Symmetric operations

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1 Introduction

The aim of this paper is to construct certain cohomological operations acting in algebraic cobordisms and Chow groups of smooth projective varieties over a field k of characteristic 0. As an application we will give a new proof of the theorem describing the size of binary direct summands in the motives of quadrics. It has the following advantage: all our constructions are detected on the level of topological realization, while the two previously known proofs used the torsion elements in the Chow-groups.

These operations are also used (and it is what they were constructed for) in the proof of the Theorem describing all possible dimensions of anisotropic forms in I^n, but this will be presented in a separate text [9].

We also give some new way to look at the Steenrod operations in Chow groups. Such operations were originally constructed by V.Voevodsky [10] in the context of arbitrary motivic cohomology, and then a simpler construction was given by P.Brosnan [1] for the case of usual Chow groups.

I'm very grateful to M.Rost for many discussions concerning his degree formula and other topics which influenced my research a lot. Also, I want to thank A.Lazarev for very useful conversations on cobordisms, and A.Merkurjev and V.Voevodsky for many essential remarks. This text was written while I was visiting École Polytechnique Fédérale de Lausanne, and I would like to express my gratitude to this institution and especcially to Prof. Eva Bayer-Fluckiger for the support, excellent working conditions, and very encouraging atmosphere.

2 Steenrod operations via proper transform

In [10] V.Voevodsky has defined the action of the motivic Steenrod algebra (which is a certain extension of the usual topological Steenrod algebra) on motivic cohomology with $\mathbf{Z}/l$ -coefficients. These operations played a key role in his proof of the Milnor's conjecture. Later, in [1] P.Brosnan has given a more elementary construction for the particular case of the action of the topological Steenrod algebra in the usual Chow groups $(mod\, l)$. In this section we will show yet another way to define Steenrod operations in Chow groups, although only for smooth proper varieties and for $\mathbf{Z}/2$ -coefficients. Still, there is some feeling that this method works in a more general situation.

First of all, I should mention that our constructions are very close to the construction used by M.Rost for the purposes of his *Degree formula* (see [7, Theorem 6.1]). We just consider things in

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greater generality (the main difference is that we study not only 0 -cycles but cycles of arbitrary dimension), which permits to get new results, and to look at some results of M.Rost from the new standpoint.

Let X be a smooth proper variety over a field k of characteristic different from 2 . Denote by $\square(X)$ the Cartesian square of X , and by $\widetilde{\square}(X)$ the blow-up of $\square(X)$ along the diagonal $X \stackrel{\Delta}{\to} \square(X)$. The natural $\mathbf{Z}/2$ -action on $\square(X)$ lifts naturally to the action on $\widetilde{\square}(X)$, and the fixed-point set of the latter is a smooth divisor $D \cong \mathbb{P}(T_X)$ on $\widetilde{\square}(X)$. So, the quotient-variety $(\widetilde{\square}(X))/(\mathbf{Z}/2)$ will be smooth and proper. We will denote it by $\widetilde{C}^2(X)$. We get the diagram:

$$\begin{array}{ccccc} \mathbb{P}(T_X) & \xrightarrow{j} & \widetilde{\square}(X) & \xrightarrow{p} & \widetilde{C}^2(X) \\ \varepsilon\downarrow & & \downarrow\pi & & \\ X & \xrightarrow[\Delta]{} & \square(X) & & \end{array}$$

Let V be a subvariety of X of codimension d . Denote by $\widetilde{\square}(V)$ the proper preimage of $\square(V)$ with respect to the morphism $\pi : \widetilde{\square}(X) \to \square(X)$. Also denote by $\widetilde{C}^2(V)$ the quotient $\widetilde{\square}(V)/(\mathbf{Z}/2)$ - a subvariety on $\widetilde{C}^2(X)$.

According to [2, Theorem 6.7], we have an equality in $\mathrm{CH}^*(\widetilde{\square}(X))$:

$$\pi^*([\square(V)]) = [\widetilde{\square}(V)] + j_*(c(\varepsilon^*(T_X) - \mathcal{O}(-1)) \cap \varepsilon^* s(V, \square(V)))_{2\dim(V)},$$

where $c(-)$ and $s(-,-)$ are Chern and Segre classes respectively.

In particular, if $V \subset X$ is a regular embedding, we get:

$$j^*([\widetilde{\square}(V)]) = \sum_{l=0}^{d} \rho^l \cdot \varepsilon^*(c_{d-l}(N_{V\subset X})[V]),$$

where $\rho = c_1(\mathcal{O}(1))$.

Let now $Z = \sum_i (-1)^{r_i}[V_i]$ be an arbitrary cycle from $\mathcal{Z}^d(X)$, where V_i are irreducible subvarieties. We define the map $\widetilde{\square} : \mathcal{Z}^d(X) \to \mathcal{Z}^{2d}(\widetilde{\square}(X))$ by the formula:

$$\widetilde{\square}(Z) := \sum_{i,j} (-1)^{r_i + r_j} [\widetilde{V_i \times V_j}],$$

where $\widetilde{V_i \times V_j}$ is a proper transform of $V_i \times V_j$ under π .

In general, $\widetilde{\square}$ does not preserve rational equivalence between cycles. But it preserves it modulo 2 .

Proposition 2.1 *We get a well-defined map* $\widetilde{\square} : \mathrm{CH}^d(X)/2 \to \mathrm{CH}^{2d}(\widetilde{\square}(X))/2$.

Proof: For $Z = \sum_i (-1)^{r_i}[V_i] \in \mathcal{Z}^d(X)$ let us denote by $\widetilde{C}^2(Z) \in \mathcal{Z}^{2d}(\widetilde{C}^2(X))$ the element $\sum_i [\widetilde{C}^2(V_i)] + \sum_{i<j} (-1)^{r_i+r_j} p_*([\widetilde{V_i \times V_j}])$. Since p is finite surjective morphism of smooth varieties, it is flat. Then $p^*(\widetilde{C}^2(Z)) = \widetilde{\square}(Z)$. Let $\mathcal{Z}^d_{dom}(X \times \mathbb{A}^1)$ denote the subgroup of cycles, generated by the classes of subvarieties dominant over $\mathbb{A}^1$. Suppose $\alpha, \beta \in \mathcal{Z}^d(X)$ and $\gamma \in \mathcal{Z}^d_{dom}(X \times \mathbb{A}^1)$ be such that $\gamma|_{X\times\{0\}} = \alpha$ and $\gamma|_{X\times\{1\}} = \beta$. Then $(\gamma \times_{\mathbb{A}^1} \gamma)|_{X\times X\times\{0\}} = \alpha \times \alpha$, $(\gamma \times_{\mathbb{A}^1} \gamma)|_{X\times X\times\{1\}} = \beta \times \beta$, and $\gamma \times_{\mathbb{A}^1} \gamma$ is stable under the natural $(\mathbf{Z}/2)$ -action on $X \times X \times \mathbb{A}^1$. Thus, $\alpha \times \alpha$ and $\beta \times \beta$ are $(\mathbf{Z}/2)$ -equivariantly rationally equivalent. Then $\alpha \times \alpha|_{(X\times X\backslash\Delta(X))}$ is $(\mathbf{Z}/2)$ -equivariantly rationally equivalent to $\beta \times \beta|_{(X\times X\backslash\Delta(X))}$. Hence,

in $\mathcal{Z}^{2d}((X\times X\backslash\Delta(X))/(\mathbf{Z}/2))$, the cycle $(\alpha\times\alpha|_{(X\times X\backslash\Delta(X))})/(\mathbf{Z}/2)$ is rationally equivalent to $(\beta\times\beta|_{(X\times X\backslash\Delta(X))})/(\mathbf{Z}/2)$. In other words, $\widetilde{C}^2(\alpha)|_{(\widetilde{C}^2(X)\backslash D)}$ is rationally equivalent to $\widetilde{C}^2(\beta)|_{(\widetilde{C}^2(X)\backslash D)}$ (compare [1, Proposition 3.4]). So,

$$(\widetilde{C}^2(\alpha)-\widetilde{C}^2(\beta))\in\operatorname{image}((p\circ j)_*).$$

Then

$$\widetilde{\Box}(\alpha)-\widetilde{\Box}(\beta)=p^*(\widetilde{C}^2(\alpha)-\widetilde{C}^2(\beta))\in\operatorname{image}(p^*\circ p_*\circ j_*)=2\cdot\operatorname{image}(j_*),$$

since p is ramified over $(p\circ j)(D)$ with ramification index 2 .

