F_\Omega(c_1(\mathcal{U}^{-1}(1)), c_1(\mathcal{U}^{-1}(1)))[1_{\mathbb{P}(\mathcal{V}^\wedge)}] = \frac{[2]_{F_\Omega}(t)}{t}(c_1(\mathcal{U}^{-1}))[1_Z].$$

□

5.5 Excess Intersection Formula

Consider cartesian square

$$\begin{array}{ccc} D & \xrightarrow{f'} & C \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow[f]{} & A \end{array}$$

with f, f' - regular embeddings, and $(g')^*(\mathcal{N}_{B\subset A})/\mathcal{N}_{D\subset C} =: \mathcal{M}$ the vector bundle of dimension d.

Theorem 5.19

$$g^* f_*(v) = f'_*(c_d(\mathcal{M}) \cdot (g')^*(v));$$

If g is projective, then also:

$$f^* g_*(u) = g'_*(c_d(\mathcal{M}) \cdot (f')^*(u)).$$

Proof: Consider particular case of our diagram. Namely,

$$\begin{array}{ccc} D & \xrightarrow{j'} & \mathbb{P}_D(\mathcal{N}_{D\subset C} \oplus \mathcal{O}) \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle \tilde{g}} \\ B & \xrightarrow[j]{} & \mathbb{P}_B(\mathcal{N}_{B\subset A} \oplus \mathcal{O}). \end{array}$$

Lemma 5.20 *Theorem 5.19 is valid for this diagram.*

Proof: Let $0 \to \mathcal{L} \to \mathcal{N} \to \mathcal{M} \to 0$ be the short exact sequence of vector bundles. Then the class $[\mathbb{P}(\mathcal{L}) \to \mathbb{P}(\mathcal{N})]$ is given by $c_{\dim(\mathcal{M})}(\mathcal{M} \otimes \mathcal{O}(1))$.

Let $\mathcal{N}_{B\subset A}$ and $\mathcal{N}_{D\subset C}$ have dimensions d_B and d_D, respectively. Denoting as $\widetilde{\varepsilon}_B : \mathbb{P}_B(\mathcal{N}_{B\subset A} \oplus \mathcal{O}) \to B$ and $\widetilde{\varepsilon}_D : \mathbb{P}_D(\mathcal{N}_{D\subset C} \oplus \mathcal{O}) \to D$ the natural projections, we get:

$$\widetilde{g}^* j_*(v) = \widetilde{g}^*(\widetilde{\varepsilon}_B^*(v) \cdot c_{d_B}(\mathcal{N}_B \otimes \mathcal{O}(1))) = \widetilde{g}^*\widetilde{\varepsilon}_B^*(v) \cdot \widetilde{g}^*(c_{d_B}(\mathcal{N}_B \otimes \mathcal{O}(1))) =$$
$$\widetilde{\varepsilon}_D^*(g')^*(v) \cdot c_{d_B}((g')^*(\mathcal{N}_B) \otimes \mathcal{O}(1)) = \widetilde{\varepsilon}_D^*(g')^*(v) \cdot c_{d_D}(\mathcal{N}_D \otimes \mathcal{O}(1)) \cdot c_d(\mathcal{M} \otimes \mathcal{O}(1)) =$$
$$j'_*(g')^*(v) \cdot c_d(\mathcal{M} \otimes \mathcal{O}(1)) = j'_*((g')^*(v) \cdot (j')^*(c_d(\mathcal{M} \otimes \mathcal{O}(1)))) = j'_*((g')^*(v) \cdot c_d(\mathcal{M})).$$

In a similar way,

$$j^*\widetilde{g}_*(u) = (\widetilde{\varepsilon}_B)_*(\widetilde{g}_*(u)\cdot c_{d_B}(\mathcal{N}_B\otimes\mathcal{O}(1))) = (\widetilde{\varepsilon}_B)_*\widetilde{g}_*(u\cdot \widetilde{g}^*(c_{d_B}(\mathcal{N}_B\otimes\mathcal{O}(1)))) =$$
$$g'_*(\widetilde{\varepsilon}_D)_*(u\cdot c_{d_D}(\mathcal{N}_D\otimes\mathcal{O}(1))\cdot c_d(\mathcal{M}\otimes\mathcal{O}(1))) = g'_*(j')^*(u\cdot c_d(\mathcal{M}\otimes\mathcal{O}(1))) = g'_*((j')^*(u)\cdot c_d(\mathcal{M})).$$

□

Consider the diagram

$$\begin{array}{ccccc}
B & \xrightarrow{j} & \mathbb{P}_B(\mathcal{N}_B\oplus\mathcal{O}) & \xleftarrow{\widetilde{g}} & \mathbb{P}_D(\mathcal{N}_D\oplus\mathcal{O}) \\
 & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau'} \\
 & & A & \xleftarrow[g]{} & C
\end{array}$$

Lemma 5.21

$$g^*\tau_*j_* = \tau'_*\widetilde{g}^*j_* \quad \textit{and} \quad j^*\tau^*g_* = j^*\widetilde{g}_*(\tau')^*.$$

Proof: Consider the standard deformation to the normal cone construction.

Let $\widehat{A} := Bl_{A\times\mathbb{P}^1, B\times\{0\}}$, $\widehat{B} := Bl_{B\times\mathbb{P}^1, B\times\{0\}} = B\times\mathbb{P}^1$, $R_A := \mathbb{P}_B(\mathcal{N}_B\oplus\mathcal{O})$, and $S_A := Bl_{A\times\{0\}, B\times\{0\}} = Bl_{A,B}$. Analogously, let $\widehat{C} := Bl_{C\times\mathbb{P}^1, D\times\{0\}}$, $\widehat{D} := Bl_{D\times\mathbb{P}^1, D\times\{0\}} = D\times\mathbb{P}^1$, $R_C := \mathbb{P}_D(\mathcal{N}_D\oplus\mathcal{O})$, and $S_C := Bl_{C\times\{0\}, D\times\{0\}} = Bl_{C,D}$. Let $\pi_A : \widehat{A}\to A$, $\pi_B : \widehat{B}\to B$, $\pi_C : \widehat{C}\to C$, $\pi_D : \widehat{D}\to D$, $\eta_A : \widehat{A}\to\mathbb{P}^1$, and $\eta_C : \widehat{C}\to\mathbb{P}^1$ be the natural projections.

