Topics Related to Combinatorial Designs

1. Designs. A *combinatorial design* $\mathcal{D}$ consists of a nonempty finite set $S = \{p_1, \ldots, p_v\}$ of *points* or *varieties*, and a nonempty family of subsets $B_1, \ldots, B_b$ of S called *blocks* or *lines*. $\mathcal{D}$ is *balanced* or *λ-linked* if every pair of points is contained in exactly λ blocks. $\mathcal{D}$ is a *balanced incomplete-block design* (*BIBD*) (F. Yates, 1936), *2-design* or *2-(v,k,λ) design* if $\mathcal{D}$ is λ-linked and $|B_i| = k$ for each block B_i, where $1 < k < v$.

Theorem 1.1. Every point in a BIBD lies in the same number r of the blocks, where $bk = vr$, $r(k-1) = \lambda(v-1)$, and $0 < \lambda < r < b$.

Proof. Exercise. //

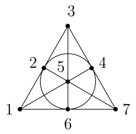
The *parameters* of $\mathcal{D}$ are (b, v, r, k, λ) (or (v, b, r, k, λ)!).

Examples. (1) The set of all k-subsets of S forms an *unreduced design* with parameters $\left(\binom{v}{k}, v, \binom{v-1}{k-1}, k, \binom{v-2}{k-2}\right)$.

(2) A *finite projective plane* (*FPP*) of *order* n is a 2-design with $b = v = n^2 + n + 1$, $r = k = n + 1$ and $\lambda = 1$. E.g., if $n = 2$ we have the *Fano configuration* (G. Fano, 1892):

Points $1, 2, \ldots, 7$,

Blocks (lines) $\{1,2,3\}$, $\{1,4,5\}$, $\{1,6,7\}$,
$\{2,4,6\}$, $\{2,5,7\}$,
$\{3,4,7\}$, $\{3,5,6\}$.



(3) A *Steiner triple system* (J. Steiner, 1853) is a 2-$(v,3,1)$ design with $b = \frac{1}{3}\binom{v}{2}$ and $r = \frac{1}{2}(v-1)$. T. Kirkman proved in 1847 that they exist iff $v \equiv 1$ or $3 \pmod 6$.

(4) See handout.

Fisher's inequality (R. A. Fisher, 1940) is that $b \geqslant v$ for every BIBD. We will prove a stronger result. A λ-linked design is *nontrivial* if $\lambda > 0$ and no block contains all the points; note that a

BIBD is nontrivial. The *incidence matrix* of a design $\mathcal{D}$ is the $v \times b$ matrix A in which $[A]_{ij} = 1$ if $p_i \in B_j$ and 0 otherwise.

Theorem 1.2. (K. N. Majumdar, 1953.) Let $\mathcal{D}$ be a nontrivial λ-linked design. Then $b \geqslant v$.

Proof. (R. C. Bose, 1949.) Let $r_i := \#\{j : p_i \in B_j\}$ $(i = 1, \ldots, v)$. Then $r_i > \lambda$ since $\mathcal{D}$ is nontrivial. $AA^\top$ is the $v \times v$ matrix $\begin{bmatrix} r_1 & \lambda & \cdots & \lambda \\ \lambda & r_2 & \cdots & \lambda \\ \vdots & \vdots & \ddots & \vdots \\ \lambda & \lambda & \cdots & r_v \end{bmatrix}$. So $\det AA^\top = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ \lambda & r_1 & \lambda & \cdots & \lambda \\ \lambda & \lambda & r_2 & \cdots & \lambda \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda & \lambda & \lambda & \cdots & r_v \end{vmatrix} =$

$$= \begin{vmatrix} 1 & -1 & -1 & \cdots & -1 \\ \lambda & r_1-\lambda & 0 & \cdots & 0 \\ \lambda & 0 & r_2-\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda & 0 & 0 & \cdots & r_v-\lambda \end{vmatrix} = \begin{vmatrix} 1+\sum_{i=1}^v \frac{\lambda}{r_i-\lambda} & 0 & 0 & \cdots & 0 \\ \lambda & r_1-\lambda & 0 & \cdots & 0 \\ \lambda & 0 & r_2-\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda & 0 & 0 & \cdots & r_v-\lambda \end{vmatrix}$$

$= \left(1 + \sum_{i=1}^v \frac{\lambda}{r_i-\lambda}\right) \prod_{i=1}^v (r_i-\lambda) \neq 0$. So $AA^\top$ is nonsingular. Hence $b \geqslant \operatorname{rank} A \geqslant \operatorname{rank} AA^\top = v$. //

A design is called *square* (or, if a BIBD, *symmetric*) if $b = v$. The *dual* $\mathcal{D}^\top$ of $\mathcal{D}$ is the design with incidence matrix $A^\top$.

