Algebraic cobordisms of a Pfister quadric *

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1 Introduction

The aim of this paper is to compute algebraic cobordisms of a Pfister quadric Q_α. We present two different approaches to this problem. The first one uses the motivic spectrum MGL (constructed by V.Voevodsky in [27]), and is reduced to the computation of $MGL^{2*,*}(Q_\alpha)$, and the second one uses the definition of algebraic cobordism $\Omega^*(Q_\alpha)$ due to M.Levine and F.Morel (see [7],[8],[9]). Consequently, the methods of computations are quite different. The more pleasant is the fact that the results do agree, especially, since, in general, it is not proven that $MGL^{2i,i}(X) = \Omega^i(X)$ (of course, it hardly could be any other way). We hope, that our two independent computations will enable the reader to see the problem from different angles. The case of a Pfister quadric is one of the few noncellular varieties for which algebraic cobordism ring is completely computed (the others are $B\mathbf{Z}/p$ and certain reductive groups - see [31]).

Certainly, the fact that Q_α is *geometrically cellular* helped here a lot. But another ingredient which permitted such computation to happen is the fact that the change of constants map to the algebraic closure appears to be injective in our case. In general (even for geometrically cellular varieties), it is not so. Nevertheless, one could expect that the same could happen for certain classes of varieties, say, for *generically cellular* ones (that is, ones which become cellular over own generic point). The computation of algebraic cobordisms for such varieties would be the next natural task.

*Mathematics Subject Classification (2000): 11E04, 14C15, 14F42, 19E15, 55N20, 55N22

†supported by the Weyl Fund and by RFBR grant 22005

We would like to thank F.Morel for suggesting that the statement like Corollary 2.8 should be true, and A.Nenashev and K.Zainoulline for many questions and remarks which substantially improved the exposition in Section 2. Also, we are very grateful to B.Kahn, I.Panin and B.Totaro for very useful discussions. This text was written while the first author was a member of the Institute for Advanced Study at Princeton and visitor at the Institut des Hautes Études Scientifiques, and he would like to express his gratitude to both institutions for the support and excellent working conditions. The support of the Weyl Fund is deeply appreciated. We would like to thank the Referee for the important corrections and suggestions.

2 Cobordism motives

Everywhere in this text we assume that the base field k has characteristic 0. Let $A^*(\mathrm{X})$ be any oriented (generalised) cohomology theory on the category of smooth algebraic varieties over the field k (see [8]).

An oriented cohomology theory is a contravariant functor from the category of smooth varieties (over k) to the category of graded rings which is equipped with the structure of push-forward maps for projective equidimensional morphisms (with appropriate shift in the grading) satisfying the axioms (A1)-(A4) in [8]. The algebraic cobordim theory $\Omega^*(X)$ (see [8, 9, 7]) is defined as the universal theory among the oriented cohomology theories.

Following [11] we define the category of A-motives denoted $\mathcal{M}_A$ as follows. We start with the category of A-correspondences (of degree 0) Cor_A^0 whose objects are the classes $[X]$ of smooth projective varieties over k, and $Mor_{Cor_A^0}([X],[Y]) = A^{dim(Y)}(X\times Y)$. The composition of morphisms is defined in a usual way: $\varphi\circ\psi := p_{X\times Z*}(p^*_{X\times Y}(\varphi)\cdot p^*_{Y\times Z}(\psi))$, where $p_{X\times Y}, p_{Y\times Z}$ and $p_{X\times Z}$ are projection from $X\times Y\times Z$ onto $X\times Y$, $Y\times Z$ and $X\times Z$, respectively. Then, $\mathcal{M}_A$ is a Karoubian envelope of Cor_A^0. It is naturally a tensor additive category. Typical object of $\mathcal{M}_A$ is the pair $([X],p)$, where X is smooth projective variety over k and $p\in End_{Cor_A^0}([X])$ is some projector ($p\circ p = p$). Hence, for each $M\in\mathcal{M}_A$, we can define the $A^*(Spec(k))$-module $A^*(M)$

In the case of a Chow groups, $\mathcal{M}_{CH}$ is nothing else, but the well-known category of effective Chow-motives $Chow^{eff}(k)$. Consider now the case when $A^*(X)$ is the algebraic cobordism theory $\Omega^*(X)$ constructed by M.Levine and F.Morel (see [8, 9, 7]), or $MGL^{2*,*}(X)$ constructed by V.Voevodsky. In both cases we have natural morphism $fog_A : A^*(X) \to CH^*(X)$ of generalized cohomology theories. It induces functor $\mathcal{M}_A \stackrel{fog}{\to} \mathcal{M}_{CH}$.

The results about $\mathcal{M}_\Omega$ or $\mathcal{M}_{MGL}$ we need will follow from some general statements about additive categories. So, we pass to those.

We say that we are in the situation $(\star)$, if we have an additive functor $F : \mathcal{C} \to \mathcal{D}$ between the additive categories such that the following three conditions are satisfied:

0) F is surjective on the isomorphism classes of objects;

1) F is surjective on morphisms;

2) For any $X \in Ob(\mathcal{C})$, the kernel of the ring homomorphism $F_X : End_{\mathcal{C}}(X) \to End_{\mathcal{D}}(F(X))$ consists of nilpotents.

Lemma 2.1 *Let us be in the situation $(\star)$. Suppose $\alpha \in Hom_{\mathcal{C}}(X, Y)$ is a morphism such that $F(\alpha)$ is an isomorphism. Then so is α.*

Proof: Since F is surjective on morphisms, there exists $\beta \in Hom_{\mathcal{C}}(Y, X)$ such that $F(\beta) \circ F(\alpha) = 1_{F(X)}$ and $F(\alpha) \circ F(\beta) = 1_{F(Y)}$. Then $\beta \circ \alpha = 1_X + x$ and $\alpha \circ \beta = 1_Y + y$, where x and y belong to the kernel of F_X and F_Y, respectively, and so, are nilpotent.

Use induction on the maximum of nilpotence exponents of x and y. If such exponent is 1, we are done. Otherwise, consider $\beta' := \beta \circ (1_Y - y)$. Notice, that $\alpha \circ x = y \circ \alpha$ and $x \circ \beta = \beta \circ y$. Then $\alpha \circ \beta' = (1_Y + y) \circ (1_Y - y) = 1_Y - y^2$ and $\beta' \circ \alpha = \beta \circ \alpha - \beta \circ y \circ \alpha = 1_X + x - x \circ (1_X + x) = 1_X - x^2$. The maximum of nilpotence exponents of $-x^2$ and $-y^2$ is smaller than that of x and y (since the latter was greater than 1). By induction, we are done. $\square$

As a corollary, we immediately get:

Proposition 2.2 *Let us be in the situation $(\star)$. Then the functor F defines a 1-to-1 correspondence $Iso(\mathcal{C}) \to Iso(\mathcal{D})$ between the set of isomorphism classes of objects of $\mathcal{C}$ and $\mathcal{D}$.*

Proof: By condition 0), our map $Iso(\mathcal{C}) \to Iso(\mathcal{D})$ is surjective, and by Lemma 2.1 and condition 1), it is injective. $\square$

Proposition 2.3 *Suppose we are in the situation* $(\star)$. *Then* F *is surjective on projectors. That is, for any* $X \in Ob(\mathcal{C})$ *and any projector* $p_{F(X)} \in End_{\mathcal{D}}(F(X))$ *there exists projector* $p_X \in End_{\mathcal{C}}(X)$ *such that* $F(p_X) = p_{F(X)}$.

Proof: Since F is surjective on morphisms, there exists $p \in End_{\mathcal{C}}(X)$ such that $F(p) = p_{F(X)}$.

Consider polynomials $\phi_n(t)$, $n \geqslant 1$ defined inductively as follows: $\phi_1(t) = (2t - t^2)^2$, $\phi_{n+1}(t) = \phi_1(\phi_n(t))$. Now, it is sufficient to apply the following lemma to the homomorphism $F_X : End_{\mathcal{C}}(X) \to End_{\mathcal{D}}(F(X))$.

Lemma 2.4 *Let* $\pi : A \to B$ *be a homomorphism of associative rings, such that* $ker(\pi)$ *consists of nilpotents. Let* $a \in A$, $b \in B$ *be such that* $\pi(a) = b$ *and* b *is an idempotent. Then for sufficiently large* n, $\phi_n(a)$ *is an idempotent, and* $\pi(\phi_n(a)) = b$.

Proof: Really, for arbitrary $c \in A$ consider $x(c) := c^2 - c$. Then $x(a)$ is nilpotent (b is idempotent). But $x(\phi_1(c))$ is divisible by $x(c)^2$ (in the commutative subring of A generated by c). Thus, $x(\phi_n(a))$ is divisible by $x(a)^{2^n}$, and so, is zero, for large n. $\square$

Notice, that, after all, p_X can be expressed as a polynomial of p (with $\mathbb{Z}$-coefficients) without the constant term. $\square$

In general, such lifting p_X is not unique. Nevertheless, the different liftings give isomorphic objects of the Karoubian envelope of $\mathcal{C}$.

Consider Karoubian envelopes $\mathcal{PA}(\mathcal{C})$ and $\mathcal{PA}(\mathcal{D})$ of the categories $\mathcal{C}$ and $\mathcal{D}$. We have natural functor $\mathcal{PA}(F) : \mathcal{PA}(\mathcal{C}) \to \mathcal{PA}(\mathcal{D})$.

Proposition 2.5 *Suppose the functor* $F : \mathcal{C} \to \mathcal{D}$ *satisfies the conditions* $(\star)$. *Then so does the functor* $\mathcal{PA}(F) : \mathcal{PA}(\mathcal{C}) \to \mathcal{PA}(\mathcal{D})$.

Proof: Condition 0): Any object of $\mathcal{PA}(\mathcal{D})$ has the form (U, p_U), where $U \in Ob(\mathcal{D})$ and $p_U \in End_{\mathcal{D}}(U)$ is a projector. By condition 0), there exists $X \in Ob(\mathcal{C})$ such that U is isomorphic to $F(X)$. Then there exists projector $p_{F(X)} \in End_{\mathcal{D}}(F(X))$ such that $(F(X), p_{F(X)})$ is isomorphic to (U, p_U) in $\mathcal{PA}(\mathcal{D})$. By Proposition 2.3, there exists projector $p_X \in End_{\mathcal{C}}(X)$ such that $(F(X), p_{F(X)}) = \mathcal{PA}(F)((X, p_X))$.