Thus, $\widetilde{\Box}$ gives a well-defined map $\mathrm{CH}^d(X)\to\mathrm{CH}^{2d}(\widetilde{\Box}(X))/2$, and from bilinearity, $2\cdot\mathrm{CH}^d(X)\mapsto 0$. $\Box$

If $V\to X$ is a regular embedding, then, as we saw above, $j^*([\widetilde{\Box}(V)])$ considered $(\mathit{mod}\,2)$ is exactly $\sum_{l=0}^{d}\rho^{d-l}S^l([V])$, where S^l are *upper* Steenrod operations in Chow groups $(\mathit{mod}\,2)$, defined by P.Brosnan (see [1]), they correspond to the operations Sq^{2l} of V.Voevodsky (see [10]). Then for arbitrary Z from the subgroup of $\mathrm{CH}^d(X)/2$, generated by the classes of smooth subvarieties, we also have: $(j^*\circ\widetilde{\Box})(Z)=\sum_{l=0}^{d}\rho^{d-l}S^l(Z)$, as the operation $j^*\circ\widetilde{\Box}$ is clearly additive $(\mathit{mod}\,2)$. Or, equivalently, for such Z , for arbitrary $r\geqslant 0$,

$$\varepsilon_*(\rho^r\cdot(j^*\circ\widetilde{\Box})(Z))=S_{d+r-n+1}(Z),$$

where $n=\dim(X)$ and S_i are *lower* Steenrod operations of P.Brosnan.

In the case of the field of characteristic 0 we have:

Theorem 2.2 *For arbitrary* $Z\in\mathrm{CH}^d(X)/2$,

$$(j^*\circ\widetilde{\Box})(Z)=\sum_{l=0}^{d}\rho^{d-l}S^l(Z).$$

Proof: As was mentioned, it is sufficient to prove that for arbitrary $Z\in\mathrm{CH}^d(X)$, and arbitrary $r\geqslant 0$, $\varepsilon_*(c_1(\mathcal{O}(1))^r\cdot(j^*\circ\widetilde{\Box})(Z))=S_{d+r-n+1}(Z)$, where $n=\dim(X)$ and S_t - lower Steenrod operation of P.Brosnan.

Indeed, let $(j^*\circ\widetilde{\Box})(Z)=\sum_{l\leqslant d}c_1(\mathcal{O}(1))^{d-l}Z_l$ for some cycles Z_l . Treating separately the case $\dim(Z)=0$, where both sides of the formula in question are 0 , we can assume that $d<n$. Have: $\varepsilon_*(c_1(\mathcal{O}(1))^r\cdot(j^*\circ\widetilde{\Box})(Z))=\sum_{l\leqslant d}c_{d-l+r-n+1}(-T_X)Z_l$. So, if this expression is equal to $S_{d+r-n+1}(Z)=\sum_{l=0}^{d}c_{d-l+r-n+1}(-T_X)S^l(Z)$ for all $r\geqslant 0$, then

$$Z_l=\sum_r S_{d+r-n+1}(Z)c_{l-d-r+n-1}(T_X)=$$

$$\sum_r\sum_m c_{d-m+r-n+1}(-T_X)c_{l-d-r+n-1}(T_X)S^m(Z)=S^l(Z).$$

Now we use:

Proposition 2.3 *The operation*

$$\varepsilon_*(\rho^r\cdot(j^*\circ\widetilde{\Box})(-)):\mathrm{CH}_a(X)\to\mathrm{CH}_{2a-r-1}(X)/2$$

commutes with the push-forward map for morphisms of smooth proper varieties.

Proof: 1) Let $i : X \to Y$ be regular embedding. Then there is commutative diagram:

$$\begin{array}{ccccc} Y & \xleftarrow{\varepsilon_Y} & \mathbb{P}(T_Y) & \xrightarrow{j_Y} & \widetilde{\square}(Y) \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle \mathbb{P}(T)(i)} & & \uparrow{\scriptstyle \widetilde{\square}(i)} \\ X & \xleftarrow[\varepsilon_X]{} & \mathbb{P}(T_X) & \xrightarrow[j_X]{} & \widetilde{\square}(X) \end{array}$$

where the right square is cartesian with $\mathbb{P}(T)(i)^*(N_{j_Y}) = \mathbb{P}(T)(i)^*(\mathcal{O}(-1)) = \mathcal{O}(-1) = N_{j_X}$. Also, notice that $\mathbb{P}(T)(i)^*(\rho_Y) = \rho_X$. Then for $[V] \in \mathrm{CH}^d(X)/2$,

$$\begin{aligned} i_*((\varepsilon_X)_*(\rho_X^r \cdot j_X^*[\widetilde{\square}_X(V)])) &= (\varepsilon_Y)_* \circ (\mathbb{P}(T)(i))_*(\rho_X^r \cdot j_X^*[\widetilde{\square}_X(V)]) = \\ \text{(by the projection formula)} \ &= (\varepsilon_Y)_*(\rho_Y^r \cdot (\mathbb{P}(T)(i)_* \circ j_X^*[\widetilde{\square}_X(V)])) = \\ \text{(by [2, Theorems 6.2,6.3])} \ &= (\varepsilon_Y)_*(\rho_Y^r \cdot (j_Y^* \circ \widetilde{\square}(i)_*[\widetilde{\square}_X(V)])) = \\ &\qquad (\varepsilon_Y)_*(\rho_Y^r \cdot j_Y^*[\widetilde{\square}_Y(V)]). \end{aligned}$$

2) Let now $W = U \times X$ with X and U smooth proper, and $h : W \to X$ be natural projection.

Since, $(\mathit{mod}\, 2)$, the operation $(j^* \circ \widetilde{\square})(\sum_i (-1)^{l_i}[V_i]) = \sum_i j^*([\widetilde{\square}(V_i)])$ is linear, it is sufficient to consider the case of the class of an irreducible subvariety.

For $R \to S$, let $\square(R/S) := R \times_S R$. Let us also denote $\mathrm{Bl}_{P,Q}$ as (P, Q) , and for the subvariety $R \subset P$, we denote as $\tilde{\chi}(R)$ the proper preimage of R if R is not contained in Q , and the preimage if it is contained (χ here is the projection $(P, Q) \to P$). For the regular embedding j we denote as N_j the normal bundle.

We have projections: $W \to X \to \mathrm{Spec}(k)$, and respectively, embeddings:

$$\square(W) = \square(W/\operatorname{Spec}(k)) \supset \square(W/X) \supset \square(W/W) = W.$$

Let us abbreviate it as: $A \supset B \supset C$. Let (A, C) and (A, B) be the corresponding blowing up, with the natural maps: $(A, C) \stackrel{f}{\to} A \stackrel{g}{\leftarrow} (A, B)$. Blowing further $\tilde{f}(B)$ and $\tilde{g}(C)$ respectively, we get the diagram:

$$(\tilde{f}(A), \tilde{f}(B)) \stackrel{\alpha}{\to} (A, C) \stackrel{f}{\to} A \stackrel{g}{\leftarrow} (A, B) \stackrel{\beta}{\leftarrow} (\tilde{g}(A), \tilde{g}(C)),$$

where $(\tilde{f}(A), \tilde{f}(B))$ is naturally identified with $(\tilde{g}(A), \tilde{g}(C))$.

We have commutative diagram:

$$\begin{array}{ccccccc} (A, C) & \xleftarrow{\alpha} & (\tilde{f}(A), \tilde{f}(B)) & = & (\tilde{g}(A), \tilde{g}(C)) & \xrightarrow{\beta} & (A, B) \\ {\scriptstyle j_1}\uparrow & & \uparrow{\scriptstyle j_1'} & & {\scriptstyle j_2'}\uparrow & & \uparrow{\scriptstyle j_2} \\ \tilde{f}(C) & \xleftarrow[\alpha']{} & (\tilde{f}(C), \tilde{f}(B) \cap \tilde{f}(C)) & & (\tilde{g}(B), \tilde{g}(C)) & \xrightarrow[\beta']{} & \tilde{g}(B) \end{array}$$

Let us denote: $E_1 := (\tilde{f}(C), \tilde{f}(C) \cap \tilde{f}(B))$ and $E_2 := (\tilde{g}(B), \tilde{g}(C))$.