Theorem 1.3. (R. C. Bose, 1939.) If $\mathcal{D}$ is a square BIBD then $\mathcal{D}^\top$ is a square BIBD with the same parameters.

Proof. Let I be the $v \times v$ identity matrix and J be the $v \times v$ matrix of 1's, so that (since $bk = vr$ and so $k = r$)

$$AA^\top = \begin{bmatrix} k & \lambda & \cdots & \lambda \\ \lambda & k & \cdots & \lambda \\ \vdots & \vdots & \ddots & \vdots \\ \lambda & \lambda & \cdots & k \end{bmatrix} = (k-\lambda)I + \lambda J.$$

We want to prove that $A^\top A = (k-\lambda)I + \lambda J$. Now, $AJ = kJ = JA$. And A is nonsingular by Theorem 1.2, so that $A^{-1}J = A^{-1}(k^{-1}AJ) = k^{-1}J$. Hence

$$\begin{aligned} A^\top A = A^{-1}(AA^\top)A &= (k-\lambda)A^{-1}IA + \lambda A^{-1}JA \\ &= (k-\lambda)I + \lambda k^{-1}kJ \\ &= (k-\lambda)I + \lambda J. \end{aligned}$$

The result follows. //

In fact, a stronger result is true:

Theorem 1.4. Let $\mathcal{D}$ be a nontrivial square λ-linked design such that every point lies in the same number r of the blocks. Then $\mathcal{D}^\top$ is a square BIBD (and hence, by Theorem 1.3, so is D).

Proof. As in Theorem 1.3, $AA^\top = (r-\lambda)I + \lambda J$, and it suffices to prove that $A^\top A = (r-\lambda)I + \lambda J$, since then $\mathcal{D}^\top$ is a square BIBD. We know that $AJ = rJ$, and the result will follow as in Theorem 1.3 if we can prove that $JA = rJ$. As before, $A^{-1}J = A^{-1}(r^{-1}AJ) = r^{-1}J$. Also $J^2 = vJ$, so that

$$\begin{aligned} AA^\top J &= (r-\lambda)IJ + \lambda J^2 = (r-\lambda+\lambda v)J, \\ A^\top J &= r^{-1}(r-\lambda+\lambda v)J, \end{aligned}$$

and

$$JA = (A^\top J)^\top = r^{-1}(r-\lambda+\lambda v)J.$$

Now,

$$\begin{aligned} (JA)J &= r^{-1}(r-\lambda+\lambda v)J^2 \\ &= J(AJ) = J(rJ) = rJ^2, \end{aligned}$$

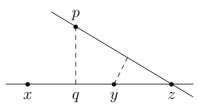
so that $r = r^{-1}(r-\lambda+\lambda v)$ and $JA = rJ$ as required. //

Theorem 1.5. Suppose that $\mathcal{D}$ is a nontrivial λ-linked design and $\mathcal{D}^\top$ is λ'-linked. Then $b = v$, and either $\lambda = \lambda' = 1$ or $\mathcal{D}$ is a square BIBD.

Proof. $\mathcal{D}$ has at least two blocks that are not disjoint, so $\lambda' \neq 0$. And if some point lies in every block then every other point lies in exactly λ blocks, and the *same* λ blocks, which is impossible since $\mathcal{D}$ is nontrivial. So $\mathcal{D}^\top$ is nontrivial, and $b = v$ by Theorem 1.2 applied to $\mathcal{D}$ and to $\mathcal{D}^\top$. If some block contains k_i points, then $\binom{k_i}{2} + (v-1)\binom{\lambda'}{2} = \lambda\binom{k_i}{2}$; this determines k_i uniquely (and so $\mathcal{D}$ is a square BIBD) unless $\lambda = 1$. Also, $\lambda' = 1 \implies \lambda = 1$. By the same argument for $\mathcal{D}^\top$, $\lambda = 1 \implies \lambda' = 1$. //

2. Analogues of Fisher's inequality.

Lemma 2.1.1. (Conj. J. J. Sylvester, 1893; proved T. Gallai, 1944.) If n points in the plane are not all collinear, then there is a line that passes through exactly two of them.