Conditions 1) and 2) are satisfied since $Hom_{\mathcal{PA}(\mathcal{E})}$ is naturally a quotient object and a subobject of $Hom_{\mathcal{E}}$. $\square$

Corollary 2.6 *Let the functor $F : \mathcal{C} \to \mathcal{D}$ satisfies the conditions ($\star$). Then the functor $\mathcal{PA}(\mathcal{F})$ defines a 1-to-1 correspondence $Iso(\mathcal{PA}(\mathcal{C})) \to Iso(\mathcal{PA}(\mathcal{D}))$ between the set of isomorphism classes of objects of the categories $\mathcal{PA}(\mathcal{C})$ and $\mathcal{PA}(\mathcal{D})$.*

Proof: It follows immediately from Propositions 2.5 and 2.2. □

Let $A^*(X) = A_{dim(X)-*}(X)$ be a generalized oriented (co)homology theory, and $fog_A : A^* \to \mathrm{CH}^*$ be a morphism of oriented theories. As usually, $A_* := A_*(\mathrm{Spec}(k))$ will denote the coefficient ring. We say that the pair (A, fog_A) satisfies the conditions $(\star, \star)$, if

(0) $A_*(X)$ is zero in negative dimensions;

(1) the maps

$$(fog_A)_0(\mathrm{Spec}(k)) : A_0 \to \mathbf{Z}, \quad \text{and}$$

$$fog_A \otimes_{A_*} \mathbb{Z} : A_*(X) \otimes_{A_*} \mathbb{Z} = A_*(X)/A_{>0} \cdot A_*(X) \xrightarrow{\cong} \mathrm{CH}_*(X).$$

are isomorphisms.

When $A^*(X) = \Omega^*(X)$ the algebraic cobordism defined by Morel and Levine, the condition $(\star, \star)$ is satisfied by [2,Theorem 14.1]. For $A^*(X) = MGL^{2*,*}(X)$ the condition is satisfied by Corollary 5.3. Note that $A_*(X) = MGL^{2d-2*,d-*}(X)$ for $d = dim(X)$, but we do not assume $A_*(X) \cong MGL_{2*,*}(X)$ in this paper.

The morphism of oriented cohomology theories induces the functor between respective motivic categories, which we still denote by the same name $fog_A : Cor^0_A \to Cor^0_{CH}$.

Proposition 2.7 *If the pair $(A^*(X), fog_A)$ satifies $(\star, \star)$, then the functor $fog_A : Cor^0_A \to Cor^0_{CH}$ satisfies the conditions ($\star$).*

Proof: Sets of objects of both categories are identified with the set of smooth projective varieties over k. Thus, the condition $(\star)(0)$ is satisfied.

By definition, we can identify: $Hom_{Cor^0_A}([X],[Y]) = A_{dim(X)}(X \times Y)$, $Hom_{Cor^0_{CH}}([X],[Y]) = CH_{dim(X)}(X \times Y)$, and fog_A corresponds to the natural projection $A_{dim(X)}(X \times Y) \to CH_{dim(X)}(X \times Y)$. By $(\star, \star)(1)$, such projection is surjective, and the condition $(\star)(1)$ is satisfied.

The kernel $ker(End_{Cor^0_A}([X]) \to End_{Cor^0_{CH}}([X]))$ is identified with the kernel $ker(A_{dim(X)}(X \times X) \to CH_{dim(X)}(X \times X))$, and the latter ideal coinsides with the dimension $= dim(X)$ part of $A_{\geqslant 1} \cdot A_*(X \times X)$. Notice, that the composition product on $A_*(X \times X)$ is A_*-linear. Let $N > dim(X)$. Then $ker^{\circ N}$ is contained in the dimension $= dim(X)$ part of $A_{>dim(X)} \cdot A_*(X \times X)$. But this part is zero, since $A_d(Z) = 0$, for any $d < 0$ from condition $(\star, \star)(0)$. Thus, the condition $(\star)(2)$ is satisfied as well. $\square$

Proposition 2.7 and Corollary 2.6 imply:

Corollary 2.8 *Let the pair* $(A^*(X), fog_A)$ *satisfies* $(\star, \star)$. *Then the functor* fog_A *defines the natural* 1*-to-*1 *correspondence* $Iso(\mathcal{M}_A(k)) \to Iso(\mathcal{M}_{CH}(k))$ *between the set of isomorphism classes of objects of* $\mathcal{M}_A(k)$ *and* $\mathcal{M}_{CH}(k)$. *In particular, this is so for* $A^* = \Omega^*$ *and* $MGL^{2*,*}$ *with natural forgetful maps.*

Proof: We only need to remind that $\mathcal{M}_A$ and $\mathcal{M}_{CH}$ are just $\mathcal{PA}(Cor^0_A)$ and $\mathcal{PA}(Cor^0_{CH})$, respectively. $\square$

Let us call the corresponding motive in $\mathcal{M}_A$ by the same name we used to call it's projection to $\mathcal{M}_{CH}$. Thus, in $\mathcal{M}_A$ we have Tate-motives, Rost-motives, etc. ... . Let us denote the lifting of the Tate-motive $\mathbf{Z}(n)[2n]$ as $\mathbf{A}(n)[2n]$. If U is arbitrary object of $\mathcal{M}_A(k)$, we denote $U \otimes \mathbf{A}(n)[2n]$ as $U(n)[2n]$. There is canonical up to sign identification: $A^m(U(n)[2n]) = A^{m-n}(U)$.

Corollary 2.9 *Let* $(A^*(X), fog_A)$ *be the pair satisfying* $(\star, \star)$. *Let* X *be smooth projective cellular variety. That is, there is a filtration* $X = X_0 \supset X_1 \supset \ldots \supset X_n$ *by closed subschemes such that for each* i, $X_i \backslash X_{i+1}$ *is a disjoint union* $\coprod_j \mathbb{A}^{r_j}$ *of some affine spaces. Then* $M^A(X)$ *is a direct sum of* $(\mathbf{A}$-$)$ *Tate-motives.*

Proof: It is well-known that $M^{CH}(X)$ is a direct sum of (Chow) Tate-motives. Indeed, by the result of V.Voevodsky, $\mathcal{M}_{CH}(k)$ is a full additive subcategory of the triangulated category $DM_{gm}(k)$ (see [26]). There is a "motive with compact support" functor $M_c : Schm(k) \to DM_{gm}(k)$ from the category of schemes of finite type over k (with proper morphisms) to the category of geometric motives such that for smooth projective variety Y,

$M_c(Y) = M^{CH}(Y)$ and $M_c(\mathbb{A}^m) = \mathbf{Z}(m)[2m]$. According to [26], we have exact triangles in $DM_{gm}(k)$:

$$M_c(X_{i+1}) \to M_c(X_i) \to M_c(X_i \backslash X_{i+1}) \to M_c(X_{i+1})[1].$$

Now using the fact that there is no hom's from $\mathbf{Z}(s)[2s]$ to $\mathbf{Z}(t)[2t+1]$ one easily proves by induction on i that $M_c(X_i)$ is a direct sum of Tate-motives. Then, so is $M^{CH}(X) = M_c(X_0)$. Corollary 2.8 implies now that the same is true for $M^A(X)$. $\square$

3 Pfister quadric and algebraic cobordisms

Let $\alpha \in K_n^M(k)/2$ be a pure symbol, and Q_α be the corresponding Pfister quadric of dimension $2^n - 2$.

In the next two sections we will compute the ring of algebraic cobordisms $\Omega^*(Q_a)$ of Q_α. The main (and, basically, the only) tool we are using, is the fact that the motive of this variety can be decomposed into rather simple parts. Actually, one can notice, that we could as easily start just from a quadric of height one (the one which is split over own generic point), and use the results from [22].

We recall that the *Pfister neighbor* of Q_α is any subquadric of Q_α of dimension $\geqslant 2^{n-1} - 1 = \dim(Q_\alpha)/2$. The neighbour of dimension $2^{n-1} - 1$ is called the *minimal* one.

It appears that the motive of a Pfister neighbor can be decomposed as follows:

Theorem 3.1 *(Rost [17], Hoffmann [4]) Let ξ be a subform of the Pfister form q_α of $dim(\xi) = 2^{n-1} + s, s > 0$ (i.e., ξ is a neighbour of q_α). Let η be the complementary form ($q_\alpha = \xi \perp \eta$). Then*

$$M^{CH}(X_\xi) = M_\alpha \otimes M^{CH}(\mathbb{P}^{s-1}) \oplus M^{CH}(X_\eta)(s)[2s],$$

where M_α is, so-called, Rost motive motive such that

$$M_\alpha|_{\overline{k}} = \mathbf{Z} \oplus \mathbf{Z}(2^{n-1} - 1)[2^n - 2].$$

Proof: Rost proved in [17, Proposition 2.4] that the result holds if there is a form η over k such that $\eta|_{k(q_\alpha)} = (\xi|_{k(q_\alpha)})_{aniso.}$. One could take η to be a complementary form since $q_\alpha|_{k(q_\alpha)}$ is hyperbolic. Here we see $\eta|_{k(q_\alpha)}$ is anisotropic by the following result of Hoffmann (Theorems 1 in [4]): if q, q' are anisotropic forms over k such that $dim(q) > 2^{n-1} \geqslant dim(q')$, then $q'|_{k(q)}$ is anisotropic. $\square$

In particular,

$$M^{CH}(Q_\alpha) \cong M_\alpha \otimes M^{CH}(\mathbb{P}^{2^{n-1}-1}). \tag{1}$$

The Rost motive have natural maps

$$\phi : M_\alpha \to \mathbf{Z} \quad \text{and} \quad \psi : \mathbf{Z}(2^{n-1}-1)[2^n-2] \to M_\alpha,$$

which over the algebraic closure give the decompostion of M_α into the direct sum of two Tate-motives. If we consider M_α as a direct summand in $M^{CH}(P_\alpha)$ for a minimal Pfister neighbor P_α, then $\phi = \pi \circ i_{M_\alpha}$, $\psi = p_{M_\alpha} \circ \pi^\vee$, where i_{M_α} and p_{M_α} are the maps defining M_α as a direct summand of $M^{CH}(P_\alpha)$, π is induced by the projection $P_\alpha \to \mathrm{Spec}(k)$, and $\vee$ denotes the dual morphism.

