Problem class 8

1. View ${}^{\mathbb{Q}^\times}/_{\square}$ as a vector space over $\mathbb{F}_2$. Determine the dimension of the subgroup generated by $\{6, -15, -10, 7\}$ when viewed as a subspace.

2. Let E be an elliptic curve over $\mathbb{Q}$ of the form $y^2 = (x - e_1)(x - e_2)(x - e_3)$ and let κ be the Kummer map. Show that the image of κ is contained in the subgroup $\big\{(a, b, c) \;\big|\; abc = \square\big\}$ of ${}^{\mathbb{Q}^\times}/_{\square} \times {}^{\mathbb{Q}^\times}/_{\square} \times {}^{\mathbb{Q}^\times}/_{\square}$.

3. Let $E : y^2 = (x + 4)(x + 1)(x - 5)$ which contains the following rational points

$$\begin{array}{ccc} T_1 = (-4, 0) & T_2 = (-1, 0) & T_3 = (5, 0) \\ P_1 = (-3, 4) & P_2 = (8, 18) & P_3 = (23, 108) \end{array}$$

   Compute the image under the Kummer map of all these six points. Prove that the image of these six points generate a group isomorphic to $\big({}^{\mathbb{Z}}/_{2\mathbb{Z}}\big)^3$. Use it to find relations among the points.

4. Let E be an elliptic curve over $\mathbb{R}$ with positive discriminant Δ. We know that we can write it as $y^2 = (x - e_1)(x - e_2)(x - e_3)$ with $e_1 < e_2 < e_3$. Determine the Kummer map $\kappa\colon E(\mathbb{R}) \to {}^{\mathbb{R}^\times}/_{\square} \times {}^{\mathbb{R}^\times}/_{\square} \times {}^{\mathbb{R}^\times}/_{\square}$ and conclude that

$$2\,E(\mathbb{R}) = \{O\} \cup \big\{(x_P, y_P) \in E(\mathbb{R}) \;\big|\; x_P \geqslant e_3\big\}.$$