We have commutative diagram with all squares proper cartesian (we will need this property only for the left and the right ones):

$$\begin{array}{ccccccc}
B & \xrightarrow{j} & R_A & \xleftarrow{\widetilde{g}} & R_C & \xleftarrow{j'} & D \\
\downarrow{\scriptstyle k_B} & & \downarrow{\scriptstyle i_{R_A}} & & \downarrow{\scriptstyle i_{R_C}} & & \downarrow{\scriptstyle k_D} \\
\widehat{B} & \xrightarrow{\widehat{f}} & \widehat{A} & \xleftarrow{\widehat{g}} & \widehat{C} & \xleftarrow{\widehat{f}'} & \widehat{D}, \\
\uparrow{\scriptstyle i_B} & & \uparrow{\scriptstyle i_A} & & \uparrow{\scriptstyle i_C} & & \uparrow{\scriptstyle i_D} \\
B & \xrightarrow[f]{} & A & \xleftarrow[g]{} & C & \xleftarrow[f']{} & D
\end{array}$$

where the upper objects live over $\{0\}$, and lower ones over $\{1\}$.

Since $\eta_A^*(\mathcal{O}(1)) = \mathcal{O}(R_A)\otimes\mathcal{O}(S_A)$, and $\eta_C^*(\mathcal{O}(1)) = \mathcal{O}(R_C)\otimes\mathcal{O}(S_C)$, the differences $\delta_A := (c_1(\eta_A^*\mathcal{O}(1)) - c_1(\mathcal{O}(R_A)))$ and $\delta_C := (c_1(\eta_C^*\mathcal{O}(1)) - c_1(\mathcal{O}(R_C)))$ are supported on S_A and S_C, respectively.

Notice, that S_A does not meet $\widehat{B}$, and S_C does not meet $\widehat{D}$.

This implies that $\delta_A\cdot\widehat{f}_*(z) = 0$; $\delta_C\cdot\widehat{g}^*\widehat{f}_*(z) = 0$; $\widehat{f}^*(\delta_A\cdot z) = 0$; and $\widehat{f}^*\widehat{g}_*(\delta_C\cdot z) = 0$. Using the first two equalities, we have:

$$g^*\tau_*j_*(v) = g^*\tau_*(i_{R_A})^*\widehat{f}_*\pi_B^*(v) = g^*(\pi_A)_*(i_{R_A})_*(i_{R_A})^*\widehat{f}_*\pi_B^*(v) = g^*(\pi_A)_*(c_1(\mathcal{O}(R_A))\cdot\widehat{f}_*\pi_B^*(v)) =$$
$$g^*(\pi_A)_*(c_1(\eta_A^*\mathcal{O}(1))\cdot\widehat{f}_*\pi_B^*(v)) = g^*i_A^*\widehat{f}_*\pi_B^*(v) = i_C^*\widehat{g}^*\widehat{f}_*\pi_B^*(v) = (\pi_C)_*(c_1(\eta_C^*\mathcal{O}(1))\cdot\widehat{g}^*\widehat{f}_*\pi_B^*(v)) =$$
$$(\pi_C)_*(c_1(\mathcal{O}(R_C))\cdot\widehat{g}^*\widehat{f}_*\pi_B^*(v)) = (\pi_C)_*(i_{R_C})_*(i_{R_C})^*\widehat{g}^*\widehat{f}_*\pi_B^*(v) = (\tau')_*\widetilde{g}^*(i_{R_A})^*\widehat{f}_*\pi_B^*(v) = (\tau')_*\widetilde{g}^*j_*(v).$$

Similarly, using the last two, we get:

$$j^*\tau^*g_*(u) = j^*i_{R_A}^*\pi_A^*g_*(u) = (\pi_B)_*\widehat{f}^*(c_1(\mathcal{O}(R_A))\cdot\pi_A^*g_*(u)) = (\pi_B)_*\widehat{f}^*(c_1(\eta_A^*(\mathcal{O}(1)))\cdot\pi_A^*g_*(u)) =$$
$$(\pi_B)_*\widehat{f}^*(i_A)_*(i_A)^*\pi_A^*g_*(u) = (\pi_B)_*\widehat{f}^*(i_A)_*g_*(u) = (\pi_B)_*\widehat{f}^*\widehat{g}_*(i_C)_*(u) = (\pi_B)_*\widehat{f}^*\widehat{g}_*(c_1(\eta_C^*(\mathcal{O}(1)))\cdot\pi_C^*(u)) =$$
$$(\pi_B)_*\widehat{f}^*\widehat{g}_*(c_1(\mathcal{O}(R_C))\cdot\pi_C^*(u)) = (\pi_B)_*\widehat{f}^*\widehat{g}_*(i_{R_C})_*(i_{R_C})^*\pi_C^*(u) = (\pi_B)_*\widehat{f}^*(i_{R_A})_*\widetilde{g}_*(\tau')^*(u) = j^*\widetilde{g}_*(\tau')^*(u).$$

□

The Theorem now follows, since, by Lemmas 5.20 and 5.21

$$g^* f_*(v) = g^* \tau_* j_*(v) = \tau'_* \widetilde{g}^* j_*(v) = \tau'_* j'_*(c_d(\mathcal{M}) \cdot (g')^*(v)) = (f')_*(c_d(\mathcal{M}) \cdot (g')^*(v)).$$

Analogously,

$$f^* g_*(u) = j^* \tau^* g_*(u) = j^* \widetilde{g}_*(\tau')^*(u) = g'_*(c_d(\mathcal{M}) \cdot (j')^*(\tau')^*(u)) = g'_*(c_d(\mathcal{M}) \cdot (f')^*(u)).$$

□

5.6 Blow-up formula

The aim of this subsection is to prove Proposition 5.27.

Let us start with some preliminary results.

Let $\mathcal{N}$ be a vector bundle of dimension d on some variety B, and $\varepsilon : \mathbb{P}(\mathcal{N}) \to B$ be the corresponding projective bundle. Let $\mathcal{K} = \mathcal{N}/\mathcal{O}(-1)$ be the natural quotient.

Proposition 5.22 *The class of the diagonal on $\mathbb{P}(\mathcal{N}) \times_B \mathbb{P}(\mathcal{N})$ is equal to*

$$c_{d-1}(\mathcal{K}_1 \otimes \mathcal{O}(1)_2) = c^{\Omega}(\mathcal{K}_1)(c_1(\mathcal{O}(1)_2)) = c_{d-1}(\mathcal{K}) \times 1 + \sum_{i \geqslant 1} \gamma_{d-1-i} \times (c_1(\mathcal{O}(-1)))^i,$$

for certain $\gamma_j \in \Omega^j(\mathbb{P}(\mathcal{N}))$.