In Voevodsky triangulated category of motives $DM^{eff}(k)$, M_α can be presented as an extension

$$\mathrm{Cone}[-1](M(\check{C}(Q_\alpha)) \to M(\check{C}(Q_\alpha))(2^{n-1}-1)[2^n-1]), \tag{2}$$

where $M(\check{C}(Q_\alpha))$ is a motive of a smooth simplicial scheme associated to the morphism $Q_\alpha \to \mathrm{Spec}(k)$ - it is an object of $DM^{eff}(k)$ which becomes isomorphic to a trivial Tate motive as soon as Q_α acquires a point (that is, the symbol α splits) - see [27]. The respective maps $M_\alpha \to M(\check{C}(Q_\alpha))$ and $M(\check{C}(Q_\alpha))(2^{n-1}-1)[2^n-2] \to M_\alpha$ are obtained from ϕ and ψ above.

By Corollary 2.8, $M^\Omega(Q_\alpha) \cong M^\Omega_\alpha \otimes M^\Omega(\mathbb{P}^{2^{n-1}-1})$. We still call M^Ω_α the Rost-motive, although, now it is an object of $\mathcal{M}_\Omega(k)$. Let $\mathbb{L} = \Omega_*(Spec(k))$ be the Lazard ring (see [8]). Denote as $\mathbf{L}(n)[2n]$ the lift to $\mathcal{M}_\Omega(k)$ of the Tate motive $\mathbf{Z}(n)[2n]$. Clearly,

$$M^\Omega_\alpha|_{\overline{k}} = \mathbf{L} \oplus \mathbf{L}(2^{n-1}-1)[2^n-2].$$

Then $\Omega^*(M^\Omega_\alpha(i)[2i]|_{\overline{k}})$ considered as a direct summand of $\Omega^*(Q_\alpha|_{\overline{k}})$ can be identified with $e^{2^{n-1}-1-i} \cdot \mathbb{L} \oplus e_i \cdot \mathbb{L}$, where $\{e^i, e_i\}_{i=0,\dots,2^{n-1}-1}$ form an $\mathbb{L}$-basis of $\Omega_*(Q_\alpha|_{\overline{k}})$ with $fog(e_i) = l_i$, $fog(e^i) = h^i$ (here l_i and h^i are the classes of a projective subspace of dimension j, and a plane section of codimension j, respectively).

Let $h^j \in \Omega^j(Q_\alpha|_{\overline{k}})$ be the plane section of codimension j, and ρ_j be the projector defining $M_\alpha^\Omega(2^{n-1}-1-j)[2^n-2-2j]$ inside $M^\Omega(Q_\alpha)$. Since $\rho(h^j) = e^j + \lambda \cdot e_{2^{n-1}-1-j}$, for certain $\lambda \in \mathbb{L}$, we can choose e^j equal to $\rho_j(h^j)$, and it will be defined over the base field.

Consider the base-extention map $ac^\Omega : \Omega^*(U) \to \Omega^*(U|_{\overline{k}})$. Let us denote it's image as $\overline{\Omega}^*(U)$.

Statement 3.2 *$\overline{\Omega}^*(Q_\alpha) = (\oplus_{i=0}^{2^{n-1}-1}\mathbb{L}e^i) \oplus (\oplus_{i=0}^{2^{n-1}-1}Je_i)$, where $J \subset \mathbb{L}$ is some ideal.*

Proof: Really, since e^i are defined over k, $\overline{\Omega}^*(M_\alpha^\Omega) = \mathbb{L}e^{2^{n-1}-1} \oplus Je_0$, for some ideal $J \subset \mathbb{L}$. Then $\overline{\Omega}^*(M_\alpha^\Omega(i)[2i]) = \mathbb{L}e^{2^{n-1}-1-i} \oplus Je_i$ and the statement follows. $\square$

We need now to compute J.

Let us remind some facts about the Lazard ring $\mathbb{L}$ ([15],[10]). Let X be smooth projective variety over k. Its characteristic numbers of degree d are parametrized by partitions of d. That is, by sequences $\psi = (\psi_1, ..., \psi_r)$, where $\psi_1 \geqslant \psi_2 \geqslant \ldots \geqslant \psi_r$ and $\psi_1 + \ldots + \psi_r = |\psi| = d$. To each ψ one assigns smallest symmmetric polynomial $\sum_{(i_1,...,i_r)} x_{i_1}^{\psi_1}...x_{i_r}^{\psi_r} = R_\psi(\sigma_1, ..., \sigma_d)$ containing the monomial $x_1^{\psi_1}....x_r^{\psi_r}$. Set

$$c_\psi(-T_X) = R_\psi(c_1, ..., c_d) \in CH^d(X)$$

where c_i is the i-th Chern class of the (virtual) normal bundle $-T_X$ on X.

The partitions (of all degrees) parametrize a $\mathbf{Z}$-basis of the polynomial ring $\mathbf{Z}[b_1, b_2, ...] = \mathbf{Z}[\mathbf{b}]$ by the rule $b_\psi := b_{\psi_1}...b_{\psi_r}$. If we assign degree s to b_s, then the degree b_ψ will be $|\psi|$.

There is an embedding of rings $\mathbb{L} \overset{j}{\hookrightarrow} \mathbf{Z}[\mathbf{b}]$ defined by

$$X \mapsto \sum_{|\psi|=dim(X)} deg(c_\psi(-T_X) \cap [X])b_\psi$$

where $[X] \in \mathrm{CH}^0(X)$ is the fundamental class of our manifold ([1, 15]).

Denote as $I(p)$ the preimage under this embedding of the prime ideal $p\mathbf{Z}[\mathbf{b}]$. In other words, it is an ideal consisting of classes of varieties, all characteristic numbers of which are divisible by p. It is clearly a prime ideal. Denote as $I(p,n)$ the subideal of $I(p)$ generated by the elements of dimension $\leqslant p^n-1$. On $\mathbb{L}$ we have the action of Landweber-Novikov operations. By the theorem of Landweber [6], the only prime ideals stable under such operations are $I(p,n)$, for all prime p and all $0 \leqslant n \leqslant \infty$ (here $I(p,\infty) = I(p)$).

There is an isomorphism $\mathbb{L} \cong \mathbf{Z}[x_1, x_2, ...]$, where x_i can be chosen in such a way, that

$$(*) \quad j(x_i) = \begin{cases} p(b_i + \textit{decomposable terms}) & , \; if \;\; i = p^t - 1 \\ b_i + \textit{decomposable terms} & , \;\; otherwise \end{cases}$$

(where p are prime numbers). Thus, the ideal $I(p,n)$ is generated by x_{p^i-1}, $0 \leqslant i \leqslant n$:

$$I(p,n) = (p, x_{p-1}, ..., x_{p^n-1})$$

(see Landweber [6]). In Section 5.1 bellow we will use notation v_i for x_{p^i-1} (when prime p is fixed).

Let X be smooth projective variety over k. Following V.Voevodsky, M.Rost, A.Merkurjev, M.Levine and F.Morel, we define the ideal $I(X) \subset \mathbb{L}$ as the image of $\pi_* : \Omega_*(X) \to \Omega_*(\mathrm{Spec}(k)) = \mathbb{L}$.

If X has rational point, then, clearly, $I(X) = \mathbb{L}$.

From $(*)$ one can also obtain:

Statement 3.3 *The ideal $I(2,r) \subset \mathbb{L}$ is generated by $[Q_{2^i-1}]$, $0 \leqslant i \leqslant r$, where Q_s is quadric of dimension s.*

Now we can compute $I(Q)$ for anisotropic quadric Q.

Theorem 3.4 *Let Q be an anisotropic quadric such that $2^m - 1 \leqslant \dim(Q) < 2^{m+1} - 1$. Then $I(Q) = I(2,m)$.*

Proof: Since Q is anisotropic, $I(Q) \subset I(2)$. Really, if there is a map $X \to Q$, and some characteristic number of X is odd, then on X there is a zero cycle of odd degree. Then such cycle exists on Q, which contradicts anisotropity, by the Theorem of Springer.

On the other hand, $[Q_s] \in I(Q)$, for all $0 \leqslant s \leqslant dim(Q)$ (plane sections), and thus, $I(2,m) \subset I(Q)$. But, by the result of M.Levine and F.Morel [7, Theorem 4.1] ([9, Theorem 3.1]), $\Omega^*(Q)$ as $\mathbb{L}$-module is generated by the elements of dimension $\leqslant \dim(Q)$ and, at the same time, by the Statement

3.3, $I(2)_{<2^{m+1}-1} = I(2,m)_{<2^{m+1}-1}$. Since $I(2,m) \subset I(Q) \subset I(2)$, it follows that $I(Q) = I(2,m)$. $\square$

Let us study the ideal J. Together with the quadric Q_α we can consider quadric P_α given by the smallest neighbour p_α of the Pfister form q_α. By Theorem 3.1, M_α is a direct summand of $M^{CH}(P_\alpha)$. By Corollary 2.8, M_α^Ω is a direct summand of $M^\Omega(P_\alpha)$.

By the Statement 3.2, the ideal $I(2,n-1) = I(Q_\alpha)$ is generated by J and the images of e^i, $0 \leqslant i \leqslant 2^{n-1}-1$ (notice that $\Omega_*(\mathrm{Spec}(k)) = \Omega_*(\mathrm{Spec}(\overline{k}))$). Since the dimensions of e^i are greater or equal $2^{n-1}-1$, $J \supset I(2,n-2)$.

On the other hand, by [7, Theorem 4.1]([9, Theorem 3.1]), $\Omega^*(X)/\mathbb{L}_{>0} \cdot \Omega^*(X) \cong CH^*(X)$, and thus, there is a surjection

$$CH^*(X) \twoheadrightarrow \overline{\Omega}^*(X)/\mathbb{L}_{>0} \cdot \overline{\Omega}^*(X).$$

In particular, for $j \geqslant 2^{n-1}-1$, we get a surjection

$$CH_j(M_\alpha) \twoheadrightarrow (\mathbb{L}/\mathbb{L}_{>0} \cdot \mathbb{L})_{j-2^{n-1}+1} \cdot e^{2^{n-1}-1} \oplus (J/\mathbb{L}_{>0} \cdot J)_j \cdot e_0.$$

Since $CH_j(M_\alpha)$ is a direct summand of $CH^{2^{n-1}-1-j}(P_\alpha)$, and the latter group is either cyclic, or trivial, we get: $(J/\mathbb{L}_{>0} \cdot J)_j = 0$, for any $j \geqslant 2^{n-1}-1$. Hence, $J \subset I(2,n-2)$, which implies $J = I(2,n-2)$.