Proof. (L. M. Kelly, 1948.) Clearly $n \geqslant 3$. Suppose that every line through two of the points has another point on it. Choose three of the points, p, x, y, such that p is not on xy but the perpendicular distance pq from p to xy is as small as possible. There is another of the n points on xy, say z. Some two of x, y, z—w.l.o.g. y and z—lie on the same side of q. Let the order be q, y, z (possibly $q = y$). Then the perpendicular distance from y to pz is less than that from p to xy, $\Rightarrow\Leftarrow$. //

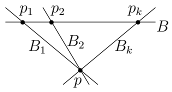
L. M. Kelly and W. O. J. Moser (1958) proved there are at least $\frac{3}{7}n$ such lines, and G. A. Dirac (1951) conjectured there are at least $\frac{1}{2}n$ if $n \neq 7$. D. W. Crowe and T. A. McKee (1968) disproved this when $n = 13$. J. Csima and E. T. Sawyer (1993) proved there are at least $\frac{6}{13}n$ if $n \neq 7$. The $\frac{1}{2}n$ conjecture remains uncontradicted for $n \neq 7, 13$.

Theorem 2.1. (R. Steinberg, 1944.) If n points in the plane are not all collinear, then they determine at least n lines.

Proof by induction on n. Clearly $n \geqslant 3$, and the result holds if $n = 3$; so suppose $n \geqslant 4$. By Lemma 2.1.1, there is a line passing through exactly two points, say p and q. Remove p; if the remaining points are all collinear, then remove q instead. Then the remaining $n-1$ points are not all collinear and, by the induction hypothesis, they determine at least $n-1$ lines. Restoring the nth point and the line pq gives at least n lines, as required. //

Conjecture. (P. Erdős.) $\exists\, c > 0$ s.t. if every line has $\leqslant k$ points then there are $\geqslant cn(n-k)$ lines.

Theorem 2.2. (N. G. de Bruijn and P. Erdős, 1948.) For a nontrivial 1-linked design, $b \geqslant v$.



Proof. Let $B = \{p_1, \ldots, p_k\}$ be a largest block and let $p \in S \setminus B$. Let $\Pi := B \cup \{p\}$, and let Λ denote the set of $k+1$ distinct blocks $\{B, B_1, \ldots, B_k\}$, where B_i is the unique block containing p and p_i. Each element of $S \setminus \Pi$ is contained, with $p_1, \ldots, p_k$, in k distinct blocks, at least $k-1$ of which are not in Λ; and each of these $k-1$ blocks contains at most $k-1$ elements of $S \setminus \Pi$. Thus

$$b \geqslant |\Lambda| + \tfrac{k-1}{k-1}|S \setminus \Pi| = (k+1) + (v-k-1) = v. \quad //$$

Let $S = \{p_1, \ldots, p_n\}$ be a set of n points in a Euclidean or projective space that determine W_2 lines, W_3 planes, W_4 3-spaces, $\ldots$, $W_r = 1$ $(r-1)$-spaces, and set $W_0 := 1$, $W_1 := n$. $W_0, \ldots, W_r$ are known as *Whitney numbers*.

Conjecture A. (G.-C. Rota.) The sequence (W_i) is *unimodal*: $i \leqslant j \leqslant k \implies W_j \geqslant \min\{W_i, W_k\}$.

Conjecture B. (J. H. Mason and D. J. A. Welsh, early 1970s.) The sequence (W_i) is *log-concave*: $W_i^2 \geqslant W_{i-1}W_{i+1}$ (i.e., $\log W_i \geqslant \frac{\log W_{i-1} + \log W_{i+1}}{2}$) $(1 \leqslant i \leqslant r-1)$.

Conjecture C. (J. H. Mason.) $\left(\frac{W_i}{\binom{n}{i}}\right)^2 \geqslant \left(\frac{W_{i-1}}{\binom{n}{i-1}}\right)\left(\frac{W_{i+1}}{\binom{n}{i+1}}\right)$ $\left(\text{i.e., } W_i^2 \geqslant \frac{i+1}{i}\frac{n-i+1}{n-i}W_{i-1}W_{i+1}\right)$ $(1 \leqslant i \leqslant r-1)$, with equality iff both sides are 1. (Clearly $W_i \leqslant \binom{n}{i}$ for each i.)

Note that C $\Longrightarrow$ B $\Longrightarrow$ A.

For n points in $\mathbb{R}^3$ determining l lines and π planes, B and C say that $l^2 \geqslant n\pi$ and $l^2 \geqslant \frac{3}{2}\frac{n-1}{n-2}n\pi$. It is not even known whether there is a constant $c > 0$ such that $l^2 > cn\pi$ (the *points–lines–planes conjecture*).