Proof: The diagonal $\Delta(\mathbb{P}(\mathcal{N}))$ is just $\mathbb{P}_{\mathbb{P}(\mathcal{N})_1}(\mathcal{O}(-1))$ for the natural subbundle $\mathcal{O}(-1) \subset \mathcal{N}|_{\mathbb{P}(\mathcal{N})_1}$. Thus, the respective class in $\Omega^{d-1}(\mathbb{P}_{\mathbb{P}(\mathcal{N})_1}(\mathcal{N}))$ is equal to $c_{d-1}((\mathcal{N}_1/\mathcal{O}(-1)_1) \otimes \mathcal{O}(1)_2) = c^{\Omega}(\mathcal{K}_1)(c_1(\mathcal{O}(1)_2))$. The leading term will be $c_{d-1}(\mathcal{K}) \times 1$. Finally, we can rewrite the power series in terms of $c_1(\mathcal{O}(-1))$ instead of $c_1(\mathcal{O}(1))$. □

Corollary 5.23 *There are such $\gamma_j \in \Omega^j(\mathbb{P}(\mathcal{N}))$, $j = d-2, \ldots, -\infty$, such that for any $u \in \Omega^*(\mathbb{P}(\mathcal{N}))$ we have the following identities:*

$$u = c_{d-1}(\mathcal{K}) \cdot \varepsilon^* \varepsilon_*(u) + \sum_{j \geqslant 1} \gamma_{d-1-j} \varepsilon^* \varepsilon_*(u \cdot c_1(\mathcal{O}(-1))^j);$$

$$u = \varepsilon^* \varepsilon_*(u \cdot c_{d-1}(\mathcal{K})) + \sum_{j \geqslant 1} c_1(\mathcal{O}(-1))^j \varepsilon^* \varepsilon_*(u \cdot \gamma_{d-1-j}).$$

Proof: Consider the diagram: $\mathbb{P}(\mathcal{N}) \overset{\varepsilon_1}{\leftarrow} \mathbb{P}(\mathcal{N}) \otimes_B \mathbb{P}(\mathcal{N}) \overset{\varepsilon_2}{\rightarrow} \mathbb{P}(\mathcal{N})$. Clearly, $(\varepsilon_1)_*(\Delta \cdot (\varepsilon_2)^*(u)) = u = (\varepsilon_2)_*(\Delta \cdot (\varepsilon_1)^*(u))$. Now it remains to apply Proposition 5.22 and the projection formula. □

Let $f : B \to A$ be a regular embedding of smooth quasiprojective varieties, and $\mathcal{N}$ be its normal bundle. Then we have the following cartesian diagram:

$$\begin{array}{ccc} \mathbb{P}(\mathcal{N}) & \xrightarrow{j} & Bl_{A,B} \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle\pi} \\ B & \xrightarrow[f]{} & A. \end{array}$$

Proposition 5.24 *The following sequence is split exact:*

$$0 \to \Omega^*(\mathbb{P}(\mathcal{N})) \stackrel{(j_*,\varepsilon_*)}{\to} \Omega^*(Bl_{A,B}) \oplus \Omega^*(B) \stackrel{(\pi_*,-f_*)}{\to} \Omega^*(A) \to 0.$$

Proof: Since the diagram above is commutative, our sequence is a complex. Let us construct a contracting homotopy for it.

We denote respective maps as:

$$\lambda_1 : \Omega^*(B) \to \Omega^*(\mathbb{P}(\mathcal{N}));\ \ \lambda_2 : \Omega^*(Bl_{A,B}) \to \Omega^*(\mathbb{P}(\mathcal{N}));\ \ \lambda_3 : \Omega^*(A) \to \Omega^*(B);\ \ \lambda_4 : \Omega^*(A) \to \Omega^*(Bl_{A,B}).$$

Take: $\lambda_4 = \pi^*$; $\lambda_3 = \beta \cdot f^*$, with $\beta = \widetilde{\varepsilon}_*\left(\frac{c_1(\mathcal{O}(1))}{c_1(\mathcal{O}(-1))}\right)$, where $\widetilde{\varepsilon} : \mathbb{P}_B(\mathcal{N} \oplus \mathcal{O}) \to B$ is the projection; $\lambda_1 = c_{d-1}(\mathcal{K}) \cdot \varepsilon^*$, and, finally, $\lambda_2 = F \circ j^*$, where $F(u) = \sum_{j\geqslant 0} \gamma_{d-2-j} \cdot \varepsilon^*\varepsilon_*(u \cdot c_1(\mathcal{O}(-1))^j)$. Let us check that this is indeed a contracting homotopy.

Here $\left(\frac{c_1(\mathcal{O}(1))}{c_1(\mathcal{O}(-1))}\right)$ just means the evaluation at $u = c_1(\mathcal{O}(1))$ of the expression $\left(\frac{u}{\text{-}_\Omega u}\right)$. The same applies to other similar expressions in the text.

Recall the following formula of M.Levine-F.Morel:

Proposition 5.25 ([6, Lemma 1.6]) *Let $f : Z \subset X$ be a regular embedding. We have the natural maps*

$$Z \stackrel{\widetilde{\varepsilon}}{\leftarrow} \mathbb{P}_Z(N_{Z\subset X} \oplus \mathcal{O}) \ \textit{and}\ Bl_{X,Z} \stackrel{\pi}{\to} X.$$

Then

$$\pi_*(1_{Bl_{X,Z}}) = 1_X + f_*\widetilde{\varepsilon}_*\left(\frac{c_1(\mathcal{O}(1))}{c_1(\mathcal{O}(-1))}\right)$$

1) From Proposition 5.25, $d \circ \lambda_3 + d \circ \lambda_4$ is equal to the identity.

2) $(d \circ \lambda_1 + \lambda_3 \circ d)(v) = v \cdot (\varepsilon_*(c_{d-1}(\mathcal{K})) - c_d(\mathcal{N}) \cdot \widetilde{\varepsilon}_*\left(\frac{c_1(\mathcal{O}(1))}{c_1(\mathcal{O}(-1))}\right))$. By the Excess Intersection Formula (Theorem 5.19), $\varepsilon_*(c_{d-1}(\mathcal{K})) = \varepsilon_*(c_{d-1}(\mathcal{K}) \cdot j^*(1)) = f^*\pi_*(1) = f^*(1 + f_*\widetilde{\varepsilon}_*\left(\frac{c_1(\mathcal{O}(1))}{c_1(\mathcal{O}(-1))}\right)) = 1 + c_d(\mathcal{N}) \cdot \widetilde{\varepsilon}_*\left(\frac{c_1(\mathcal{O}(1))}{c_1(\mathcal{O}(-1))}\right)$. Thus, our expression is equal to v.