Theorem 3.5 *Let $\alpha \in K_n^M(k)/2$ be nonzero pure symbol, and Q_α be the corresponding Pfister quadric. Then $\overline{\Omega}^*(Q_\alpha)$ coinsides with*

$$(\oplus_{0 \leqslant i \leqslant 2^{n-1}-1} \mathbb{L} \cdot e^i) \oplus (\oplus_{0 \leqslant j \leqslant 2^{n-1}-1} I(2,n-2) \cdot e_j),$$

where e^i is the class of the plane section of codimension i, and e_j is the class of the projective subspace of dimension j in $Q_\alpha|_{\overline{k}}$.

Proof: We already know that $\overline{\Omega}^*$ has this kind of decomposition, but about e^i and e_j we know only that $fog(e_j) = l_j$, $fog(e^i) = h^i$, and classes e^i are defined over k. Let us denote classes of plane sections and projective subspaces in $\Omega^*(Q_\alpha|_{\overline{k}})$ by the same symbols h^j and l_i.

Since e_i differs from l_i by some $\mathbb{L}$-linear combination of $l_0, \ldots, l_{i-1}$, we have: $I(2,n-2)((\oplus_{0 \leqslant m < i} l_m) \oplus e_i) = I(2,n-2)(\oplus_{0 \leqslant m \leqslant i} l_m)$, and one can assume that $e_i = l_i$.

The difference $e^i - h^i$ is equal to $\sum_{i<j\leqslant 2^{n-1}-1} \lambda_j h^j + \sum_{0\leqslant m \leqslant 2^{n-1}-1} \mu_m l_m$ and is defined over the base field k. Then $\mu_m \in I(2,n-2)$ and we can substitute e^i by h^i. $\square$

4 Injectivity

The aim of this section is to show that the map $ac : \Omega^*(Q_\alpha) \to \Omega^*(Q_\alpha|_{\overline{k}})$ is injective, which together with the Theorem 3.5 will give a computation of $\Omega^*(Q_\alpha)$.

Remind, that we have natural maps $\mathbf{L} \stackrel{\phi}{\leftarrow} M^\Omega_\alpha \stackrel{\psi}{\leftarrow} \mathbf{L}(2^{n-1}-1)[2^n-2]$, which, restricted to $\overline{k}$, give the splitting $M^\Omega_\alpha|_{\overline{k}} = \mathbf{L} \oplus \mathbf{L}(2^{n-1}-1)[2^n-2]$.

Lemma 4.1 *Let P be arbitrary subquadric of P_α. Then the maps* $M^\Omega(P) \stackrel{id\otimes\phi}{\longleftarrow} M^\Omega(P)\otimes M^\Omega_\alpha$ *and* $M^\Omega(P)\otimes M^\Omega_\alpha \stackrel{id\otimes\psi}{\longleftarrow} M^\Omega(P)(2^{n-1}-1))[2^n-2]$ *have splittings, and*

$$M^\Omega(P)\otimes M^\Omega_\alpha \cong M^\Omega(P) \oplus M^\Omega(P)(2^{n-1}-1)[2^n-2].$$

Proof: Denote $d_\alpha := dim(P_\alpha) = 2^{n-1}-1$, and $d := dim(P)$. Let $p_{M_\alpha} : M^\Omega(P_\alpha) \to M^\Omega_\alpha$ and $i_{M_\alpha} : M^\Omega_\alpha \to M^\Omega(P_\alpha)$ be the maps defining the Rost motive as the direct summand of the motive of P_α. The splitting $s_P : M^\Omega(P) \to M^\Omega(P)\otimes M^\Omega_\alpha$ is given by the composition:

$$M^\Omega(P) \xrightarrow{\Delta} M^\Omega(P\times P) \stackrel{id\otimes emb}{\longrightarrow} M^\Omega(P\times P_\alpha) \stackrel{id\otimes p_{M_\alpha}}{\longrightarrow} M^\Omega(P)\otimes M^\Omega_\alpha.$$

It follows just from the fact that the composition $M^\Omega(P_\alpha) \stackrel{p_{M_\alpha}}{\to} M^\Omega_\alpha \stackrel{\phi}{\to} \mathbf{L}$ is induced by the projection $P_\alpha \to \mathrm{Spec}(k)$. Analogously, the splitting $t_P : M^\Omega(P)\otimes M^\Omega_\alpha \to M^\Omega(P)(d_\alpha)[2d_\alpha]$ is given by the composition:

$$\begin{aligned} M^\Omega(P)\otimes M^\Omega_\alpha \stackrel{id\otimes i_{M_\alpha}}{\longrightarrow} M^\Omega(P\times P_\alpha) \stackrel{(id\otimes emb)^\vee}{\longrightarrow} M^\Omega(P\times P)(d_\alpha-d)[2(d_\alpha-d)] \\ \stackrel{\Delta^\vee}{\longrightarrow} M^\Omega(P)(d_\alpha)[2d_\alpha], \end{aligned}$$

where for $\varphi : M(X) \to M(Y)(r)[2r]$ represented by the map $W \to X\times Y$, the dual map $\varphi^\vee : M(Y) \to M(X)(d+r)[2(d+r)]$ is represented by the composition $W \to X\times Y \stackrel{s_{1,2}}{\to} Y\times X$ (here $d = dim(Y) - dim(X)$).

Since $(id\otimes\phi)\circ(id\otimes\psi) = 0$, $M^\Omega(P)\otimes M^\Omega_\alpha$ contains direct summand $M^\Omega(P)\oplus M^\Omega(P)(d_\alpha)[2d_\alpha]$. Let N be the complimentary summand. In view of Corollary 2.8, to prove that $N=0$, it is sufficient to show that $fog(N) \in \mathcal{M}_{CH}(k)$ is zero.

Since $M^{CH}_\alpha|_{\overline{k}} = \mathbf{Z}\oplus\mathbf{Z}(2^{n-1}-1)[2^n-2]$, the restriction $fog(N)|_{\overline{k}}$ is zero. But it follows from the Rost Nilpotence Theorem (see [18, Proposition 9]) that the motive of the product of quadrics does not contain *phantom* direct summands, that is, nonzero direct summands which become zero when restricted to $\overline{k}$. Thus, $N = 0$. $\square$

Remark: In [18, Proposition 9] the Nilpotence Theorem is formulated only for the case of motives of quadrics, but (some modification of) the proof works as well for the motives of products of quadrics (cf. the proof of [3, Theorem 8.2]). Another proof of Lemma 4.1 can be easily obtained from the presentation (2) in Section 3.

Lemma 4.2 *Let R be smooth projective variety of dimension $d < 2^{n-1} - 1$ such that the projection $M^{\Omega}(R) \leftarrow M^{\Omega}(R) \otimes M^{\Omega}_{\alpha}$ has a splitting s and $M^{\Omega}(R) \otimes M^{\Omega}_{\alpha} \cong M^{\Omega}(R) \oplus M^{\Omega}(R)(2^{n-1}-1)[2^n-2]$. Then such splitting is unique.*

Proof: The uniqueness follows from the fact that

$$Hom_{\mathcal{M}_{\Omega}}(M^{\Omega}(R), M^{\Omega}(R)(2^{n-1}-1)[2^n-2]) = \Omega^{d+2^{n-1}-1}(R \times R) = 0,$$

by the dimensional considerations. $\square$

Let Q_{2^i-1} be a subquadric of P_α of dimension $2^i - 1$, where $0 \leqslant i \leqslant n-2$. Let $\pi_i : M^{\Omega}(Q_{2^i-1}) \otimes M^{\Omega}_{\alpha} \to M^{\Omega}_{\alpha}$ be the projection, and $s_i : M^{\Omega}(Q_{2^i-1}) \to M^{\Omega}(Q_{2^i-1}) \otimes M^{\Omega}_{\alpha}$ be the splitting from Lemma 4.1. Applying Lemma 4.2 to the case $R = Q_{2^i-1}$, we get that s_i is unique.

Denote as c_i the element $(\pi_i)_*(s_i)_*([1_{Q_{2^i-1}}]) \in \Omega_{2^i-1}(M^{\Omega}_{\alpha})$. Notice, that c_i is just $p_{M_\alpha *}([Q_{2^i-1} \subset P_\alpha])$, where $[Q_{2^i-1} \subset P_\alpha] \in \Omega_{2^i-1}(P_\alpha)$ is the class of our subquadric. These classes satisfy the following equations:

Proposition 4.3 $c_i \cdot [Q_{2^j-1}] = c_j \cdot [Q_{2^i-1}]$, *where $[Q_r] \in \Omega_r(\mathrm{Spec}(k)) = \mathbb{L}$ is the class of the quadric of dimension r.*

Proof:
Consider the diagram:

$$\begin{array}{ccc} M^{\Omega}(Q_{2^i-1} \times Q_{2^j-1}) \otimes M^{\Omega}_{\alpha} & \xrightarrow{\pi'_i} & M^{\Omega}(Q_{2^j-1}) \otimes M^{\Omega}_{\alpha} \\ {\scriptstyle \pi'_j}\downarrow & & \downarrow{\scriptstyle \pi_j} \\ M^{\Omega}(Q_{2^i-1}) \otimes M^{\Omega}_{\alpha} & \xrightarrow[\pi_i]{} & M^{\Omega}_{\alpha}. \end{array}$$

From Lemma 4.1 it follows that the variety $R = Q_{2^i-1} \times Q_{2^j-1}$ satisfies the conditions of Lemma 4.2. (Indeed, if variety X satisfies these conditions, then variety $X \times Y$ satisfies them too (for any Y) - just multiply the old splitting by id_Y.) It follows from Lemma 4.2 that the splittings $s_i \otimes id_{M^\Omega(Q_{2^j-1})}$ and $s_j \otimes id_{M^\Omega(Q_{2^i-1})}$ coinside and are equal to the unique splitting s for the variety $Q_{2^i-1} \times Q_{2^j-1}$. Consequently, $(\pi'_j)^*(s_i)_*(1) = s_*(1) = (\pi'_i)^*(s_j)_*(1)$, and

$$\begin{aligned} c_i \cdot [Q_{2^j-1}] = (\pi_j)_*(\pi_j)^*(c_i) = (\pi_j)_*(\pi_j)^*(\pi_i)_*(s_i)_*(1) = \\ (\pi_j)_*(\pi'_i)_*(\pi'_j)^*(s_i)_*(1) = (\pi_j)_*(\pi'_i)_*(\pi'_i)^*(s_j)_*(1) = \\ (\pi_j)_*(s_j)_*(1) \cdot [Q_{2^i-1}] = c_j \cdot [Q_{2^i-1}]. \end{aligned}$$

□

Let $A \subset \Omega^*(M_\alpha^\Omega)$ be the ideal generated by the elements c_i, $0 \leqslant i \leqslant n-2$ and by $e^{2^{n-1}-1}$ (the generic cycle of the Rost motive). Consider $\overline{A} \subset \Omega^*(M_\alpha^\Omega|_{\overline{k}}) = \mathbb{L} \cdot e_0 \oplus \mathbb{L} \cdot e^{2^{n-1}-1}$ - the image of A under the natural extension of scalars map ac. Since c_i is just $p_{M_\alpha *}([Q_{2^i-1} \subset P_\alpha])$, clearly, $ac(c_i) = [Q_{2^i-1}] \cdot e_0$. Since $j([Q_{2^i-1}]) = \lambda \cdot b_{2^i-1} + \textit{decomposable terms}$, where $\lambda \in \mathbf{Z}$ is odd, the elements $[Q_{2^i-1}], 0 \leqslant i \leqslant n-2$ form a regular sequence in $\mathbb{L}$. It follows from Proposition 4.3 that the map $ac : A \to \overline{A}$ is an isomorphism.