Since $\tilde{f}(B)$ intersects $\tilde{f}(C)$ transversally (that is, $N_{\tilde{f}(B) \subset \tilde{f}(A)}|_{\tilde{f}(B) \cap \tilde{f}(C)} = N_{\tilde{f}(C) \cap \tilde{f}(B) \subset \tilde{f}(C)}$), the first square is cartesian with $\alpha'^*(N_{j_1}) = N_{j_1'}$ and so, by [2, Theorems 6.2,6.3],

$$\alpha^*(j_{1*}[\tilde{f}(C)]) = j'_{1*}[E_1].$$

Analogously, it is easy to see (for example, with the help of [2, Theorem 6.7]) that

$$\beta^*(j_{2*}[\tilde{g}(B)]) = j'_{1*}[E_1] + j'_{2*}[E_2].$$

Remembering that $(A,B)=(\square(W),\square(W/X))$ and $\tilde{g}(B)=\mathbb{P}(\square(h/id)^*T_X)$, where $\square(h/id):\square(W/X)\to\square(X/X)=X$ is the natural map, we get a cartesian square with $\gamma'^*(N_{j_X})=N_{j_2}$:

$$\begin{array}{ccc} (A,B) & \stackrel{j_2}{\longleftarrow} & \tilde{g}(B) \\ {\scriptstyle\gamma}\big\downarrow & & \big\downarrow{\scriptstyle\gamma'} \\ (\square(X),X) & \underset{j_X}{\longleftarrow} & \mathbb{P}(T_X). \end{array}$$

The map $\varepsilon_W:\tilde{f}(C)=\mathbb{P}(T_W)\to W$ is equal to the composition:
$\tilde{f}(C)\stackrel{j_1}{\to}(A,C)\stackrel{f}{\to}A=W\times W\stackrel{pr_1^W}{\to}W$. Therefore,

$$(\varepsilon_W)_*(c_1(\mathcal{O}(1))^r\cdot(j_1^*\circ\widetilde{\square}_W)([V]))=(-1)^r\cdot(pr_1^W)_*\circ f_*((j_{1*}[\tilde{f}(C)])^{r+1}\cdot\widetilde{\square}_W([V])).$$

Let now $\widetilde{\widetilde{\square}}(V)$ be the proper preimage of the variety $\widetilde{\square}_W(V)$ under the map α . Then, clearly, $\alpha_*[\widetilde{\widetilde{\square}}(V)]=[\widetilde{\square}_W(V)]$, and so, by the projection formula,

$$(\varepsilon_W)_*(c_1(\mathcal{O}(1))^r\cdot j_1^*([\widetilde{\square}_W(V)]))=(-1)^r\cdot(pr_1^W)_*\circ f_*\circ\alpha_*((j'_{1*}[E_1])^{r+1}\cdot[\widetilde{\widetilde{\square}}(V)]).$$

Analogously, since $\gamma_*\circ\beta_*[\widetilde{\widetilde{\square}}(V)]=\widetilde{\square}_X[h_*(V)]$, we have:

$$\begin{aligned}(\varepsilon_X)_*(c_1(\mathcal{O}(1))^r\cdot(j_X^*\circ\widetilde{\square}_X)(h_*[V]))&=\\ (-1)^r\cdot(pr_1^X)_*\circ(\pi_X)_*((j_{X*}[\mathbb{P}(T_X)])^{r+1}\cdot(\gamma_*\circ\beta_*)[\widetilde{\widetilde{\square}}(V)])&=\\ (\text{by the projection formula and [2, Theorems 6.2,6.3]})&=\\ (-1)^r\cdot(pr_1^X)_*\circ(\pi_X)_*\circ\gamma_*((j_{2*}[\tilde{g}(B)])^{r+1}\cdot\beta_*[\widetilde{\widetilde{\square}}(V)])&=\\ (\text{again by the projection formula})&=\\ (-1)^r\cdot(pr_1^X)_*\circ(\pi_X)_*\circ\gamma_*\circ\beta_*((j'_{1*}[E_1]+j'_{2*}[E_2])^{r+1}\cdot[\widetilde{\widetilde{\square}}(V)]).&\end{aligned}$$

Since $h\circ pr_1^W\circ f\circ\alpha=pr_1^X\circ\pi_X\circ\gamma\circ\beta$, we can rewrite the expression

$$h_*(\varepsilon_W)_*(c_1(\mathcal{O}(1))^r\cdot(j_W^*\circ\widetilde{\square}_W)[V])\quad\text{as}\quad(-1)^r\cdot(pr_1^X)_*\circ(\pi_X)_*\circ\gamma_*\circ\beta_*((j'_{1*}[E_1])^{r+1}\cdot[\widetilde{\widetilde{\square}}(V)]).$$

So, to prove that our operation commutes with h_* we need only to check that all terms involving $j'_{2*}[E_2]$ in the formula for $(\varepsilon_X)_*(c_1(\mathcal{O}(1))^r\cdot(j_X^*\circ\widetilde{\square}_X)(h_*[V]))$ are divisible by 2 .

Any such term has the form:

$$(-1)^r\cdot(pr_1^X)_*\circ(\pi_X)_*\circ\gamma_*\circ\beta_*((j'_{1*}[E_1])^s\cdot(j'_{2*}[E_2])^{r+1-s}\cdot[\widetilde{\widetilde{\square}}(V)]),$$

where $0\leqslant s\leqslant r$.

Consider the quotient variety $\widetilde{C}^2(W/X):=\widetilde{\square}(W/X)/(\mathbf{Z}/2)$. It is a smooth proper variety (since the set of fixed points of the $\mathbf{Z}/2$ -action is a smooth divisor $\mathbb{P}(\square(h/id)^*T_X)$) with the natural embedding $\widetilde{C}^2(W/X)\subset\widetilde{C}^2(W)$. Moreover, $\widetilde{\square}(W/X)=p_W^{-1}(\widetilde{C}^2(W/X))$, where $p_W:\widetilde{\square}(W)\to\widetilde{C}^2(W)$ is the natural projection. Thus, the $\mathrm{Bl}_{\widetilde{C}^2(W),\widetilde{C}^2(W/X)}$ is a quotient-variety of the natural $\mathbf{Z}/2$ -action on $(\tilde{f}(A),\tilde{f}(B))=\mathrm{Bl}_{\widetilde{\square}(W),\widetilde{\square}(W/X)}$.

The natural $\mathbf{Z}/2$ -action on $(\tilde{f}(A), \tilde{f}(B)) = (\tilde{g}(A), \tilde{g}(C))$ extends to the action on the divisor $E_2 = (\tilde{g}(B), \tilde{g}(C))$ with the set of fixed points - the smooth divisor $\tilde{\beta}'(\tilde{g}(C))$. so, we get smooth quotient Y and a cartesian square

$$\begin{array}{ccc} \mathrm{Bl}_{\widetilde{\square}(W),\widetilde{\square}(W/X)} & \xleftarrow{j_2'} & E_2 \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle q'} \\ \mathrm{Bl}_{\widetilde{C}^2(W),\widetilde{C}^2(W/X)} & \xleftarrow{i} & Y. \end{array}$$

Notice that $(pr_1^X) \circ \pi_X \circ \gamma \circ \beta \circ j_2' = \varepsilon_X \circ \gamma' \circ \beta'$ and the latter composition factors as $\theta \circ q'$ (since it is stable under $\mathbf{Z}/2$ -action on E_2). Thus,

$$(-1)^r \cdot (pr_1^X)_* \circ (\pi_X)_* \circ \gamma_* \circ \beta_*((j_{1*}'[E_1])^s \cdot (j_{2*}'[E_2])^{r+1-s} \cdot [\widetilde{\widetilde{\square}}(V)]) =$$
$$(-1)^r \cdot \theta_* \circ q_*' \circ j_2'^*((j_{1*}'[E_1])^s \cdot (j_{2*}'[E_2])^{r-s} \cdot [\widetilde{\widetilde{\square}}(V)]).$$

But the cycle $[\widetilde{\widetilde{\square}}(V)]$ is the pull-back q^* of the cycle $[\widetilde{\widetilde{\widetilde{C}}}{}^2(V)] \in \mathrm{CH}^*(\mathrm{Bl}_{\widetilde{C}^2(W),\widetilde{C}^2(W/X)})$. Similarly, $j_{1*}'[E_1] = \alpha^*(c_1(\mathcal{O}(-1)))$, and so, it is equal to $q^*u^*(c_1(\mathcal{L}^\vee))$, where $u : \mathrm{Bl}_{\widetilde{C}^2(W),\widetilde{C}^2(W/X)} \to \widetilde{C}^2(W)$ is the blowing down map and $\mathcal{L}$ is the natural linear bundle on $\widetilde{C}^2(W)$ - see [7, Theorem 6.1]. Finally, $j_{2*}'[E_2] = q^*i_*[Y]$.