3) $(\lambda_2 \circ d + \lambda_1 \circ d)(u) = F(u \cdot c_1(\mathcal{O}(-1))) + c_{d-1}(\mathcal{K}) \cdot \varepsilon^*\varepsilon_*(u) = (\sum_{j\geqslant 0} \gamma_{d-2-j} \cdot \varepsilon^*\varepsilon_*(u \cdot c_1(\mathcal{O}(-1))^{j+1})) + c_{d-1}(\mathcal{K}) \cdot \varepsilon^*\varepsilon_*(u) = u$, by Corollary 5.23.

4) From the commutative diagram with exact rows

$$\begin{array}{ccccccc}
\Omega^*(\mathbb{P}(\mathcal{N})) & \xrightarrow{j_*} & \Omega^*(Bl_{A,B}) & \xrightarrow{\overline{i}^*} & \Omega^*(Bl_{A,B}\backslash\mathbb{P}(\mathcal{N})) & \longrightarrow & 0 \\
 & & \pi^*\uparrow & & \| & & \\
 & & \Omega^*(A) & \xrightarrow[i^*]{} & \Omega^*(A\backslash B) & \longrightarrow & 0
\end{array}$$

we see that the map $(j_*, \pi^*) : \Omega^*(\mathbb{P}(\mathcal{N})) \oplus \Omega^*(A) \to \Omega^*(Bl_{A,B})$ is surjective. So, we can check our condition separately for elements of the form $j_*(u)$, and of the form $\pi^*(v)$.

$$(d \circ \lambda_2 + \lambda_4 \circ d)(j_*(u)) = j_*F(u \cdot c_1(\mathcal{O}(-1))) + \pi^*\pi_*j_*(u) = j_*(u - c_{d-1}(\mathcal{K}) \cdot \varepsilon^*\varepsilon_*(u)) + \pi^*f_*\varepsilon_*(u),$$

by Corollary 5.23. By the Excess Intersection Formula, $\pi^*f_*\varepsilon_*(u) = j_*(c_{d-1}(\mathcal{K}) \cdot \varepsilon^*\varepsilon_*(u))$, and our expression is equal to $j_*(u)$.

$$(d \circ \lambda_2 + \lambda_4 \circ d)(\pi^*(v)) = j_*(F(j^*\pi^*(v))) + \pi^*\pi_*\pi^*(v) = j_*(F(\varepsilon^*f^*(v))) + \pi^*(v + v \cdot f_*\widetilde{\varepsilon}_*\left(\frac{c_1(\mathcal{O}(1))}{c_1(\mathcal{O}(-1))}\right)),$$

and by the Excess Intersection Formula and the definition of F, this is equal to

$$\pi^*(v)(1 + j_*(c_{d-1}(\mathcal{K}) \cdot \varepsilon^*\widetilde{\varepsilon}_*\left(\frac{c_1(\mathcal{O}(1))}{c_1(\mathcal{O}(-1))}\right) + \sum_{j\geqslant 0} \gamma_{d-2-j} \cdot \varepsilon^*\varepsilon_*(c_1(\mathcal{O}(-1))^j))).$$

Notice, that by Proposition 5.22,

$$\sum_{j\geqslant 0} \gamma_{d-2-j} \cdot \varepsilon^*\varepsilon_*(c_1(\mathcal{O}(-1))^j) = (\varepsilon_1)_*\left(\frac{c^{\Omega}(\mathcal{K}_1)(t) - c_{d-1}(\mathcal{K}_1)}{t}(c_1(\mathcal{O}(-1)_2))\right).$$

Let us denote $\mathbb{P}(\mathcal{N})_1$ as D with the natural maps: $\eta : \mathbb{P}_D(\mathcal{N}) \to D$, $\widetilde{\eta} : \mathbb{P}_D(\mathcal{N} \oplus \mathcal{O}) \to D$. Then

$$c_{d-1}(\mathcal{K}) \cdot \varepsilon^*\widetilde{\varepsilon}_*\left(\frac{c_1(\mathcal{O}(1))}{c_1(\mathcal{O}(-1))}\right) + \sum_{j\geqslant 0} \gamma_{d-2-j} \cdot \varepsilon^*\varepsilon_*(c_1(\mathcal{O}(-1))^j) =$$
$$\widetilde{\eta}_*(c_{d-1}(\mathcal{K}) \cdot \left(\frac{c_1(\mathcal{O}(1))}{c_1(\mathcal{O}(-1))}\right)) + \eta_*(\frac{c^{\Omega}(\mathcal{K})(t) - c_{d-1}(\mathcal{K})}{t}(c_1(\mathcal{O}(-1)_2))).$$

By the Quillen's formula (Theorem 5.35), this is equal to

$$\operatorname*{Res}_{t=0}\left(\frac{(c^{\Omega}(\mathcal{K})(t) - c_{d-1}(\mathcal{K}))(-\omega)}{t \cdot c^{\Omega}(\mathcal{N})}\right) + \operatorname*{Res}_{t=0}\left(\frac{(-_{\Omega}\, t) \cdot c_{d-1}(\mathcal{K})(-\omega)}{t \cdot c^{\Omega}(\mathcal{N}) \cdot (-_{\Omega}\, t)}\right) =$$
$$\operatorname*{Res}_{t=0}\left(\frac{(-\omega)}{t \cdot (c_1(\mathcal{O}(-1)) -_{\Omega} t)}\right) = \operatorname*{Res}_{t=0}\left(\frac{\frac{(-_{\Omega}\, t)}{t}(-\omega)}{(-_{\Omega}\, t) \cdot (c_1(\mathcal{O}(-1)) -_{\Omega} t)}\right)$$

(notice, that we have $-\omega = (-_{\Omega})^*(\omega)$ here since our parameter is $c_1(\mathcal{O}(-1))$ and not $c_1(\mathcal{O}(1))$). And the latter expression (by the same Quillen's formula) is just $\mu_*\left(\frac{c_1(\mathcal{O}(1))}{c_1(\mathcal{O}(-1))}\right)$, where $\mu : \mathbb{P}_D(\mathcal{O}(-1) \oplus \mathcal{O}) \to D$ is the projection. But D is a smooth divisor on $X = Bl_{A,B}$, and by Proposition 5.25, $j_*\mu_*\left(\frac{c_1(\mathcal{O}(1))}{c_1(\mathcal{O}(-1))}\right)$ is equal to $[Bl_{X,D} \to X] - [id : X \to X]$, which is zero.