The Chow groups of M_α^{CH} were computed by M.Rost in [17]. His result is:

$$CH^*(M_\alpha^{CH}) = \mathbb{Z} \cdot fog(c_0) \oplus \mathbb{Z} \cdot e^{2^{n-1}-1} \oplus \left(\oplus_{i=1}^{n-2} \mathbb{Z}/2 \cdot fog(c_i) \right). \qquad (3)$$

This shows that the composition $A \to \Omega^*(M_\alpha^\Omega) \to \mathrm{CH}^*(M_\alpha^{CH})$ is surjective. Since $CH^*(M_\alpha^{CH}) = \Omega^*(M_\alpha^\Omega)/\mathbb{L}^{<0}\Omega^*(M_\alpha^\Omega)$, and $\Omega^*(M_\alpha^\Omega)$ is a positively (by dimension) graded module, we get $A = \Omega^*(M_\alpha^\Omega)$. So, we have proved:

Proposition 4.4 *The map $ac : \Omega^*(Q_\alpha) \to \Omega^*(Q_\alpha|_{\overline{k}})$ is injective, and Theorem 3.5 describes $\Omega^*(Q_\alpha)$.*

Remark: It should be noticed, that an alternative proof of Proposition 4.4 can be obtained with the help of *symmetric operations* (see [24] and [23]). It has an advantage of being independent from the computations of [17]. In turn, it gives a new way to compute the Chow groups of a Pfister quadric.

Recall the motivic cobordism theory $MGL^{*,*}(X)$ defined by V.Voevodsky. Notice, that $MGL^{2*,*}(X)$ is an oriented generalised cohomology theory. Moreover, by the universality of $\Omega^*(X)$ for oriented cohomologies, there is the natural map ([8])

$$\rho_{MGL} : \Omega^*(X) \to MGL^{2*,*}(X)$$

which is epic from Corollary 5.2. It follows then from the naturality of the realization map $t_{\mathbb{C}}$, and [8, Theorem 12.6] that the natural maps

$$\Omega^{-*}(pt.) \stackrel{\rho_{MGL}}{\to} MGL^{-2*,-*}(pt.) \stackrel{t_{\mathbb{C}}}{\to} \mathbb{L}_*$$

are isomorphisms. In particular, from Corollary 2.9, the map ρ_{MGL} is isomorphic for $X = Q_\alpha|_{\bar{k}}$. Hence we easily see

Corollary 4.5 $\Omega^*(Q_\alpha) \cong MGL^{2*,*}(Q_\alpha)$.

Infact, almost all arguments in Sections 3,4 work also for $MGL^{2*,*}(X)$. However, in the next sections we will give another method of computing this ring by means of the Atiyah-Hirzebruch spectral sequence.

5 Atiyah-Hirzebruch spectral sequence

In this section we consider the Atiyah-Hirzebruch spectral sequences (AHss) for genealized motivic cohomologies related to the spectrum MGL constructed by Voevodsky ([25]). Since we are using the cohomology operations, below we mainly work with cohomology theories.

In the current section, p is a fixed prime number. Let k be a subfield of the complex number field $\mathbb{C}$, and $t_{\mathbb{C}}$ be the induced realization map (see [25]). As explained in the preceding sections, there are maps of oriented cohomology theories

$$\Omega^*(X) \xrightarrow{\rho_{MGL}} MGL^{2*,*}(X) \xrightarrow{t_{\mathbb{C}}} MU^{2*}(X(\mathbb{C}))$$

where $MU^{2*}(X(\mathbb{C}))$ is the complex cobordim ring of the manifold of $\mathbb{C}$-rational points of X. Recall that the complex cobordism ring of one point ([15],[12],[20]) is

$$\mathbb{L}_{-*} \cong MU^{2*}(pt.) = MU^{2*} \cong \mathbf{Z}[x_1, x_2, ...], \quad deg(x_i) = |x_i| = -2i.$$

So we have the map $i : \mathbb{L}_{-*} \cong MU^{2*} \to MGL^{2*,*}(pt.)$ such that $t_{\mathbb{C}}i$ is the identity on MU^{2*}. Moreover $MU^{2*} \cong MGL^{2*,*}(pt.)$ by Corollary 5.2 bellow and the naturality of $t_{\mathbb{C}}$.

The following results taken from §3 of [31]. Let us denote as AMU the spectrum $MGL_{(p)}$ representing the motivic cobordism theory. Of course, $t_{\mathbb{C}}(AMU) = MU_{(p)}$. Since $MGL^{*,*}(X)$ is a multiplicative cohomology theory, we know it is an $MGL^{*,*}(pt.)$-module, and hence, an MU^*-module. Given a regular sequence $S_n = (s_1, \ldots, s_n)$ with $s_i \in MU^*_{(p)}$, we can inductively construct the AMU-module spectrum (for the details of this construction see §4 in [2] or [5]) by the cofibering of spectra

$$\mathbb{T}^{-1/2|s_i|} \wedge AMU(S_{i-1}) \xrightarrow{\times s_i} AMU(S_{i-1}) \to AMU(S_i)$$

where $\mathbb{T} = S^1_s \wedge S^1_t$ so that $M(\mathbb{T}) = \mathbf{Z}(1)[2]$. For the original (topological) constructions, see [16] or [19]. Here we note that we do not assume $AMU(S)$ are AMU-ring spectra.

Hence we get the generalized cohomology theory $AMU(S_n)^{*,*}(X)$ such that

$$t_{\mathbb{C}}(AMU(S_n)) \cong MU(S_n) \quad \textit{with } MU(S_n)^* = MU^*/(Ideal(S_n)).$$

This way, we can construct spectra $ABP, Ak(n), AH\mathbf{Z}, AH\mathbf{Z}/p$ so that $t_{\mathbb{C}}(Ah) \cong h$ for $h = BP, k(n), \ldots$. Recall that $BP = MU(x_i | i \neq p^j - 1)_{(p)}$ and $BP^* = \mathbf{Z}_{(p)}[v_1, v_2, ...]$, where we denote x_{p^i-1} as v_i. (See [16],[19] for their topological cases.)

Remark. It follows from the Statement 3.3 , that in the case $p = 2$ we can take $v_i = [Q_{2^i-1}]$.

One of important results for these cohomology theories is that in the $\mathbb{A}^1$-stable homotopy category, we get $H_{\mathbf{Z}/p} \cong AH\mathbf{Z}/p = AMU(p, x_1, x_2, ...)$, e.g., $AH\mathbf{Z}/p^{*,*}(X) \cong H^{*,*}(X; \mathbf{Z}/p)$ (Theorem 3.3 in [31]).

If $S \subset (p, x_1, ..., x_n, ...)$, then we have the natural map of spectra

$$AMU(S) \to AMU(p, ..) = H\mathbf{Z}/p \quad (AMU(S) \to H\mathbf{Z}_{(p)} \ \ if\ p \notin S).$$

In such a case, we can construct the AHss. (The MGL case of this spectral sequence was constructed by Hopkins and Morel.)

Theorem 5.1 (Theorem 3.5 in [31]) *Let $Ah = AMU(S)$ for some regular sequence of generators $S = (a_1, \ldots)$, $a_j = x_{i_j} \in MU^*_{(p)}$. Then there is the (natural $MU^*_{(p)}$-module) Atiyah-Hirzebruch spectral sequence*

$$E(Ah)_2^{(m,n,2n')} = H^{m,n}(X; h^{2n'}) \Longrightarrow Ah^{m+2n',n+n'}(X)$$

with the differential $\quad d_{2r+1} : E_{2r+1}^{(m,n,2n')} \to E_{2r+1}^{(m+2r+1,n+r,2n'-2r)}$.

Remarks.

1) The cohomology $H^{m,n}(X, h^{2n'})$ here is the usual motivic cohomology with coefficients in the abelian group $h^{2n'}$, e.g., if $h^{2n'}$ is $\mathbf{Z}/p$-module, then $H^{m,n}(X; h^{2n'}) \cong H^{m,n}(X; \mathbf{Z}/p) \otimes h^{2n'}$. In particular, if X is smooth, then $E_r^{m,n,2n'} \cong 0 \quad \mathit{for}\ m > 2n$.

2) The convergence in AHss means that there is the filtration

$$Ah^{*,*'}(X) = F_0^{*,*'} \supset F_1^{*,*'} \supset F_2^{*,*'} \supset ...$$

such that $F_i^{*,*'}/F_{i+1}^{*,*'} \cong E_\infty^{*+2i,*'+i,-2i}$.

3) Let $S = (x_{i_1}, ...) \subset R = (x_{j_1}, ...)$. Then the induced map $AMU(S) \to AMU(R)$ of spectra induces the natural $MU^*_{(p)}$- module map of AHss

$$E(AMU(S))_r^{*,*,*} \to E(AMU(R))_r^{*,*,*}.$$

Let us denote $first.deg$ as $[deg]$, the $second.deg$ as (deg), and, finally, denote

$$w(x) = -[deg](x) + 2(deg)(x),$$

so that $w(x) \geq 0$ for nonzero element x in the spectral sequences (or cohomology theories) for a smooth X. Moreover, note that $w(d_r) = -1$ and $third.deg(d_r) = -r + 1 < 0$.