Therefore, $(-1)^r \cdot \theta_* \circ q_*' \circ j_2'^*((j_{1*}'[E_1])^s \cdot (j_{2*}'[E_2])^{r-s} \cdot [\widetilde{\widetilde{\square}}(V)]) = (-1)^r \cdot \theta_* \circ q_*' \circ j_2'^* \circ q^*(\mathit{something}) = (-1)^r \cdot \theta_* \circ q_*'q'^* \circ i^*(\mathit{something}) = 2 \cdot (-1)^r \cdot \theta_* i^*(\mathit{something})$.

Thus, we have proved that all terms involving $j_{2*}'[E_2]$ are divisible by 2 in $\mathrm{CH}^*(X)$, and so:

$$h_*((\varepsilon_W)_*(\rho_W^r \cdot (j_W^* \circ \widetilde{\square}_W)[V])) = (\varepsilon_X)_*(\rho_X^r \cdot (j_X^* \circ \widetilde{\square}_X)(h_*[V])) \ (\mathit{mod}\, 2).$$

As any morphism of smooth proper varieties is a composition of morphisms of the type 1) and 2) the statement follows.

□

We know that the statement is true for the classes of regularily embedded subvarieties. On the other hand, since $char(k) = 0$, by the Theorem of H.Hironaka ([3]), for each irreducible subvariety $\overline{V} \subset X$ there exists a morphism $h : W \to X$ and a smooth subvariety $V \subset W$ such that $h|_V : V \to \overline{V}$ is birational. Proposition above (together with the similar result of P.Brosnan on $S_\bullet$ - see [1]) shows that the statement is true for the class of an arbitrary subvariety, and hence, for an arbitrary cycle. □

Remark: 1) Notice, that in the case $char(k) \neq 0$ we still can define the *upper* Steenrod operations $S^\bullet$ by the formula: $(j^* \circ \widetilde{\square})(Z) = \sum_{l=0}^{d} \rho^{d-l} S^l$, and then the *lower* ones: $S_\bullet := S^\bullet \cdot c_\bullet(-T_X)$ will commute with the push-forwards for the morphisms of smooth proper varieties.

2) The rule above works only when $\dim(Z) > 0$, since the basis of $\mathrm{CH}^*(\mathbb{P}(T_X))$ over $\mathrm{CH}^*(X)$ consists of $\rho^i, 0 \leqslant i \leqslant \dim(X) - 1$. In particular, $\rho^{\dim(X)} \cdot [Z] = 0$, for $\dim(Z) = 0$. But this is not a problem, since for 0 -cycles we can just define: $S_0 = S^0 = id$, and $S^i = S_i = 0$, for arbitrary $i > 0$.

3 Divisibility of the Landweber-Novikov operations

Let k be a field of characteristic 0 , and X be a smooth projective variety over k .

In [5],[6] M.Levine and F.Morel have defined the ring of Algebraic cobordisms $\Omega^*(X)$. We will briefly remind this definition.

Let $\mathcal{M}^*(X)$ denote the set of isomorphism classes of projective morphisms of pure codimension from a smooth quasiprojective variety Y to X graded by codimension. The disjoint union provides the structure of monoid on $\mathcal{M}^*(X)$. Let $\mathcal{M}^*(X)^+$ be it's group completion. The classes $f : Y \to X$ and $g : Z \to X$ are called *elementary cobordant* if there exists a projective morphism h from a smooth quasiprojective variety W to $X \times \mathbb{A}^1$ which is transversal to $X \times \{0\}$ and $X \times \{1\}$ and such that it's restrictions $h^{-1}(X \times \{0\}) \to X$ and $h^{-1}(X \times \{1\}) \to X$ are isomorphic to f and g , respectively. The quotient of $\mathcal{M}^*(X)^+$ by such relations is denoted $\text{Pre - }\Omega^*(X)$.

If $\mathcal{L}$ is an invertible sheaf on X generated by it's global sections, one can define the action $c_1(\mathcal{L}) : \text{Pre - }\Omega^n(X) \to \text{Pre - }\Omega^{n+1}(X)$ as follows. For $Y \to X$ we choose the section $s : Y \to E(\mathcal{L}|_Y)$ transversal to the zero-section, and put: $c_1(\mathcal{L}) \cdot [Y \to X] := [Z \to X]$, where $Z \subset Y$ is a smooth subvariety defined by the equation $s = 0$.

Let $F_U(-,-)$ be the *universal formal group law* having the coefficients in the Lazard ring $\mathbb{L}$. The topological realization functor provides the surjection $\text{Pre - }\Omega^*(\text{Spec}(k)) \twoheadrightarrow \mathbb{L}$. Let $\tilde{a}_{i,j}$ be any lifting of the universal coefficients $a_{i,j} \in \mathbb{L}$ satisfying $\tilde{a}_{i,j} = \tilde{a}_{j,i}$. Let $\tilde{\Omega}^*$ be the quotient of $\text{Pre - }\Omega^*(X)$ by the relations $c_1(\mathcal{L}\otimes\mathcal{M}) = F_{\tilde{U}}(c_1(\mathcal{L}), c_1(\mathcal{M}))$ for all pairs $\mathcal{L}, \mathcal{M}$ of invertible sheaves generated by the global sections (notice that the action of $c_1(-)$ is nilpotent). On the ring $\tilde{\Omega}^*$ there is natural action of $c_1(\mathcal{N})$ for arbitrary invertible sheaf $\mathcal{N}$ and such an action satisfy the abovementioned rule with respect to tenzor product of sheaves. Finally, the ring of algebraic cobordisms $\Omega^*(X)$ is the quotient of $\tilde{\Omega}^*$ by the relations generated by $[Y \to X] = c_1(\mathcal{N}_{Y\subset X})[Id_X]$, where Y is a smooth codimension 1 subvariety of X and $\mathcal{N}_{Y\subset X}$ is a normal bundle.

We have a natural ring homomorphism $pr_X : \Omega^*(X) \to \text{CH}^*(X)$ which is surjective, and its kernel is equal to $\ker(pr_{\text{Spec}(k)}) \cdot \Omega^*(X)$, where $\ker(pr_{\text{Spec}(k)})$ coincides with $\Omega^{<0}(\text{Spec}(k))$ - the ideal of elements of negative codimension (that is, of positive dimension). The topological realization functor defines an isomorphism $\Omega^*(\text{Spec}(k)) \cong \mathbb{L}$ for arbitrary field k of characteristic 0 .

In this section we will give some new construction for the Chow trace of certain Landweber-Novikov operations, which, in particular, shows that such Chow traces are divisible by 2 on certain cobordism groups.

Let $v : V \to X$ be a morphism of smooth projective varieties. Let U be an arbitrary smooth projective variety over k such that there is a regular embedding $V \to U$. Denote: $W := U \times X$. Clearly, the map v decomposes into the composition $V \xrightarrow{g} W \xrightarrow{f} X$, where g is a regular embedding and f is a natural projection.

Consider the diagram
$$\begin{array}{ccccc} \mathbb{P}(T_W) & \xrightarrow{j} & \widetilde{\square}(W) & \xrightarrow{p} & \widetilde{C}^2(W) \\ \varepsilon\downarrow & & \downarrow\pi & & \\ W & \xrightarrow{\Delta} & \square(W) & & \end{array}$$

We have the smooth projective subvariety $\widetilde{\square}(V)$ of $\widetilde{\square}(W)$. Clearly it is $p^{-1}(\widetilde{C}^2_g(V))$ for the smooth projective subvariety $\widetilde{C}^2_g(V)$ of $\widetilde{C}^2(W)$. The line bundle $\mathcal{O}(1)$ on $\widetilde{\square}(W)$ is $p^*(\mathcal{L})$ for some line bundle $\mathcal{L}$ on $\widetilde{C}^2(W)$ - see [7, Theorem 6.1]. Let us denote: $\varrho := c_1(\mathcal{L})$. Finally, we have the natural correspondence $\widetilde{C}^2(f)$ from $\widetilde{C}^2(W)$ to X given by the smooth projective subvariety $\widetilde{C}^2(W/X) := \text{Bl}_{W\times_X W, \Delta(W)}/(\mathbf{Z}/2)$ of $\widetilde{C}^2(W) \times X$ (notice that this variety has a natural projection to X and a natural embedding into $\widetilde{C}^2(W)$).

Definition 3.1 *Define* $\Phi^{d+r}([V]) := \widetilde{C}^2(f)_*(\varrho^r \cdot [\widetilde{C}^2_g(V)]) \in \Omega^{2d+r}(X)$.