Thus,$(d \circ \lambda_2 + \lambda_4 \circ d)(\pi^*(v)) = \pi^*(v)$, and the proposition is proven. □

Remark: Notice, that the exactness of our sequence would follow automatically, if we knew that Ω^* extends to the generalized cohomology theory in the sense of [11] (with the localization axiom, etc.). Such theory is expected to be $MGL^{*,*'}$, but to my knowledge, currently it is not known. But another reason to have the above proof is to obtain the exact form of the contracting homotopy. For example, we use it in the following:

Corollary 5.26 *The map* $(\pi)_* \oplus (j)^* : \Omega^*(Bl_{A,B}) \to \Omega^*(A) \oplus \Omega^*(\mathbb{P}(\mathcal{N}))$ *is injective.*

Proof: We just need to mention that $id_{Bl_{A,B}} = \pi^* \circ \pi_* + j_* \circ F \circ j^*$. □

Suppose that we have a cartesian diagram $\begin{array}{ccc} B & \xrightarrow{f} & A \\ \uparrow{\scriptstyle g'} & & {\scriptstyle g}\uparrow \\ D & \xrightarrow[f']{} & C, \end{array}$ where f, f', g, g' are regular embeddings of smooth quasi-projective varieties. It induces the commutative diagram

$$\begin{array}{ccccccc} A & \xleftarrow{\pi_A} & Bl_{A,B} & \xleftarrow{j_A} & \mathbb{P}_B(\mathcal{N}_{B\subset A}) & \xrightarrow{\varepsilon_B} & B \\ g\uparrow & & g_{Bl}\uparrow & & \uparrow\overline{g} & & \uparrow g' \\ C & \xleftarrow{\pi_C} & Bl_{C,D} & \xleftarrow{j_C} & \mathbb{P}_D(\mathcal{N}_{D\subset C}) & \xrightarrow[\varepsilon_D]{} & D, \end{array}$$

where the middle square is proper cartesian. Let also

$$\begin{array}{ccccc} \mathbb{P}_D(\mathcal{N}_{D\subset C}) & \xrightarrow{k} & \mathbb{P}_D((g')^*\mathcal{N}_{B\subset A}) & \xrightarrow{h} & \mathbb{P}_B(\mathcal{N}_{B\subset A}) \\ \varepsilon_D\downarrow & & \varepsilon\downarrow & (2) & \downarrow\varepsilon_B \\ D & = & D & \xrightarrow[g']{} & B \end{array}$$

be the natural maps.

Proposition 5.27

$$(g_{Bl})_*\pi_C^* = \pi_A^* g_* + (j_A)_* h_* \left(\frac{c_d(\mathcal{M}\otimes\mathcal{O}(1)) - c_d(\mathcal{M})}{c_1(\mathcal{O}(-1))} \cdot \varepsilon^*(f')^* \right),$$

where $\mathcal{M} = (g')^*(\mathcal{N}_{B\subset A})/\mathcal{N}_{D\subset C}$*, and* $d = \dim(\mathcal{M}) = \dim(A) + \dim(D) - \dim(B) - \dim(C)$*.*

Proof: Denote as $\widetilde{k}, \widetilde{h}, \widetilde{\varepsilon_D}, \widetilde{\varepsilon}, \widetilde{\varepsilon_B}$ the maps analogous to $k, h, \varepsilon_D, \varepsilon, \varepsilon_B$, but with $\mathcal{N}$ changed everywhere to $\mathcal{N}\oplus\mathcal{O}$.

Denote also the element $\left(\frac{c_1(\mathcal{O}(1))}{c_1(\mathcal{O}(-1))}\right)$ as δ.

We want to show that the values of $(\pi_A)_*$ and $(j_A)^*$ on both parts of our equation are the same. It follows from Proposition 5.25 that

$$(\pi_A)_*(g_{Bl})_*(\pi_C)^*(v) = g_*(\pi_C)_*(\pi_C)^*(v) = g_*(v\cdot(1+(f')_*(\widetilde{\varepsilon_D})_*(\delta))).$$

Analogously,

$$(\pi_A)_*(\pi_A)^* g_*(v) = g_*(v)\cdot(1+f_*(\widetilde{\varepsilon_B})_*(\delta)).$$

Thus,

$$(\pi_A)_*((g_{Bl})_*(\pi_C)^*(v) - (\pi_A)^*g_*(v)) = f_*g'_*(\widetilde{\varepsilon_D})_*((\widetilde{\varepsilon_D})^*(f')^*(v)\cdot\delta) - f_*(\widetilde{\varepsilon_B})_*((\widetilde{\varepsilon_B})^*f^*g_*(v)\cdot\delta)$$

By the Excess Intersection Formula (Theorem 5.19),

$$f^*g_*(v) = g'_*(c_d(\mathcal{M})\cdot(f')^*(v)).$$

Also, $(\widetilde{\varepsilon_B})^*g'_* = \widetilde{h}_*\widetilde{\varepsilon}^*$, $g'_*(\widetilde{\varepsilon_D})_* = (\widetilde{\varepsilon_B})_*\widetilde{h}_*\widetilde{k}_*$, and $k_*(\widetilde{\varepsilon_D})^* = c_d(\mathcal{M}\otimes\mathcal{O}(1))\cdot\widetilde{\varepsilon}^*$. Thus,

$$\begin{aligned} &(\pi_A)_*((g_{Bl})_*(\pi_C)^*(v) - (\pi_A)^*g_*(v)) = \\ &f_*(\widetilde{\varepsilon_B})_*\widetilde{h}_*\widetilde{k}_*((\widetilde{\varepsilon_D})^*(f')^*(v)\cdot\delta) - f_*(\widetilde{\varepsilon_B})_*\widetilde{h}_*(\widetilde{\varepsilon}^*(c_d(\mathcal{M})\cdot(f')^*(v))\cdot\delta) = \\ &f_*(\widetilde{\varepsilon_B})_*\widetilde{h}_*\left(\frac{(c_d(\mathcal{M}\otimes\mathcal{O}(1)) - c_d(\mathcal{M}))c_1(\mathcal{O}(1))}{c_1(\mathcal{O}(-1))}\cdot\widetilde{\varepsilon}^*(f')^*(v)\right). \end{aligned}$$

The latter expression is clearly equal to

$$f_*(\varepsilon_B)_*h_*\left(\frac{c_d(\mathcal{M}\otimes\mathcal{O}(1))-c_d(\mathcal{M})}{c_1(\mathcal{O}(-1))}\cdot\varepsilon^*(f')^*(v)\right)=(\pi_A)_*(j_A)_*h_*\left(\frac{c_d(\mathcal{M}\otimes\mathcal{O}(1))-c_d(\mathcal{M})}{c_1(\mathcal{O}(-1))}\cdot\varepsilon^*(f')^*(v)\right).$$