Corollary 5.2 *If $h^0 \cong \mathbf{Z}_{(p)}$, then for any smooth X, the natural map*

$$Ah^{2*,*}(X) \otimes_{MU^*_{(p)}} \mathbf{Z}_{(p)} \cong CH^*(X)_{(p)}$$

is an isomorphism

Proof: First note, that $E_2^{*,*',0} \cong H^{*,*'}(X)_{(p)} \otimes h^0 \cong H^{*,*'}(X)_{(p)}$. In particular, we can identify $E_2^{2*,*,0} \cong CH^*(X)_{(p)}$. For element $x \in E^{2*,*,*''}$ (i.e., $w(x) = 0$), we have $d_r(x) = 0$ since $w(d_r(x)) = -1 < 0$. Thus, $E_r^{2*,*,*''}$ is a quotient of $E_2^{2*,*,*''}$. But we also know that $Im(d_r) \subset E_r^{*,*',<0}$. Hence,

$$E_2^{2*,*,0} = E_\infty^{2*,*,0} \cong F_0^{2*,*}/F_1^{2*,*} \subset grAh^{2*,*}(X),$$

where we identify $E_\infty^{*,*,*}$ with $grAh^{*,*}(X)$. The isomorphism $h^{<0}E_\infty^{2*,*,0} = E_\infty^{2*,*,<0}$ is written $h^{<0}(F_0^{2*,*}/F_1^{2*,*}) = \oplus_{i\geq 1} F_i^{2*,*}/F_{i+1}^{2*,*}$ by the definition of the filtration (2). Of couese $F_i \supset F_{i+1}$, we see $h^{<0}F_0^{2*,*} = F_1^{2*,*}$. Thus we have $Ah^{2*,*}(X) \otimes_{MU^*} \mathbf{Z}_{(p)} \cong F_0^{2*,*}/F_1^{2*,*} \cong CH^*(X)_{(p)}$. $\square$

Corollary 5.3 *Let $Ah^{2*,*}(X)$ be the generalized cohomology theory from Theorem 5.1. Moreover, suppose that $Ah^{*,*}(X)$ is multiplicative and $h^0 \cong \mathbf{Z}_{(p)}$ is satisfied. Then the condition $(\star,\star)$ from Section 2 is satified for $A^*(X) = Ah^{2*,*}(X)$ and the natural functor $fog_A : A^* \to \mathrm{CH}^*_{(p)}$.*

Proof: The condition $(\star,\star)(0)$ is trivial, and the condition $(\star,\star)(1)$ follows from the fact that $Ah^{2*,*}(X) \otimes_{MU^*_{(p)}} \mathbf{Z}_{(p)} \to CH^*(X)_{(p)}$ can be decomposed into the composition $Ah^{2*,*}(X) \otimes_{MU^*_{(p)}} \mathbf{Z}_{(p)} \xrightarrow{\pi} Ah^{2*,*}(X) \otimes_{h^*} \mathbf{Z}_{(p)} \to CH^*(X)_{(p)}$, where π is surjective. $\square$

Hence $MGL^{2*,*}(X)_{(p)}$ satisfy $(\star,\star)$ and so does $ABP^{2*,*}(X)$ by the following reason. (In general $AMU(S)$-theory is not a multiplicative cohomology theory.) We can define ([31]) the multiplicative projection $\Phi : AMU \to AMU$ such that $\Phi(x_i) = x_i$ if $i = p^j - 1$ and $\Phi(x_i) = 0$ otherwise. Then we can show ((4.5) in [31]) $ABP \cong \Phi AMU$. Moreover we have

Lemma 5.4 *(Lemma 4.1 in [31]) For all X, we have a BP^*-algebra isomorphism*

$$MGL^{*,*}(X)_{(p)} \cong ABP^{*,*}(X) \otimes_{BP^*} MU^*_{(p)}.$$

To see the differentials of AHss, we recall the cohomology operations in mod p motivic cohomology. In this cohomology, we have the Bockstein homomorphism β and the reduced powers operations P^i which commute with the realization map $t_{\mathbf{C}}$. Moreover, we have the Milnor operation

$$Q_i : H^{*,*}(X;\mathbf{Z}/p) \to H^{*+2p^i-1,*+p^i-1}(X;\mathbf{Z}/p)$$

with $Q_0 = \beta$ and $Q_{i+1} = [Q_i, P^{p^i}]\ mod(\rho)$ where $\rho = -1 \in k^*/(k^*)^p = H^{1,1}(pt.;\mathbf{Z}/p)$ (see [28] for details). We note $Q_i^2 = 0$ and $Q_iQ_j = -Q_jQ_i$. But Q_i is not a derivation when $\rho \neq 0$. We also note that $w(P^i) = 0$ and $w(Q_i) = -1$.

V.Voevodsky (in particular, Lemma 2.2 in [29]) and G.Powell show that the mod p motivic Steenrod algebra $A_p^{*,*}$ is generated as an $H^{*,*}(pt,\mathbf{Z}/p)$-module by the products of P^i and β. In particular, they also prove that as a left $H^{*,*}(pt;\mathbf{Z}/p)$-module,

$$A_p^{*,*} \cong H^{*,*}(pt;\mathbf{Z}/p) \otimes RP \otimes \Lambda(Q_0, Q_1, ...) \tag{4}$$

where RP is the $\mathbf{Z}/p$-module generated by products of reduced powers $P^{i_1}....P^{i_n}$ (without the Bockstein).

Now we recall the connected Morava K-theory, $k(n)^*(X)$ with the coefficient ring

$$k(n)^* \cong \mathbf{Z}/p[v_n] \cong BP^*/(p, v_1, ..., \hat{v}_n, ...).$$

We consider the algebraic connected Morava K-theory

$$Ak(n)^{*,*'}(X) = ABP(p, v_1, ..., \hat{v}_n, ...)^{*,*'}(X).$$

Since $|v_n|(= deg(v_n)) = -2(p^n - 1)$, we know that $k(n)^* = 0$ for $* \neq 0 \ mod(2p^n - 2)$. So we have

$$E_2^{*,*',*''} = H^{*,*'}(X; \mathbf{Z}/p) \otimes k(n)^{*''} = 0 \quad for \ *'' \neq 0 \ mod(2p^n - 2).$$

Hence the first nonzero differential is d_{2p^n-1}.

By the naturality, the first nonzero differential

$$d_{2p^n-1} : H^{*,*}(X; \mathbf{Z}/p) \to H^{*+2p^n-1,*+p^n-1}(X; \mathbf{Z}/p) \otimes v_n$$

must be a cohomology operation of the motivic cohomology. (Cohomology operations are just natural transformations of the cohomology theory.) We know ([30, 31]) $d_{2p^n-1}(x) = v_n \otimes Q_n(x)$ for AHss of (the topological) $k(n)^*$-theory. From (4), we have

Lemma 5.5 *((4.6) in [31]) The first nonzero differential in the AHss for* $Ak(n)^{*,*}(X)$ *is given by*

$$d_{2p^n-1}(x) = v_n \otimes (Q_n + a_{IJ} P^I Q_J)(x)$$

where $P^I \in RP, Q_J \in \Lambda(Q_0, ..., Q_{n-1}), |J| \geq 2, \ a_{IJ} \in H^{>0,*}(pt.\mathbf{Z}/p)$.

6 $MGL^{2*,*}$ theory of Pfister quadrics

Recall that the *Čech* complex $\check{C}(X)$ is the simplicial scheme such that $\check{C}(X)_n = X^{n+1}$ and the faces and degeneracy maps are given by partial projections and diagonals respectively ([25, 27]). In the stable $\mathbb{A}^1$ homotopy category, define $\tilde{C}(X)$ by the cofiber sequence $\tilde{C}(X) \to \check{C}(X) \to Spec(k)$.

Let us write $\check{C}(Q_\alpha) = \chi_\alpha$ and $\tilde{C}(Q_\alpha) = \tilde{\chi}_a$. V.Voevodsky proved that in the triangulated category of motives DM^{eff} (see [27] for details), there exists the distinguished triangle

$$M(\chi_\alpha)(2^{n-1} - 1)[2^n - 2] \to M_\alpha \to M(\chi_\alpha) \xrightarrow{\delta_a} M(\chi_\alpha)(2^{n-1} - 1)[2^n - 1].$$

(Recall also the explanation for (2) in §3.) Using the induced long exact sequence of motivic cohomologies and the arguments from the paper [13], we can prove

Theorem 6.1 (Theorem 5.8 in [31]) *There is a $K_*^M(k)\otimes\Lambda(Q_0,...,Q_{n-1})$-modules isomorphism*

$$H^{*,*}(\tilde{\chi}_a;\mathbf{Z}/2)\cong K_*^M(k)/(Ker(\alpha))\otimes\Lambda(Q_0,...,Q_{n-1})\otimes\mathbf{Z}/2[\delta_a^2]\{\alpha'\}$$

where $deg(\delta_a^2)=(2^{n+1}-2,2^n-2)$, $deg(\alpha')=(n,n-1)$.

Corollary 6.2 (Corollary 5.9 in [31]) *There is a $K_*^M(k)$-module isomorphism for $[deg]-(deg)>0$ and $[deg]\le 2^n-2$,*

$$H^{*,*}(M_\alpha;\mathbf{Z}/2)\cong K_*^M(k)/(Ker(\alpha))\otimes\Lambda(Q_0,...,Q_{n-2})\{\alpha'\}.$$

Let τ be the nonzero element in $H^{0,1}(pt.;\mathbf{Z}/2)\cong\mathbf{Z}/2$. Then the elements α and α' are related by: $\tau\alpha'=\pi^*(\alpha)$ in $H^{n,n}(M_\alpha;\mathbf{Z}/2)$ ([31]), where we denote by π^* the composition $K_n^M(k)/2\cong H^{n,n}(pt.;\mathbf{Z}/2)\xrightarrow{\pi^*}H^{n,n}(M_\alpha;\mathbf{Z}/2)$.