Proposition 3.2 $\Phi^{d+r}([V])$ *does not depend on the choice of* U *, and the choice of a regular embedding* $q : V \to U$ *.*

Proof: It is clearly sufficient to consider the case $id : V \to V$ and arbitrary $q : V \to U$. Let us denote the corresponding W -spaces by W_1 and W_2 . We have the natural embeddings $g_1 := (id, v) : V \to W_1$, $g_2 := (q, v) : V \to W_2$, and $h := (q, id) : W_1 \to W_2$ so that $g_2 = h \circ g_1$. Then we get an embedding $\widetilde{C}^2(h) : \widetilde{C}^2(W_1) \to \widetilde{C}^2(W_2)$. Notice that the subvariety $\widetilde{C}^2(W_2/X)$ of $\widetilde{C}^2(W_2) \times X$ is transversal to $\widetilde{C}^2(h) \times id$, and

$$\widetilde{C}^2(W_2/X) \times_{(\widetilde{C}^2(W_2)\times X)} (\widetilde{C}^2(W_1) \times X) = \widetilde{C}^2(W_1/X)$$

with the morphisms to $\widetilde{C}^2(W_1) \times X$ and $\widetilde{C}^2(W_2/X)$ - the natural ones. So, we get a proper cartesian square in the sense of the [5, axiom (A3)]:

$$\begin{array}{ccc}
\widetilde{C}^2(W_2/X) & \xrightarrow{i_{f_2}} & \widetilde{C}^2(W_2) \times X \\
{\scriptstyle \widetilde{C}^2(h/id)}\uparrow & & \uparrow{\scriptstyle \widetilde{C}^2(h)\times id} \\
\widetilde{C}^2(W_1/X) & \xrightarrow[i_{f_1}]{} & \widetilde{C}^2(W_1) \times X
\end{array}$$

So, both squares in the following diagram are proper cartesian:

$$\begin{array}{ccccc}
 & & X & & \\
 & & {\scriptstyle pr}\uparrow & & \\
\widetilde{C}^2(W_2/X) & \xrightarrow{i_{f_2}} & \widetilde{C}^2(W_2) \times X & \xrightarrow{\tau_2} & \widetilde{C}^2(W_2) \\
{\scriptstyle \widetilde{C}^2(h/id)}\uparrow & & \uparrow{\scriptstyle \widetilde{C}^2(h)\times id} & & \uparrow{\scriptstyle \widetilde{C}^2(h)} \\
\widetilde{C}^2(W_1/X) & \xrightarrow[i_{f_1}]{} & \widetilde{C}^2(W_1) \times X & \xrightarrow{\tau_1} & \widetilde{C}^2(W_1) \\
 & & & & \uparrow{\scriptstyle \widetilde{C}^2(g_1)} \\
 & & & & \widetilde{C}^2(V)
\end{array}$$

Then,

$$\begin{aligned}
\Phi_1^{r+d}([V]) := (pr \circ (\widetilde{C}^2(h) \times id))_* \circ (i_{f_1})_*(i_{f_1})^* \circ \tau_1^*(\varrho_1^r \cdot [\widetilde{C}^2(V), \widetilde{C}^2(g_1)]) = \\
pr_* \circ (i_{f_2})_* \circ \widetilde{C}^2(h/id)_* \circ (i_{f_1})^* \circ \tau_1^*(\varrho_1^r \cdot [\widetilde{C}^2(V), \widetilde{C}^2(g_1)]) = \\
pr_* \circ (i_{f_2})_*(i_{f_2})^* \circ (\widetilde{C}^2(h) \times id)_* \circ \tau_1^*(\varrho_1^r \cdot [\widetilde{C}^2(V), \widetilde{C}^2(g_1)]) = \\
pr_* \circ (i_{f_2})_*(i_{f_2})^* \circ \tau_2^* \circ \widetilde{C}^2(h)_*(\varrho_1^r \cdot [\widetilde{C}^2(V), \widetilde{C}^2(g_1)]) = \\
pr_* \circ (i_{f_2})_*(i_{f_2})^* \circ \tau_2^*(\varrho_2^r \cdot [\widetilde{C}^2(V), \widetilde{C}^2(g_2)]) = \Phi_2^{r+d}([V]).
\end{aligned}$$

We used the fact that $\widetilde{C}^2(h)^*(\mathcal{L}_2) = \mathcal{L}_1$, and so, $\varrho_1 = \widetilde{C}^2(h)^*(\varrho_2)$, and that $\widetilde{C}^2(h) \circ \widetilde{C}^2(g_1) = \widetilde{C}^2(g_2)$. □

Proposition 3.3 *Let* $v_1 : V_1 \to X$ *and* $v_2 : V_2 \to X$ *are elementary cobordant. Then* $\Phi^{r+d}([V_1]) = \Phi^{r+d}([V_2])$ *. And so, we get a well defined operation:*
Φ^{r+d} : Pre - $\Omega^d(X) \to \Omega^{2d+r}(X)$.

Proof: We need the following evident lemma.

Lemma 3.4 *Let* $v_1 : V \to X$ *and* $v_2 : V_2 \to X$ *are elementary cobordant via* $t : T \to X \times \mathbb{A}^1$ *. Suppose,* $h : T \to U \times \mathbb{A}^1$ *is a map such that* $pr_2 \circ h = pr_2 \circ t$ *and* $h|_{\{0\}} = q_1 : V_1 \to U$ *,* $h|_{\{1\}} = q_2 : V_2 \to U$ *. Then* $(q_1, v_1) : V_1 \to U \times X$ *and* $(q_2, v_2) : V_2 \to U \times X$ *are elementary cobordant.*

$\square$

Since T is quasi-projective, we have an embedding $i : T \hookrightarrow \mathbb{P}^n$. Take $U := \mathbb{P}^n$ and $h := (i, pr_2 \circ t) : T \to U \times \mathbb{A}^1$. Since $pr_2 \circ t$ is projective and i is an embedding of a smooth variety, h is a regular embedding. In particular, since $pr_2 \circ h$ is transversal to $\{0\}$ and $\{1\}$, $q_1 := h|_{h^{-1}(U \times \{0\})} : V_1 \to U$, and $q_2 := h|_{h^{-1}(U \times \{1\})} : V_2 \to U$ are regular embeddings. By the previous Lemma, $g_1 := (q_1, v_1) : V_1 \to W := U \times X$ and $g_2 := (q_2, v_2) : V_2 \to W$ are elementary cobordant. Then $[\widetilde{C}^2(V_1), \widetilde{C}^2(g_1)]$ and $[\widetilde{C}^2(V_2), \widetilde{C}^2(g_2)]$ are elementary cobordant as well. Now we just observe that $\Phi^{d+r}([V]) := pr_* \circ (i_f)_*(i_f)^* \circ \tau^*(\varrho^r \cdot [\widetilde{C}^2(V), \widetilde{C}^2(g)])$, where the maps are taken from the diagram:

$$\begin{array}{ccccc} & & X & & \\ & & \uparrow{\scriptstyle pr} & & \\ \widetilde{C}^2(W/X) & \xrightarrow{i_f} & \widetilde{C}^2(W) \times X & \xrightarrow{\tau} & \widetilde{C}^2(W) \end{array}$$

$\square$

Let again $v : V \to X$ be some morphism of smooth projective varieties, and U be an arbitrary smooth projective variety together with the regular embedding $q : V \to U$. Then the variety $W := U \times X$ has the regular embedding $g := (q, v) : V \to W$ as well as the smooth projection $f : W \to X$ satisfying: $v = f \circ g$. Let us denote: $\widetilde{\Box}(W) := \mathrm{Bl}_{W \times W, \Delta(W)}$, and $\widetilde{\Box}(W/X) := \mathrm{Bl}_{W \times_X W, \Delta(W)}$. We have smooth projective subvariety $\widetilde{\Box}(V)$ of $\widetilde{\Box}(W)$, the divisor class $\rho = c_1(\mathcal{O}(1))$ on $\widetilde{\Box}(W)$, and the correspondence $\widetilde{\Box}(f)$ from $\widetilde{\Box}(W)$ to X given by the smooth projective subvariety $\widetilde{\Box}(W/X)$ of $\widetilde{\Box}(W) \times X$.

Definition 3.5 *For arbitrary* $r \geqslant 0$ *define:* $\Psi^{d+r}([V]) := \widetilde{\Box}(f)_*(\rho^r \cdot [\widetilde{\Box}(V)])$.