Now, apply $(j_A)^*$ to our equation. Since the square (2) is proper cartesian, and k_*k^* is given by the multiplication by $c_d(\mathcal{M}\otimes\mathcal{O}(1))$, we get

$$(j_A)^*(g_{Bl})_*\pi_C^*(v)=(\overline{g})_*(j_C)^*\pi_C^*(v)=(\overline{g})_*(\varepsilon_D)^*(f')^*(v)=h_*(c_d(\mathcal{M}\otimes\mathcal{O}(1))\cdot\varepsilon^*(f')^*(v)).$$

Using Excess Intersection Formula, we get:

$$(j_A)^*\pi_A^*g_*(v)=(\varepsilon_B)^*f^*g_*(v)=(\varepsilon_B)^*g'_*(c_d(\mathcal{M})\cdot(f')^*(v))=h_*\varepsilon^*(c_d(\mathcal{M})\cdot(f')^*(v)).$$

On the other hand,

$$(j_A)^*(j_A)_*h_*\left(\frac{c_d(\mathcal{M}\otimes\mathcal{O}(1))-c_d(\mathcal{M})}{c_1(\mathcal{O}(-1))}\cdot\varepsilon^*(f')^*\right)=h_*(c_d(\mathcal{M}\otimes\mathcal{O}(1))-c_d(\mathcal{M}))\cdot\varepsilon^*(f')^*(v))$$

Thus, we have proven that the value of $((\pi_A)_*,(j_A)^*)$ on the left and the right hand side of our formula is the same. It remains to apply Corollary 5.26. □

5.7 The formula of Quillen

The aim of this subsection is to prove the algebraic analog of the formula of Quillen expressing the class of the projective bundle in the cobordism ring of the base.

Let R be some commutative ring with a ring homomorphism $\mathbb{L}\to R$. Consider the ring $A:=R[[\lambda_1,\ldots,\lambda_n]]$ of power series. In the ring A we can add and subtract elements in the sense of the universal formal group law. We will denote the corresponding operations as $+_\Omega$ and $-_\Omega$. The proof of the following statement is straightforward.

Statement 5.28 *The element $(\lambda_i -_\Omega \lambda_j)$, for $i\neq j$ is not a zero-divisor in A.*

Les S be the multiplicative system in A generated by $(\lambda_i -_\Omega \lambda_j)$ for all $1\leqslant i\neq j\leqslant n$. Denote as B the localization $A[S^{-1}]$. It follows from the Statement 5.28 that the map $A\to B$ is injective.

Let $f(t)\in R[[t]]$ be arbitrary power series. Consider the following element of B:

$$p_n(f)(\lambda_1,\ldots,\lambda_n):=\sum_{i=1}^n f(\underset{\Omega}{-}\lambda_i)\cdot\prod_{j\neq i}(\lambda_j\underset{\Omega}{-}\lambda_i)^{-1}.$$

Let $X:=\times_{i=1}^n\mathbb{P}^\infty$. Let us denote as $\mathcal{L}_i$ the linear bundle $\mathcal{O}(1)_i$. Then $\Omega^*(X)\otimes_{\mathbb{L}}R$ is exactly A, if we identify $c_1(\mathcal{L}_i)$ with λ_i. Let $E:=\oplus_{i=1}^n\mathcal{O}(1)_i$ be the standard vector bundle, and $Y:=\mathbb{P}(E)\xrightarrow{\pi}X$ be its projectivization. The following result permits to compute the class of Y in the cobordism ring of X.

Proposition 5.29 *For arbitrary $f(t)\in R[[t]]$:*

(1) *The element $p_n(f)$ belongs to A.*

(2) $\pi_*(f(c_1(\mathcal{O}(1)))\cdot[1_Y])=p_n(f)\in\Omega^*(X)\otimes_{\mathbb{L}}R.$

Proof: Use induction on n. For $n=1$, the projection $\pi: Y \to X$ is an isomorphism, and $\pi^*(\mathcal{O}_X(-1)) = \mathcal{O}(1)$. Thus, $\pi_*(f(c_1(\mathcal{O}(1)))\cdot[1_Y]) = f(\mathcal{O}_X(-1)) = p_1(f)$. So, both conditions are satisfied. Suppose the statements are known for $(n-1)$.

Let us check condition (2) for n. We work inside B. Let us denote $c_1(\mathcal{O}(1))$ as μ. For arbitrary subset $I \subset \{1,\ldots,n\}$ let us denote as E_I the vector subbundle $\oplus_{i\in I}\mathcal{L}_i$ of E, and as Y_I the corresponding projective bundle. Let $u_I : Y_I \to Y$ will be respective embedding, and π_I - the composition $\pi \circ u_I$. Clearly,

$$(u_I)_*[1_{Y_I}] = \prod_{j\in\{1,\ldots,n\}\setminus I} c_1(\mathcal{O}(1)\otimes\mathcal{L}_j) = \prod_{j\in\{1,\ldots,n\}\setminus I} (\mu \underset{\Omega}{+} \lambda_i) \in \Omega^{n-\#(I)}(Y). \tag{2}$$

$$\pi_*(f(\mu)[1_Y]) = \pi_*\left(\frac{(\mu +_\Omega \lambda_n) -_\Omega (\mu +_\Omega \lambda_{n-1})}{(\lambda_n -_\Omega \lambda_{n-1})}\cdot f(\mu)[1_Y]\right) =$$

$$\pi_*\left(\frac{f(\mu)[1_Y]}{(\lambda_n -_\Omega \lambda_{n-1})}\cdot((\mu \underset{\Omega}{+} \lambda_n) + (\underset{\Omega}{-}\mu \underset{\Omega}{-} \lambda_{n-1}) + \right.$$

$$\left.\frac{(\mu +_\Omega \lambda_n) -_\Omega (\mu +_\Omega \lambda_{n-1}) - (\mu +_\Omega \lambda_n) - (-_\Omega \mu -_\Omega \lambda_{n-1})}{(\mu +_\Omega \lambda_n)(\mu +_\Omega \lambda_{n-1})}\cdot(\mu \underset{\Omega}{+} \lambda_n)(\mu \underset{\Omega}{+} \lambda_{n-1})\right).$$

Using (2), and applying the inductive assumptions to:

1) $R_1 = R$, $A_1 = R_1[[\lambda_1,\ldots,\lambda_{n-1}]]$, $f_1(t) = f(t)$;

2) $R_2 = R[[\lambda_{n-1}]]$, $A_2 = R_2[[\lambda_1,\ldots,\lambda_{n-2},\lambda_n]]$, $f_2(t) = f(t)\cdot\frac{-_\Omega(t +_\Omega \lambda_{n-1})}{(t +_\Omega \lambda_{n-1})}$, and