Note that $w(\alpha')=n-2$. Since $w(Q_i)=-1$, if we apply $n-2$-times various operations Q_i to α', then the weight of the result will be just zero. Indeed the $wt(x)=0$ parts of $H^{2*,*}(M_\alpha;\mathbf{Z}/2)$ for $0<*\le 2^n-2$ is expressed as $Q_0...\hat{Q}_i...Q_{n-2}(\alpha')$. Moreover the fact that c_i is the Bockstein($=Q_0$) image for $i>0$ implies the result of Rost - see equation (3) of Section 4.

Corollary 6.3 (Rost [18]) *We have an isomorphism*

$$CH^*(M_\alpha)\cong H^{2*,*}(M_\alpha;\mathbf{Z})\cong\mathbf{Z}\{1,c_0\}\oplus\mathbf{Z}/2\{c_1,...,c_{n-2}\}$$

where c_i is a lift of $Q_0...\hat{Q}_i....Q_{n-2}(\alpha')\in H^{2*,*}(M_\alpha;\mathbf{Z}/2)$ and $deg(c_i)=(2^n-2^{i+1},2^{n-1}-2^i)$.

Here we recall that the Rost motive M_α can be represented as a motive of some affine quadric. One of direct corollaries of Theorem 3.1 is that the motives of Pfister quadric and a codimension 1 subquadric R in it can be decomposed as:

$$M(Q_\alpha)\cong M_\alpha\otimes M(\mathbb{P}^{2^{n-1}-1}),\quad M(R)\cong M_\alpha\otimes M(\mathbb{P}^{2^{n-1}-2}).$$

Corollary 6.4 *The Rost motive M_α is represented as the motive of affine quadric $U_\alpha=Q_\alpha-R$.*

Proof: There is the distinguished triangle in $DM(k)$

$$M(U_\alpha) \to M(Q_\alpha) \to M(R)(1)[2] \to M(U_\alpha)[1].$$

Hence the corollary follows from the isomorphisms $M(Q_\alpha) \cong M_\alpha \otimes M(\mathbb{P}^{2^{n-1}-1})$ and $M(R) \cong M_\alpha \otimes M(\mathbb{P}^{2^{n-1}-2})$. Notice, that the map from $M(Q_\alpha)$ to $M(R)(1)[2]$ we have here presents $M(R)(1)[2]$ as a direct summand of $M(Q_\alpha)$ (since it is true over algebraic closure). $\square$

Let $A^{*,*'}(X)$ be any (oriented) generalized cohomology theory. Then for smooth X and regular closed $Z \subset X$ of codimension c with open complement U we have long exact sequence

$$\to A^{*-2c,*'-c}(Z) \to A^{*,*'}(X) \to A^{*,*'}(U) \to A^{*-2c+1,*'+c}(Z) \to .$$

Indeed we have the Thom isomorphism (see [14],[11])

$$A^{*,*'}(X/U) \cong A^{*,*'}(Th_X(Z)) \cong ABP^{*-2c,*'-c}(Z)$$

where $Th_X(Z)$ is the Thom space of the normal bundle for $Z \subset X$.

From Corollary 2.8 and Corollary 5.3, we know that $\mathcal{M}_{ABP}(k)$ and $\mathcal{M}_{CH}(k)$ have the same set of isomorphism classes of objects. Hence we also consider the motive M_α^{ABP} for $ABP^{2*,*}$-theory. From the above corollary and the long exact sequence, we have the isomorphism

$$ABP^{2*,*}(M_\alpha^{ABP}) \cong ABP^{2*,*}(U_\alpha).$$

Of course $ABP^{*,*'}(U_\alpha)$ is the ABP-theory of a space, so there is an AHss converging to $ABP^{*,*}(U_\alpha)$ and the cohomology operations act on $H^{*,*}(U_\alpha; \mathbf{Z}/2)$.

Let $\pi : U_\alpha \subset Q_\alpha \to C(Q_\alpha) = \chi_\alpha$ be the induced map. Then the map

$$\pi^* : H^{,*,*}(\chi_\alpha; \mathbf{Z}/2) \to H^{*,*}(U_\alpha; \mathbf{Z}/2) \cong H^{,*,*}(M_\alpha; \mathbf{Z}/2)$$

is isomorphic for $[deg] - (deg) > 0$ and $[deg] \le 2^n - 2$ from Corollary 6.2. In particular, we can identify $\alpha' \in H^{n,n-1}(U_\alpha; \mathbf{Z}/2)$ and $c_i = Q_0 ... \hat{Q}_i ... Q_{n-2}(\alpha') \in H^{2*,*}(U_\alpha; \mathbf{Z}/2)$ as in Corollary 6.2 and Corollary 6.3. From the proof of Corollary 5.2, we recall $F_1^{2*,*} = BP^{<0} F_0^{2*,*}$ for the associated filtration. Hence a lift in $ABP^{2*,*}(U_\alpha)$ of an element in $H^{2*,*}(U_\alpha; \mathbf{Z}_{(2)}) \subset grABP^{2*,*}(U_\alpha)$ is well defined modulo $I(2)$. Let $\bar{c}_i \in ABP^{2*,*}(U_\alpha)$ be lifts of c_i.

We need the following lemma.

Lemma 6.5 *Let $S = (x_{i_1}, \dots)$ be a sequence from Theorem 5.1. Then for smooth X, we have the $MU^*_{(p)}$-module isomorphism*

$$AMU(S)^{2*,*}(X) \cong AMU^{2*,*}(X)/(S).$$

Proof: Let $S = R \cup \{x_j\}$ with $x_j \notin R$. Consider the long exact sequence induced from the cofiber sequence given in Section 5

$$\to AMU(R)^{2*+2j,*+j}(X) \xrightarrow{x_j} AMU(R)^{2*,*}(X) \to MU(S)^{2*,*}(X)$$

$$\xrightarrow{\delta} MU(R)^{2*+2j+1,*+j}(X) \to .$$

Here $AMU(R)^{2*+2j+1,*+j}(X) = 0$ from Remark (1),(2) after the Theorem 5.1. By induction, we assume $AMU(R)^{2*,*}(X) \cong AMU^{2*,*}(X)/(R)$. Then we have

$$AMU(S)^{2*,*}(X) \cong AMU(R)^{2*,*}(X)/(x_j) \cong AMU^{2*,*}(X)/(R, x_j).$$

Thus we get the lemma. □

In particular, the case $AMU(S) = AH\mathbf{Z}_{(p)}$ is Corollary 5.2. Moreover, we use the case $AMU(S) = Ak(n)$,

$$(*) \quad Ak(n)^{2*,*}(X) \cong ABP^{2*,*}(X)/(p, v_1, \dots, \hat{v}_n, \dots)$$

in the lemma bellow.

Lemma 6.6 *In $ABP^{2*,*}(U_\alpha)$ there are the following relations:*

$$v_i\bar{c}_j + v_j\bar{c}_i = 0 \ \ mod(I(2)^2) \ , \quad for \ 0 \le i, j \le n-2.$$

Proof: We can clearly assume that $i \neq j$. Take $z = Q_0 \dots \hat{Q}_i \dots \hat{Q}_j \dots Q_{n-2}(\alpha')$ in $H^{*,*}(U_\alpha; \mathbf{Z}/2)$ so that $w(z) = 1$, and $Q_i(z) = c_j$, $Q_j(z) = c_i$ for the construction of c_i in Corollary 6.3. We consider the AHss for $Ak(i)$-theory

$$E_2^{*,*,*}(U_\alpha) = H^{*,*}(U_\alpha; k(i)^*) \Longrightarrow Ak(i)^{*,*}(U_\alpha).$$

The first nonzero differential is given by Lemma 5.5

$$d_{2^{i+1}-1}(z) = v_i \otimes (Q_i + \sum a_{IJ}P^I Q_J)](z).$$

Here $Q_J(z)=0$ follows from $w(Q_J(z))\le -1$ since $|J|\ge 2$. Thus we know

$$(1)\quad d_{2^{i+1}-1}(z)=v_i\otimes Q_i(z)=v_i\otimes c_j.$$

Hence $v_ic_j=0$ in $grAk(i)^{2*,*}(U_\alpha)$ and this implies

$$(2)\quad v_i\tilde{c}_j=0\quad mod(F_{-|v_i|+1}=(v_i^2)=I(2)^2)\quad \ in\ Ak(i)^{2*,*}(U_\alpha)$$

where $\tilde{c}_i$ is a lift of c_i into $Ak(i)$-theory. Since $\bar{c}_j$ is a lift of c_j in $ABP^{2*,*}(U_\alpha)$, from $(*)$, we get:

$$(3)\quad v_i\bar{c}_j=0\quad mod(I(2)^2,2,v_1,\dots,\hat{v}_i,\dots)\quad in\ ABP^{2*,*}(U_\alpha).$$

Note, that $[deg](v_i\bar{c}_j)=2^n-2^{j+1}-2^{i+1}+2$. But, if $[deg](v_s\bar{c}_t)$ is also this degree, then $(s,t)=(i,j)$ or (j,i).

From Corollary 5.2 and Corollary 6.3, we see that $ABP^{2*,*}(U_\alpha)$ is generated by $1,\bar{c}_0,...,\bar{c}_{n-2}$ as a BP^*-module. Then we can write (3) as

$$(4)\quad v_i\bar{c}_j+\lambda_jv_j\bar{c}_i=0\quad mod(I(2)^2),\quad \ in\ ABP^{2*,*}(U_\alpha),$$

where λ_j is either 0, or 1.

In a similar way, interchanging i and j, we get the equation

$$(5)\quad v_j\bar{c}_i+\lambda_iv_i\bar{c}_j=0\quad mod(I(2)^2),\quad \ in\ ABP^{2*,*}(U_\alpha).$$

Then either (4), or (5), or (4) + (5) will have the desired form. □

Remark. We can prove the following fact ([30],[21]).

> For $y_m\in ABP^{*,*}(X), 0\le m$, suppose that $\sum_m v_my_m=0$ in $ABP^{*,*}(X)$. Then there is $x\in H^{*,*}(X;\mathbf{Z}/p)$ such that $Q_m(x)=\rho(y_m)\in H^{*,*}(X;\mathbf{Z}/p)$ for all m where $\rho: ABP\to AH\mathbf{Z}/p$ is the natural (Thom) map.

In our case, $\sum v_my_m=v_i\bar{c}_j+v_j\bar{c}_i\ mod(I(2)^2)$ and $z=x$. The preceding lemma is also proved by using the above fact (and (3) in the proof of the lemma), since z with $Q_i(z)=c_j$ is uniquely determined as $z=Q_0...\hat{Q}_i...\hat{Q}_i...Q_{n-1}(\alpha')$.