Using the same arguments as in the proofs of Propositions 3.2,3.3, we get:

Proposition 3.6 $\Psi^{d+r}([V])$ *does not depend on the choice of* U *and the choice of a regular embedding* $q : V \to U$ *.*

Proposition 3.7 *If* $v_1 : V_1 \to X$ *and* $v_2 : V_2 \to X$ *are elementary cobordant, then* $\Psi^{d+r}([V_1])$ *is cobordant to* $\Psi^{d+r}([V_2])$ *. So,* Ψ^{d+r} *gives a well-defined operation:* $\text{Pre - }\Omega^d(X) \to \Omega^{2d+r}(X)$ *.*

Consider the operations:

$$pr_{\Omega^* \to \mathrm{CH}^*} \circ \Phi^{d+r}, pr_{\Omega^* \to \mathrm{CH}^*} \circ \Psi^{d+r} : \text{Pre-}\Omega^d \to \mathrm{CH}^{2d+r} .$$

We have the diagram:

$$\begin{array}{ccccccc}
 & & X & & & & \\
 & & {\scriptstyle pr}\uparrow & & & & \\
\widetilde{C}^2(W/X) & \xrightarrow{i_{\widetilde{C}^2,f}} & \widetilde{C}^2(W)\times X & \xrightarrow{\tau_{\widetilde{C}^2}} & \widetilde{C}^2(W) & \xleftarrow{\widetilde{C}^2(g)} & \widetilde{C}^2(V) \\
{\scriptstyle p_{W/X}}\uparrow & & \uparrow{\scriptstyle p_W\times id} & & \uparrow{\scriptstyle p_W} & & \uparrow{\scriptstyle p_V} \\
\widetilde{\square}(W/X) & \xrightarrow{i_{\widetilde{\square},f}} & \widetilde{\square}(W)\times X & \xrightarrow{\tau_{\widetilde{\square}}} & \widetilde{\square}(W) & \xleftarrow{\widetilde{\square}(g)} & \widetilde{\square}(V)
\end{array}$$

with all squares proper cartesian.
Since $p_W^*(\mathcal{L})=\mathcal{O}(1)$, and $pr_{\Omega^*\to\mathrm{CH}^*}((p_V)_*[\widetilde{\square}(V)])=2\cdot pr_{\Omega^*\to\mathrm{CH}^*}([\widetilde{C}^2(V)])$, we get:

Proposition 3.8 $pr_{\Omega^*\to\mathrm{CH}^*}\circ\Psi^{d+r}=2\cdot pr_{\Omega^*\to\mathrm{CH}^*}\circ\Phi^{d+r}$.

Consider the diagram:

$$\begin{array}{ccccccc}
 & & X & & & & \\
 & & {\scriptstyle pr}\uparrow & & & & \\
\square(W/X) & \xrightarrow{i_{\square,f}} & \square(W)\times X & \xrightarrow{\tau_{\square}} & \square(W) & \xleftarrow{\square(g)} & \square(V) \\
{\scriptstyle \pi_{W/X}}\uparrow & & \uparrow{\scriptstyle \pi_W\times id} & & \uparrow{\scriptstyle \pi_W} & & \uparrow{\scriptstyle \pi_V} \\
\widetilde{\square}(W/X) & \xrightarrow{i_{\widetilde{\square},f}} & \widetilde{\square}(W)\times X & \xrightarrow{\tau_{\widetilde{\square}}} & \widetilde{\square}(W) & \xleftarrow{\widetilde{\square}(g)} & \widetilde{\square}(V) \\
{\scriptstyle j_{W/X}}\uparrow & & \uparrow{\scriptstyle j_W\times id} & & \uparrow{\scriptstyle j_W} & & \uparrow{\scriptstyle j_V} \\
\mathbb{P}(T_{W/X}) & \xrightarrow{i_{\mathbb{P}(T),f}} & \mathbb{P}(T_W)\times X & \xrightarrow{\tau_{\mathbb{P}(T)}} & \mathbb{P}(T_W) & \xleftarrow{\mathbb{P}(T)(g)} & \mathbb{P}(T_V)
\end{array}$$

where $\square(Y):=Y\times Y$ and $\square(Y/X):=Y\times_X Y$. Here all the lower squares are proper cartesian.
By [2, Theorem 6.7],

$$pr_{\Omega^*\to\mathrm{CH}^*}([\widetilde{\square}(V)])=pr_{\Omega^*\to\mathrm{CH}^*}(\pi_W^*([\square(V)])-(j_W)_*(\sum_{i=0}^{a-1}c_1(\mathcal{O}(1))^{a-i-1}\varepsilon_W^*(c_i(T_W-T_V)[V]))),$$

where $a=\dim(W)-\dim(V)$.
Thus,

$$\begin{aligned}
&pr_{\Omega^*\to\mathrm{CH}^*}\circ\Psi^d([V])=\\
&pr_{\Omega^*\to\mathrm{CH}^*}\circ pr_*\circ(\pi_W\times id)_*\circ(i_{\widetilde{\square},f})_*(i_{\widetilde{\square},f})^*\circ\tau_{\widetilde{\square}}^*([\widetilde{\square}(V)])=\\
&pr_{\Omega^*\to\mathrm{CH}^*}\circ pr_*\circ(i_{\square,f})_*\circ(\pi_{W/X})_*(\pi_{W/X})^*\circ(i_{\square,f})^*\circ\tau_{\square}^*([\square(V)])-\\
&pr_{\Omega^*\to\mathrm{CH}^*}\circ pr_*\circ(\pi_W\times id)_*\circ(j_W\times id)_*\circ(i_{\mathbb{P}(T),f})_*(i_{\mathbb{P}(T),f})^*\circ\\
&\tau_{\mathbb{P}(T)}^*(\sum_{i=0}^{a-1}c_1(\mathcal{O}(1))^{a-i-1}\varepsilon_W^*(c_i(T_W-T_V)[V])).
\end{aligned}$$

Since $pr_{\Omega^*\to\mathrm{CH}^*}((\pi_{W/X})_*([\widetilde{\square}(W/X)]))=pr_{\Omega^*\to\mathrm{CH}^*}([\square(W/X)])$, we get:

$$\begin{aligned}
&pr_{\Omega^*\to\mathrm{CH}^*}\circ pr_*\circ(i_{\square,f})_*\circ(\pi_{W/X})_*(\pi_{W/X})^*\circ(i_{\square,f})^*\circ\tau_{\square}^*([\square(V)])=\\
&pr_{\Omega^*\to\mathrm{CH}^*}\circ pr_*\circ(i_{\square,f})_*(i_{\square,f})^*\circ\tau_{\square}^*([\square(V)])=(pr_{\Omega^*\to\mathrm{CH}^*}([V]))^2.
\end{aligned}$$

On the other hand,

$$pr_{\Omega^*\to \mathrm{CH}^*}((\varepsilon_{W/X})_*(c_1(\mathcal{O}(1))^{a-i-1})) = pr_{\Omega^*\to \mathrm{CH}^*}(c_{a-i-b}(f^*T_X - T_W)[W]),$$

where $b = \dim(W) - \dim(X)$. So,

$$\begin{aligned} pr_{\Omega^*\to \mathrm{CH}^*} \circ pr_* \circ (\pi_W \times id)_* \circ (j_W \times id)_* \circ (i_{\mathbb{P}(T),f})_*(i_{\mathbb{P}(T),f})^* \circ \\ \tau^*_{\mathbb{P}(T)}(\sum_{i=0}^{a-1} c_1(\mathcal{O}(1))^{a-i-1}\varepsilon^*_W(c_i(T_W - T_V)[V])) = \\ pr_{\Omega^*\to \mathrm{CH}^*} \circ f_* \circ (\sum_{i=0}^{a-1} c_i(T_W - T_V)c_{a-i-b}(f^*T_X - T_W)[V]) = \\ pr_{\Omega^*\to \mathrm{CH}^*} \circ v_*(c_d(v^*T_X - T_V)[V]), \end{aligned}$$

where $v = f \circ g : V \to X$ is the morphism defining $[V]$. Notice that $a - b = d$ and $b > 0$.

Thus, $pr_{\Omega^*\to \mathrm{CH}^*} \circ \Psi^d([V]) = pr_{\Omega^*\to \underline{\mathrm{CH}}^*}(([V]^2) - v_*(c_d(v^*T_X - T_V)[V]))$.