3) $R_3 = R[[\lambda_{n-1},\lambda_n]]$, $A_3 = R_3[[\lambda_1,\ldots,\lambda_{n-2}]]$, $f_3(t) = f(t)\cdot\frac{(t +_\Omega \lambda_n) -_\Omega (t +_\Omega \lambda_{n-1}) - (t +_\Omega \lambda_n) - (-_\Omega t -_\Omega \lambda_{n-1})}{(t +_\Omega \lambda_n)(t +_\Omega \lambda_{n-1})}$,

we can rewrite the above expression as:

$$\frac{(p_{n-1}(f_1)(\lambda_1,\ldots,\lambda_{n-1}) + p_{n-1}(f_2)(\lambda_1,\ldots,\lambda_{n-2},\lambda_n) + p_{n-2}(f_3)(\lambda_1,\ldots,\lambda_{n-2}))}{(\lambda_n -_\Omega \lambda_{n-1})} =$$

$$\frac{1}{(\lambda_n -_\Omega \lambda_{n-1})}\left(\sum_{j\neq n}\frac{f(-_\Omega \lambda_j)}{\prod_{n\neq l\neq j}(\lambda_l -_\Omega \lambda_j)} + \sum_{j\neq n-1}\frac{f(-_\Omega \lambda_j)\cdot(\lambda_j -_\Omega \lambda_{n-1})}{\prod_{l\neq j}(\lambda_l -_\Omega \lambda_j)} + \right.$$

$$\left.\sum_{(n-1)\neq j\neq n}\frac{f(-_\Omega \lambda_j)\cdot((\lambda_n -_\Omega \lambda_{n-1}) - (\lambda_n -_\Omega \lambda_j) - (\lambda_j -_\Omega \lambda_{n-1}))}{\prod_{l\neq j}(\lambda_l -_\Omega \lambda_j)}\right) = p_n(f)(\lambda_1,\ldots,\lambda_n).$$

The second statement is proven. Since $\pi_*(f(\mu)[1_Y]) \in \Omega^*(X)\otimes_{\mathbb{L}} R = A$, we get the first statement too. □

On the ring A we have the natural action of the symmetric group S_n, and the element $p_n(f)$ is stable under this action. Thus, $p_n(f)$ can be expressed as a power series $q_n(f)(\sigma_1,\ldots,\sigma_n)$, where σ_i is i-th elementary symmetric function. Now we can get the algebro-geometric analog of the formula of Quillen ([14, Theorem 1]). This algebro-gemetric analog appeared first in [13, Formula (24)]. We will provide independent proof here.

Theorem 5.30 *Let X be smooth quasiprojective variety, and V be some n-dimensional vector bundle on X. Let $Y := \mathbb{P}(V) \xrightarrow{\pi} X$, and $f(t) \in \Omega^*(X)[[t]]$ be some power series. Then:*

$$\pi_*(f(c_1(\mathcal{O}(1)))[1_Y]) = q_n(f)(c_1(V),\ldots,c_n(V)).$$

Proof: One can observe that the proof of Proposition 5.29 works for arbitrary X and $Y = \mathbb{P}(V)$ as long as $V = \oplus_i \mathcal{L}_i$, and $(c_1(\mathcal{L}_i) -_{\Omega} c_1(\mathcal{L}_j))$ is not a zero divisor in $\Omega^*(X)$ for all $i \neq j$. In particular, we get:

Lemma 5.31 *Theorem 5.30 is true for* $X = (\times_{i=1}^{n} \mathbb{P}^{\infty}) \times (\times_{j=1}^{n} \mathbb{P}^{\infty})$*, and* $Y = \mathbb{P}(V)$*, where* $V = \oplus_{i=1}^{n} (\mathcal{O}(1)_{i,1} \otimes \mathcal{O}(-1)_{i,2})$*.*

It is clear that the statement of Theorem 5.30 is sufficient to check for $f(t) = t^r$, for all r. Suppose $\varphi : A \to B$ be map of smooth quasi-projective varieties, and V be a vector bundle on B. Then the following diagram is proper cartesian.

$$\begin{array}{ccc} \mathbb{P}_A(\varphi^*(V)) & \xrightarrow{\nu} & \mathbb{P}_B(V) \\ {\scriptstyle \pi_A}\downarrow & & \downarrow{\scriptstyle \pi_B} \\ A & \xrightarrow[\eta]{} & B. \end{array}$$

Also, $\nu^*(\mathcal{O}(1)) = \mathcal{O}(1)$. Then

$$\pi_{A*}((c_1(\mathcal{O}(1)))^r \cdot [1_{\mathbb{P}_A(\eta^*(V))}]) = \pi_{A*} \circ \nu^*((c_1(\mathcal{O}(1)))^r \cdot [1_{\mathbb{P}_B(V)}]) = \eta^* \circ \pi_{B*}((c_1(\mathcal{O}(1)))^r \cdot [1_{\mathbb{P}_B(V)}]).$$

We get:

Lemma 5.32 (1) *If Theorem 5.30 is true for the pair* (B, V)*, then it is true for the pair* $(A, \eta^*(V))$*.*

(2) *If the map* $\eta^* : \Omega^*(B) \to \Omega^*(A)$ *is injective, and Theorem 5.30 is true for the pair* $(A, \eta^*(V))$*, then it is true for the pair* (B, V)*.*

For the smooth quasi-projective variety X, and a linear bundle $\mathcal{L}$ on it, there exist ample line bundles $\mathcal{L}_1, \mathcal{L}_2$ such that $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2^{\vee}$. That means, that there exists a map $\eta_{\mathcal{L}} : X \to \mathbb{P}^{\infty} \times \mathbb{P}^{\infty}$ such that $\mathcal{L} = \eta_{\mathcal{L}}^*(\mathcal{O}(1)_1 \otimes \mathcal{O}(-1)_2)$. From Lemmas 5.31 and 5.32(1) we get:

Lemma 5.33 *Theorem 5.30 is true for arbitrary smooth quasi-projective variety* X*, and* $V = \oplus_i \mathcal{L}_i$*, where* $\mathcal{L}_i$ *are linear bundles.*

Lemma 5.34 *Let* X *be smooth quasi-projective variety, and* V *be a vector bundle on* X *equipped with the filtration* $V_1 \subset \ldots \subset V_n = V$ *with linear subfactors* V_i/V_{i-1}*. Then the Theorem 5.30 is true for the pair* (X, V)*.*

Proof: Using the trick of I.Panin-A.Smirnov ([12]), we can find a composition of affine bundles $\eta : X' \to X$ such that $\eta^*(V)$ is a sum of linear bundles. From the extended homotopy property for algebraic cobordism ([5, Corollary 9.3]), the map $\eta^* : \Omega^*(X) \to \Omega^*(X')$ is an isomorphism. It remains to apply the Lemma 5.32(2). □

Now, the general case can be reduced to that of Lemma 5.34 if we consider the variety of complete flags $\mathrm{Flag}_X(V)$, and use the Lemma 5.32(2). □

Let $F(x, y)$ be any 1-dimensional (commutative) formal group law with coefficients in R. Then there exists unique invariant 1-form ω_F on $\mathrm{Spec}(R[[x]])$ such that $\omega_F(0) = dx$. Such form is given by: $\omega_F = \left(\frac{\partial F}{\partial y}|_{y=0}\right)^{-1} dx$. In the case of the universal formal group law on $\mathbb{L}$ we denote such form simply as ω.