Corollary 6.7 *If $i<j$, then $v_i\bar{c}_j=0\ mod(\bar{c}_0,...,\bar{c}_{j-1})$ in $ABP^{2*,*}(U_\alpha)$.*

Proof: From above lemma, we get this result with $mod(I(2)^2)$. For $k > j$, we see,

$$[deg](y_k\bar{c}_k) \leqslant [deg](\bar{c}_k) = 2^n - 2^k < 2^n - 2^{i+1} - 2^{j+1} + 2 = [deg](v_i\bar{c}_j).$$

Hence we have a relation:

$$\tilde{v}_{(i,j)}\bar{c}_j = 0 \quad mod\,(\bar{c}_0, \ldots, \bar{c}_{j-1}),$$

where $\tilde{v}_{(i,j)} = v_i \; mod\,(I(2)^2)$. To prove that we can assume that $\tilde{v}_{(i,j)} = v_i$, we have to use the induction on i fixing j. $\square$

From Lemma 5.4, we get the AMU-version.

Corollary 6.8 *There are relations for* $0 \leq i < j \leq n-2$,

$$v_i\bar{c}_j + v_j\bar{c}_i = 0 \;\; mod(I(2)^2) \quad in \; MGL^{2*,*}(U_\alpha).$$

Moreover, if $i < j$*, then* $v_i\bar{c}_j = 0 \; mod(\bar{c}_0, ..., \bar{c}_{j-1})$ *in* $MGL^{2*,*}(U_\alpha)$.

7 injectivity for $MGL^{2*,*}$

From Corollary 2.8 and Corollary 5.3, we know that $\mathcal{M}_{MGL}(k)$ and $\mathcal{M}_{CH}(k)$ have the same set of isomorphism classes of objects. Hence we can define M_α^{MGL} and consider $MGL^{2*,*}(M_\alpha^{MGL}) \cong MGL^{2*,*}(U_\alpha)$, let us write it simply $MGL^{2*,*}(M_\alpha)$.

Recall that $\bar{k}$ is the algebraic closure of k and $X|_{\bar{k}} = X \otimes_k \bar{k}$. Let

$$ac : MGL^{2*,*}(X) \to MGL^{2*,*}(X|_{\bar{k}})$$

be the induced map. Since $M_\alpha^{MGL}|_{\overline{k}}$ is a direct sum of (MGL) Tate-motives, we have:

$$MGL^{2*,*}(M_\alpha|_{\bar{k}}) \cong MU^* \otimes CH^*(M_\alpha|_{\bar{k}}) \cong MU^*\{1, \bar{c}\}, \tag{5}$$

where $deg(\bar{c}) = (2^n - 2, 2^{n-1} - 1)$. The element $\bar{c}$ here is the image of $e_0 \in \Omega_0(M_\alpha)$ (defined in Section 3) under the natural map $\Omega^*(X) \to MGL^{2*,*}(X)$.

Recall also that $MGL^{2*,*}(M_\alpha)$ is generated by $1, \bar{c}_0, ..., \bar{c}_{n-2}$ as a MU^*-module (compare with Theorem 3.5).

Lemma 7.1 *We have $ac(1) = 1$ and $ac(\bar{c}_i) = v_i\bar{c}\ mod(I(2, i-1))$. Hence*

$$ac(MGL^{2*,*}(M_\alpha)) = MU^*\{1\} \oplus I(2, n-2)\{\bar{c}\}.$$

Proof: Since for arbitrary E/k, the map $CH^{2^{n-1}-1}(M_\alpha) \to CH^{2^{n-1}-1}(R_\alpha)$ is an isomorphism, and for anisotropic quadric Q of dimension d, the map

$$ac^{CH} : \mathbf{Z} \cong CH^d(Q) \to CH^d(Q|_{\bar{k}}) \cong \mathbf{Z}$$

is the multiplication by 2, we get: $ac^{CH}(c_0) = 2c$, where c is the generator of $CH^{2^{n-1}-1}(M_\alpha|_{\bar{k}}) \cong \mathbf{Z}$. Since $deg(MU^*) \leq 0$, we note that $[deg](MGL^{2*,*}(M_\alpha)) \leq 2dim(R_\alpha)$ by Corollary 5.2. Hence we know

$$MGL^{2^n-2, 2^{n-1}-1}(M_\alpha) \cong CH^{2^{n-1}-1}(M_\alpha) \cong \mathbf{Z}.$$

Thus we see that $ac(\bar{c}_0) = 2\bar{c}$ also in $MGL^{2*,*}(M_\alpha|_{\bar{k}})$.

The relation $v_i\bar{c}_0 = 2\bar{c}_i\ mod(I(2)^2)$ in Corollary 6.8 implies at first $ac(\bar{c}_i) \in I(2)\bar{c}$. Since

$$(v_i\bar{c}_0 - 2\bar{c}_i) \in I(2)^2\{\bar{c}_0, ..., \bar{c}_{n-2}\}$$

(note that $[deg](v_i\bar{c}_0 - 2\bar{c}_i) > 0$ but $[deg](1) = 0$), we have $ac(v_i\bar{c}_0 - 2\bar{c}_i) \in I(2)^3\{\bar{c}\}$. Hence we get

$$ac(\bar{c}_i) = (v_i/2)ac(\bar{c}_0) = v_i\bar{c} \ \ mod(I(2)^2\overline{c})$$

from Corollary 6.8 since $MGL^{2*,*}(M_\alpha|_{\bar{k}})$ is MU^*-free. □

Compare with Proposition 4.4.

Theorem 7.2 *The map $ac : MGL^{2*,*}(M_\alpha) \to MGL^{2*,*}(M_\alpha|_{\bar{k}})$ is injective and*

$$MGL^{2*,*}(M_\alpha) \cong MU^*\{1\} \oplus I(2, n-2)\{\bar{c}\}.$$

Proof: Consider the filtration

$$(2, ..., v_{n-2}) = I(2, n-2) \supset I(2, n-3) \supset ... \supset I(2, 0) = (2).$$

Then we have the isomorphism $I(2, i)/I(2, i-1) \cong MU^*/I(2, i-1)\{v_i\}$. Thus we get the isomorphism

$$grI(2, n-2)\{\bar{c}\} \cong \oplus_{0 \leq i \leq n-2} MU^*/I(2, i-1)\{v_i\bar{c}\}.$$

Let A be the MU^*-submodule of $MGL^{2*,*}(M_\alpha)$ generated by $\{\bar{c}_0, ..., \bar{c}_{n-2}\}$, i.e., $MGL^{2*,*}(M_\alpha) \cong MU^*\{1\} \oplus A$. Then from Lemma 7.1, we have the map $g = ac|_A : A \to I(2, n-2)\{\bar{c}\}$. Let A_i be the submodule of $MGL^{2*,*}(M_\alpha)$ generated by $\{\bar{c}_0, ..., \bar{c}_i\}$. Since $ac(\bar{c}_i) = v_i\bar{c}\ mod(I(2, i-1))$, we have the induced map

$$gr(g) : grA = \oplus A_i/A_{i+1} \to \oplus_{0 \le i \le n-2} MU^*/I(2, i-1)\{v_i\bar{c}\}.$$

Here from Corollary 6.8, the module A_i/A_{i-1} is a quotient of the module

$$MU^*\{\bar{c}_0, ..., \bar{c}_i\}/(v_k\bar{c}_i - v_i\bar{c}_k, \bar{c}_0, ..., \bar{c}_{i-1}|k < i) \cong MU^*/I(2, i-1)\{\bar{c}_i\}.$$

Here note $gr(g)(A_i/A_{i-1}) = MU^*/I(2, i-1)\{v_i\bar{c}\}$. Hence $gr(g)|_{(A_i/A_{i-1})}$ is isomorphic and so are $gr(g)$ and g. □

Corollary 7.3 *There are BP^*-module isomorphisms*

$$grABP^{2*,*}(U_\alpha) \cong (BP^*\{1\} \oplus \oplus_{i=0}^{n-2} BP^*/I(2, i-1)\{c_i\}),$$

$$ABP^{2*,*}(U_\alpha) \cong (BP^*\{1, \bar{c}_0, ...\bar{c}_{n-1}\}/(v_i\bar{c}_j - v_j\bar{c}_i|i < j)).$$

Note that $ABP^{2*,*}(Q_\alpha)$ is a free $ABP^{2*,*}(U_\alpha)$-module

$$ABP^{2*,*}(Q_\alpha) \cong ABP^{2*,*}(U_\alpha)\{1, h, \ldots, h^{2^{n-1}-1}\}$$

where $deg(h) = (2, 1)$. Similar fact also holds for $grABP^{2*,*}(Q_\alpha)$. However note that $h^{2n-1} \neq 0 \in ABP^{2*,*}(Q_\alpha)$.

Let $\pi : X \to pt$ is the projection map and $I(X) = \pi_* MGL^{2*,*}(X)$. Let $I'(X)$ be the sub MU^*-module of $I(X)$ generated by π_*-images of elements of $MGL^{2*,*}(X)$ of positive degree. Recall the affine quadric Q_α with $MGL^{2*,*}(U_\alpha) \cong MGL^{2*,*}(M_\alpha)$.

Theorem 7.4 *We have $\pi_*(1) = v_n$ and $\pi_*(\bar{c}_i) = v_i$. Hence $I'(U_\alpha) \cong I(2, n-2)$ and $I(U_\alpha) \cong I(2, n-1)$.*

Proof: Since $t_{\mathbb{C}}(P_\alpha) = v_{n-1}$, we have $\pi_*(1) = v_{n-1}$. For $\pi_*(\bar{c}_0)$, we know $\pi_*(c_0) = deg(P_\alpha) = 2$ in $CH^0(pt.) \cong \mathbf{Z}$. Hence $\pi_*(\bar{c}_i) = v_i$ follows from the relation $2\bar{c}_i = v_i\bar{c}_0$ in $MGL^{2*,*}(M_\alpha)$. □

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Abstract

In this article we compute the ring of Algebraic Cobordisms of a Pfister Quadric. This is the rare example of noncellular variety where such computation is known We consider the Algebraic Cobordisms Ω^* of M.Levine-F.Morel, as well as the $MGL^{2*,*}$ of V.Voevodsky. Respectively, the methods of computation in two cases are quite different. But the result do agree (which support the expectations that the two theories, actually, coinside). We show that the restriction homomorphism in our case is injective for any field extension E/F.