In the same way, since $c_1(\mathcal{O}(-1)) \cdot [\widetilde{\square}(V)] = (j_V)_*([\mathbb{P}(T_V)])$, for arbitrary $r > 0$, we get:

$$\begin{aligned} pr_{\Omega^*\to \mathrm{CH}^*} \circ \Psi^{d+r}([V]) = \\ pr_{\Omega^*\to \mathrm{CH}^*} \circ pr_* \circ (\pi_W \times id)_* \circ (i_{\widetilde{\square},f})_*(i_{\widetilde{\square},f})^* \circ \tau^*_{\widetilde{\square}}([\widetilde{\square}(V)] \cdot c_1(\mathcal{O}(1))^r) = \\ -pr_{\Omega^*\to \mathrm{CH}^*} \circ (\varepsilon_{W/X})_*(i_{\mathbb{P}(T),f})^* \circ \tau^*_{\mathbb{P}(T)}([\mathbb{P}(T_V)] \cdot c_1(\mathcal{O}(1))^{r-1}) = \\ -pr_{\Omega^*\to \mathrm{CH}^*} \circ f_* \circ (\varepsilon_{W/X})_*(\sum_{i=0}^{a} c_1(\mathcal{O}(1))^{a-i+r-1} \cdot \varepsilon^*_{W/X}(c_i(T_W - T_V)[V])) = \\ pr_{\Omega^*\to \mathrm{CH}^*}(-v_*(c_{d+r}(v^*T_X - T_V)[V])). \end{aligned}$$

On Ω^* we have an action of the Landweber-Novikov operations. Such operations of degree s are parametrized by the partitions of s . In particular, for the partition $(1, 1, \ldots, 1)$ we will denote the corresponding operation by $S^s_{L-N,X} : \Omega^d(X) \to \Omega^{d+s}(X)$. It is defined by the rule: $[v : V \to X] \mapsto v_*(c_s(v^*T_X - T_V)[V])$.

Our computations show:

Proposition 3.9 *For arbitrary* $d \in \mathbf{Z}$ *, and* $r \in \mathbb{N}$ *we have:*

(1) $pr_{\Omega^*\to \mathrm{CH}^*}(\Psi^{d+r}) = pr_{\Omega^*\to \mathrm{CH}^*}(-S^{d+r}_{L-N}) : \Omega^d \to \mathrm{CH}^{2d+r}$ *;*

(2) $pr_{\Omega^*\to \mathrm{CH}^*}(\Psi^d) = pr_{\Omega^*\to \mathrm{CH}^*}(\square - S^d_{L-N}) : \Omega^d \to \mathrm{CH}^{2d}$ *.*

In particular, $pr_{\Omega^*\to \mathrm{CH}^*}(\Psi^{d+r})$ is well-defined on Ω^d .

Notice also, that for $d + r < 0$, $pr_{\Omega^*\to \mathrm{CH}^*} \circ \Psi^{d+r} = 0$.

Remark: It would be interesting to understand the relations between the whole operation Ψ^{d+r} and S^{d+r}_{L-N} .

Using Proposition 3.8, we get:

Theorem 3.10 *Let* $d \in \mathbf{Z}$ *, and* $r \in \mathbb{N}$ *. Then*

(1) $pr_{\Omega^*\to \mathrm{CH}^*} \circ S^{d+r}_{L-N}|_{\Omega^d}$ *is divisible by* 2 *;*

(2) $\quad pr_{\Omega^* \to \mathrm{CH}^*} \circ (\square - S^d_{L-N})|_{\Omega^d}$ *is divisible by* 2 .

This generalizes the following result (observed by M.Rost).

Theorem 3.11 [7, Theorem 6.1]) *Let* Y *be a smooth projective variety of positive dimension* n *over the field* k *. Then* $degree(c_n(-T_Y))$ *is divisible by* 2 .

Indeed, take $X = \mathrm{Spec}(k)$, $d = -n$, $r = 2n$. Then $degree(c_n(-T_Y)) = pr_{\Omega^* \to \mathrm{CH}^*} \circ S^n_{L-N,\mathrm{Spec}(k)}([Y])$ and so, it is divisible by 2 .

Remark: It should be noticed, that an alternative proof of Theorem 3.10 can be obtained with the help of the just mentioned result of M.Rost, the description of the $\mathrm{Ker}(pr_{\Omega^* \to \mathrm{CH}^*})$ by M.Levine and F.Morel, and the results of P.Brosnan on Steenrod operations.

Finally, Theorem 3.10 reflects the fact that the composition $pr_{\Omega^* \to \mathrm{CH}^* /2} \circ S^s_{L-N}$ should be equal to the composition $S^s \circ pr_{\Omega^* \to \mathrm{CH}^* /2}$. In this light, the value of the current section is reduced to the explicit construction of the halves of the Chow-traces of the Landweber-Novikov operations.

4 Operations in Chow groups

In this section the base field k is assumed to be of characteristic 0 .

Using the operations in algebraic cobordisms constructed in the previous section we can in certain cases produce the operations from the Chow groups of one variety to the Chow groups of the other which can not be reduced to the push-forward, pull-back and Steenrod operations.

Let us denote:

$$\begin{aligned} \varphi^{d+r} &:= pr_{\Omega^* \to \mathrm{CH}^*} \circ \Phi^{d+r} : \Omega^d \to \mathrm{CH}^{2d+r} /(2 - tors.); \\ \psi^{d+r} &:= pr_{\Omega^* \to \mathrm{CH}^*} \circ \Psi^{d+r} : \Omega^d \to \mathrm{CH}^{2d+r}, \end{aligned}$$

and then define:

$$\begin{aligned} \varphi_s &:= \sum_{t=d}^{s} c_{s-t}(-T_X) \cdot \varphi^t : \Omega^d(X) \to \mathrm{CH}^{d+s}(X)/(2 - tors.); \\ \psi_s &:= \sum_{t=d}^{s} c_{s-t}(-T_X) \cdot \psi^t : \Omega^d(X) \to \mathrm{CH}^{d+s}(X). \end{aligned}$$

Theorem 4.1 *Let* $X \stackrel{f}{\leftarrow} Y \stackrel{g}{\rightarrow} Z$ *be the maps of smooth projective varieties such that* g *is smooth and:*

(1) $\quad c_\bullet(T_g) \ (\,mod\, 2\,) \in im(g^*)$ *;*

(2) $\quad c_i(-T_X) \equiv 0 \ (\,mod\, 2\,)$ *, for all* $r < i \leqslant r + d$.

Then the composition $\varphi_{r+d} \circ f_* \circ g^*$ *gives a well-defined map*

$$\mathrm{CH}^{d+dim(f)}(Z) \to \mathrm{CH}^{2d+r}(X)/\left(2 + \mathrm{im}(f_* \circ g^*) + 2\text{-}tors.\right).$$

Proof: We need to show that $\varphi_{r+d}\circ f_*\circ g^*(\mathrm{Ker}(pr_{\Omega^*\to\mathrm{CH}^*}))$ belongs to the ideal generated by 2, $\mathrm{im}(f_*\circ g^*)$ and 2-torsion.

By the result of M.Levine and F.Morel (see [6, Theorem 3.1]), the kernel of the map $pr_{\Omega^*\to\mathrm{CH}^*}$ is generated by the elements of the form $V\times_{\mathrm{Spec}(k)}U$, where $v:V\to Z$, $u:U\to\mathrm{Spec}(k)$ and $\dim(U)>0$.

With the help of Propositions 3.8 and 3.9 one gets easily the following result.