We can give a formulation of Theorem 5.30 in a more familiar Quillen's form:

Theorem 5.35 *Let X be smooth quasiprojective variety, V be some n-dimensional vector bundle on X, and $\pi : \mathbb{P}_X(V) \to X$ be the corresponding projective bundle. Let $f(t) \in \Omega^*(X)[[t]]$. Then*

$$\pi_*(f(c_1(\mathcal{O}(1)))) = \operatorname*{Res}_{t=0} \frac{f(t) \cdot \omega}{\prod_i (t +_\Omega \lambda_i)},$$

where λ_i are the roots of V.

Proof: By the projection formula, it is sufficient to prove the Theorem for the case $f(t) \in \mathbb{L}[[t]]$. We know from Theorem 5.30 that

$$\pi_*(f(c_1(\mathcal{O}(1)))) = p_n(f)(\lambda_1, \ldots, \lambda_n).$$

In the formula for *Res* one should understand $\frac{1}{(t +_\Omega \lambda_i)}$ simply as

$$\frac{1}{t}(1 + \frac{\lambda_i}{t} G(t, \lambda_i))^{-1} = \frac{1}{t} \sum_{l \geqslant 0} \frac{(-1)^l \lambda_i^l}{t^l} G(t, \lambda_i)^l,$$

where $x +_\Omega y = x + y \cdot G(x, y)$. Thus, $\operatorname{Res}_{t=0} \frac{f(t) \cdot \omega}{\prod_i (t +_\Omega \lambda_i)}$ is certain universal formula on λ_i's, that is, some power series $r_n(f)(\lambda_1, \ldots, \lambda_n) \in \mathbb{L}[[\lambda_1, \ldots, \lambda_n]]$. We just need to check that $r_n(f) = p_n(f)$. Since we have an embedding $\mathbb{L}[[\lambda_1, \ldots, \lambda_n]] \subset \mathbb{L} \otimes_{\mathbf{Z}} \mathbb{Q}[[\lambda_1, \ldots, \lambda_n]]$, we can check the equality in the latter ring. But over $\mathbb{L} \otimes_{\mathbf{Z}} \mathbb{Q}$ the universal formal group law is equivalent to the additive one. That is, there exist power series $\alpha(t), \beta(t) \in \mathbb{L} \otimes_{\mathbf{Z}} \mathbb{Q}[[t]]$ such that $\alpha(t) = t + a_2 t^2 + \ldots$, $\beta(t) = t + b_2 t^2 + \ldots$, $\beta(\alpha(t)) = t$ and $\alpha(x) +_\Omega \alpha(y) = \alpha(x + y)$. Let $s = \beta(t)$. Then $\omega = ds$. Let $\frac{x}{\alpha(x)} = \gamma(x) \in \mathbb{L} \otimes_{\mathbf{Z}} \mathbb{Q}[[x]]$. Then

$$\operatorname*{Res}_{t=0} \frac{f(t) \cdot \omega}{\prod_i (t +_\Omega \lambda_i)} = \operatorname*{Res}_{s=0} \frac{f(\alpha(s)) \cdot \omega}{\prod_i \alpha(s + \beta(\lambda_i))} = \operatorname*{Res}_{s=0} \frac{f(\alpha(s)) \prod_i \gamma(s + \beta(\lambda_i)) \cdot ds}{\prod_i (s + \beta(\lambda_i))}.$$

Lemma 5.36

$$\operatorname*{Res}_{s=0} \frac{g(s) ds}{\prod_i (s + \mu_i)} = \sum_i \frac{g(-\mu_i)}{\prod_{j \neq i} (\mu_j - \mu_i)}$$

Proof: By definition, $\operatorname{Res}_{s=0} \frac{g(s) ds}{\prod_i (s + \mu_i)} = \sum_l g_{l+n-1} \cdot \sum_{l_1 + \ldots + l_n = l} \prod_i (-\mu_i)^{l_i}$. Due to the identity:

$$\sum_i \frac{(-\mu_i)^{l+n-1}}{\prod_{j \neq i} (\mu_j - \mu_i)} = \sum_{l_1 + \ldots + l_n = l} \prod_i (-\mu_i)^{l_i},$$

the latter expression is equal to $\sum_i \frac{g(-\mu_i)}{\prod_{j \neq i} (\mu_j - \mu_i)}$. □

It follows from Lemma 5.36 that our expression is equal to

$$\sum_i \frac{f(\alpha(-\beta(\lambda_i)) \prod_j \gamma(-\beta(\lambda_i) + \beta(\lambda_j))}{\prod_{j \neq i} (-\beta(\lambda_i) + \beta(\lambda_j))} = \sum_i \frac{f(-_\Omega \lambda_i)}{\prod_{j \neq i} (\lambda_j -_\Omega \lambda_i)}.$$

□

5.8 Miscellaneous

Let $o \to \mathcal{U} \to \mathcal{V} \to \mathcal{W} \to 0$ be the short exact sequence of vector bundles on X. Let $\pi : Bl_{\mathbb{P}(\mathcal{V}),\mathbb{P}(\mathcal{U})} \to \mathbb{P}(\mathcal{V})$ be the blow-down map, and $\varepsilon : \mathbb{P}(\mathcal{W}) \to X$ be the projection. The proof of the following statement is straightforward.

Statement 5.37 *$Bl_{\mathbb{P}(\mathcal{V}),\mathbb{P}(\mathcal{U})}$ is isomorphic to $\mathbb{P}_{\mathbb{P}(\mathcal{W})}(\mathcal{X})$, where the bundle $\mathcal{O}(1)$ on the latter is isomorphic to $\pi^*(\mathcal{O}(1))$ and $\mathcal{X}$ fits into the exact sequence: $0 \to \varepsilon^*(\mathcal{U}) \to \mathcal{X} \to \mathcal{O}(-1) \to 0$.*

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