Lemma 4.2 *Let* $[V]\in\Omega^a(X)$, $[U]\in\Omega^{-b}(\mathrm{Spec}(k))$, *where* $b>0$ *and* $r\geqslant 0$. *Then*

$$\varphi^{a-b+r}([V]\cdot[U])=\begin{cases}0, & \text{if } a-b+r<b;\\ \eta_2(U)\cdot pr_{\Omega^*\to\mathrm{CH}^*}(S^{a+r-2b}_{L-N,X}([V]), & \text{otherwise}\end{cases}\qquad (mod\, 2\text{-}tors.),$$

where $\eta_2(U):=degree(c_b(-T_U))/2\in\mathbf{Z}$ *is the invariant of* U *defined by M.Rost (see [7]).*

Let $[V']=g^*[V]$, then it follows from Lemma 4.2 that $\varphi_{r+d}\circ f_*\circ g^*([U]\cdot[V])$ is either 0 (if $r+d<\dim(U)$), or otherwise is equal $(mod\ 2\ \text{-}\ tors.)$ to

$$\eta_2(U)\cdot\sum_{i=0}^{r}c_i(-T_X)\cdot f_*(c_{r+d-\dim(U)-i}(-T_{V'}+f^*T_X)[V']).$$

Since $c_i(-T_X)\equiv 0\ (mod\,2)$ for arbitrary $r<i\leqslant r+d$, the later expression $(mod\,2)$ is equal to

$$\eta_2(U)\cdot\sum_{i=0}^{r+d}c_i(-T_X)\cdot f_*(c_{r+d-\dim(U)-i}(-T_{V'}+f^*T_X)[V'])=$$
$$\eta_2(U)\cdot f_*(c_{r+d-\dim(U)}(-T_{V'})[V']).$$

On V' there is an exact sequence of sheaves: $0\to T_g|_{V'}\to T_{V'}\to g^*(T_V)\to 0$. Thus, $c_\bullet(T_{V'})=c_\bullet(T_g)\cdot g^*c_\bullet(T_V)$. So, by the condition (1), $c_\bullet(-T_{V'})\in im(g^*)$, and $\varphi_{r+d}\circ f_*\circ g^*([U]\cdot[V])$ modulo 2 and 2-tors. is in the image of $f_*\circ g^*$. □

Proposition 4.3 *Under the conditions of Theorem 4.1, for arbitrary* $\beta\in\mathrm{CH}^{d+\dim(f)}(Z)$, *the class* $c_r(-T_X)\cdot(f_*\circ g^*(\beta))^2$ *is divisible by 2 modulo the image of* $(f_*\circ g^*)$ *in the group* $\mathrm{CH}^{2d+r}(X)$.

Proof: By Proposition 3.9, for $v:V\to X$ from $\Omega^d(X)$,

$$\psi_{r+d}([V])=c_r(-T_X)\cdot(pr_{\Omega^*\to\mathrm{CH}^*}([V]))^2-$$
$$pr_{\Omega^*\to\mathrm{CH}^*}(\sum_{i=d}^{d+r}c_{d+r-i}(-T_X)\cdot v_*(c_i(-T_V+v^*T_X)[V]))$$

And, by the condition (2) of Theorem 4.1, $(mod\,2)$,

$$\sum_{i=d}^{d+r}c_{d+r-i}(-T_X)\cdot v_*(c_i(-T_V+v^*T_X)[V])\equiv$$
$$\sum_{i=0}^{d+r}c_{d+r-i}(-T_X)\cdot v_*(c_i(-T_V+v^*T_X)[V])=v_*(c_{d+r}(-T_V)[V]).$$

Let now $[V]$ be equal to $f_*g^*([V'])$ for some $[V'] \in \Omega^{d+\dim(f)}(Z)$. Then, $(\,mod\,2\,)$,

$$\psi_{r+d}([V]) = \; c_r(-T_X)\cdot(pr_{\Omega^*\to \mathrm{CH}^*}\circ f_*\circ g^*[V'])^2 - pr_{\Omega^*\to \mathrm{CH}^*}\circ v_*(c_{d+r}(g^*(-T_{V'})-T_g)g^*[V']).$$

But, by the condition (1) of Theorem 4.1, $(\,mod\,2\,)$, $v_*(c_{d+r}(g^*(-T_{V'})-T_g)g^*[V'])$ belongs to the image of $f_*\circ g^*$. Thus, modulo 2 and $im(f_*\circ g^*)$,

$$\psi_{r+d}([V]) = c_r(-T_X)\cdot(pr_{\Omega^*\to \mathrm{CH}^*}\circ f_*\circ g^*[V'])^2.$$

But $\psi_{r+d}([V])$ is divisible by 2 - the quotient is $\varphi_{r+d}([V])$. Thus, modulo $im(f_*\circ g^*)$, the class $c_r(-T_X)\cdot(pr_{\Omega^*\to \mathrm{CH}^*}\circ f_*\circ g^*[V'])^2$ is divisible by 2 . The statement now follows from the fact that the projection $\Omega^*(Z)\to \mathrm{CH}^*(Z)$ is surjective. $\square$

As a corollary we can give a new proof of the Theorem describing the possible sizes of binary direct summands in the motives of quadrics.

Theorem 4.4 ([4, Theorem 6.1]) *Let Q be an anisotropic quadric, and N be a direct summand of $M(Q)$ such that $N|_{\overline{k}} = \mathbb{Z}(a)[2a]\oplus\mathbb{Z}(b)[2b]$. Then $|b-a| = 2^s-1$, for some $s\in\mathbb{N}\cup\{0\}$.*

Proof: Using the standard arguments - see [8, Corollaries 3.9, 4.14], it is easy to show that there exists a field extension E/k and some anisotropic quadric P over E such that the Tate-twist $N(-a)[-2a]|_E$ is a direct summand of $M(P)$ with $b-a=\dim(P)$. Thus, the problem can be reduced to the case where $a=0$ and $b=\dim(Q)=:n$.

The fact that in $M(Q)$ there is a direct summand N with $N|_{\overline{k}} = \mathbb{Z}\oplus\mathbb{Z}(n)[2n]$ means that the class of the cycle $[x\times Q]+[Q\times x]\in \mathrm{CH}^n(Q\times Q|_{\overline{k}})$ is defined over the base field k (x here is some rational point on $Q|_{\overline{k}}$).

Let $G(1,Q)$ be the Grassmann variety of lines on Q and $F(1,Q)$ be the variety of flags $(l_0\subset l_1)$ on Q . We have the natural (forgetful) maps

$$Q \stackrel{f}{\leftarrow} F(1,Q) \stackrel{g}{\rightarrow} G(1,Q).$$

Let $\Theta\subset F(1,Q)\times(Q\times Q)$ be the cycle $\{((y,l),(y,z))|z\in l\}$. The dimension of Θ is equal to $\dim(F(1,Q))+1$, and so it defines the map $\theta:\mathrm{CH}^n(Q\times Q)\to \mathrm{CH}^{n-1}(F(1,Q))$. Clearly, $\theta([x\times Q])=0$ and $\theta([Q\times x])$ is the class of the cycle $\{(y,l)|x\in l\}$. Thus, $\theta([Q\times x]+[x\times Q]) = g^*(\gamma)$, where $\gamma\in \mathrm{CH}^{n-1}(G(1,Q))$ is the cycle $\{l|x\in l\}$. In other words, $\gamma = g_*\circ f^*([x])$. Let $h\in \mathrm{CH}^1(Q)$ be the hyperplane section. Then $\gamma = g_*(f^*(h)\cdot g^*(\gamma))$, and hence, the class γ is defined over k .

Now, suppose that n does not have the form 2^s-1 for any s , that is, $n=2^t+m$, where $0\leqslant m\leqslant 2^t-2$. Consider the divisor $H\in \mathrm{CH}^1(G(1,Q))$ given by the cycle of lines which intersect given plane section of codimension 2 on Q . Put: $\beta := H^m\cdot\gamma\in \mathrm{CH}^{n-1+m}(G(1,Q))$.

Let $d=m+1$ and $r=2^t-m-2$. $c_\bullet(-T_Q) = (1+h)^{-(n+2)}\cdot(1+2h)\equiv(1+h)^{-(n+2)}\,(\,mod\,2\,)$. Thus, $c_i(-T_Q)\equiv\binom{-(n+2)}{i}\cdot h^i$. Suppose that $r<i\leqslant r+d$, then $\binom{-(n+2)}{i}\equiv\binom{n+1+i}{i}\equiv\binom{2^t+m+1+2^t-m-1+(i-r-1)}{i}\equiv\binom{2^{t+1}+(i-r-1)}{i}\equiv\binom{i-r-1}{i}\equiv 0\,(\,mod\,2\,)$. Thus, the condition (2) of the Theorem 4.1 is satisfied. On the other hand, $c_\bullet(T_g)\equiv 1+g^*(H)\,(\,mod\,2\,)$, so the condition (1) is satisfied as well. By Proposition 4.3, $c_r(-T_Q)\cdot(f_*g^*(\beta))^2$ is divisible by 2 in $\mathrm{CH}_0(Q)$ (since the image of $(f_*\circ g^*)$ in $\mathrm{CH}_0(Q)$ is trivial). But $c_r(-T_Q)\equiv\binom{2^{t+1}-1}{r}\cdot h^r\equiv h^r\,(\,mod\,2\,)$, and $f_*g^*(\beta)=h^{m+1}$. So, the 0 -cycle $h^{r+2(m+1)}=h^{\dim(Q)}$ is divisible by 2 in $\mathrm{CH}_0(Q)$. Thus, we get a 0 -cycle of degree 1 on Q , and by Springer's Theorem, Q is isotropic - a contradiction. So, the dimension of Q should have the form 2^s-1 . $\square$

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