Theorems 1.2, 2.1 and 2.2 give $W_2 \geqslant W_1$. *Motzkin's hyperplane inequality* (Theorem 2.3) is that $W_i \geqslant W_1 \,\forall\, i$ $(1 \leqslant i \leqslant r-1)$. T. A. Dowling and R. M. Wilson (1975) proved that

$$W_1 + W_2 + \ldots + W_t \leqslant W_{r-t} + \ldots + W_{r-2} + W_{r-1} \quad \forall\, t\ (1 \leqslant t \leqslant r-1).$$

Lemma 2.3.1. In a design $\mathcal{D}$, let $k_j := |B_j|$ and $r_i := \#\{j : p_i \in B_j\}$. Suppose $r_i > 0$ and $k_j < v$, $\forall\, i, j$, and $r_i \geqslant k_j$ whenever $p_i \notin B_j$. Then $b \geqslant v$.

Proof. Suppose $b < v$.

Case 1: the complements $\overline{B}_1, \ldots, \overline{B}_b$ of $B_1, \ldots, B_b$ have a transversal. Then we can relabel the points p_i so that $p_i \in \overline{B}_i$, i.e., $p_i \notin B_i$, and therefore $r_i \geqslant k_i$ $(i = 1, \ldots, b)$. Let α be the number of pairs (p_i, B_j) such that $p_i \in B_j$ $(1 \leqslant i \leqslant v,\ 1 \leqslant j \leqslant b)$. Then

$$\sum_{i=1}^{b} r_i \geqslant \sum_{i=1}^{b} k_i = \alpha = \sum_{i=1}^{v} r_i > \sum_{i=1}^{b} r_i, \ \Rightarrow\Leftarrow.$$

Case 2: $\overline{B}_1, \ldots, \overline{B}_b$ do not have a transversal. By Hall's theorem, there exist k and $\overline{B}_1, \ldots, \overline{B}_k$ (relabelling) such that $|\overline{B}_1 \cup \ldots \cup \overline{B}_k| < k$. Note that $k \geqslant 2$ since $\overline{B}_1 \neq \emptyset$. Choose k minimal; then $\overline{B}_1, \ldots, \overline{B}_{k-1}$ have a transversal. We can relabel the points p_i so that

$p_i \in \overline{B}_i$ and therefore $r_i \geqslant k_i$ $(i = 1, \ldots, k-1)$, and then none of $\overline{B}_1, \ldots, \overline{B}_{k-1}$ contain any points other than $p_1, \ldots, p_{k-1}$. Let β be the number of pairs $(p_i, \overline{B}_j)$ such that $p_i \in \overline{B}_j$ $(i, j = 1, \ldots, k-1)$. Then

$$\begin{array}{c|c|c|}
 & \bar{B}_1 \ldots \bar{B}_{k-1} & \bar{B}_k \ldots \bar{B}_b \\ \hline
p_1, \ldots, p_{k-1} & \beta & ? \\ \hline
p_k, \ldots, p_v & 0 & ? \\ \hline
\end{array}$$

$$\sum_{i=1}^{k-1} (b - r_i) < \sum_{i=1}^{k-1} (v - k_i) = \sum_{i=1}^{k-1} |\overline{B}_i| = \beta \leqslant \sum_{i=1}^{k-1} (b - r_i), \ \Rightarrow\Leftarrow.$$

This contradiction shows that $b \geqslant v$ after all. //

Theorem 2.3. (Th. Motzkin, 1951.) With W_t as above, $W_t \geqslant W_1$ $(t = 1, \ldots, r-1)$.

Proof by induction on t. The result holds if $t = 1$, so suppose $2 \leqslant t \leqslant r-1$. The number s_j of $(t-2)$-spaces contained in a $(t-1)$-space B_j is at least $k_j = |B_j|$, by the induction hypothesis applied to the set of points in B_j. So if $p_i \notin B_j$ then the number r_i of $(t-1)$-spaces containing p_i satisfies $r_i \geqslant s_j \geqslant k_j$. Let $v := W_1$. Since clearly $r_i > 0$ and $k_j < v$, $\forall\, i, j$, the result follows by Lemma 2.3.1. //

Theorem 2.4. (D. R. Woodall, 1975.) Suppose we are given a finite set $S = \{p_1, \ldots, p_v\}$, positive integers d and $\lambda_2, \ldots, \lambda_d$, and d families of proper subsets of S called *t-blocks* $(t = 1, 2, \ldots, d)$, such that

(i) the 1-blocks are precisely the singletons $\{p_i\}$ of S,

(ii) if $t \geqslant 2$ then, for each $(t-1)$-block B and each $p_i \in S \setminus B$, there are exactly λ_t t-blocks containing $B \cup \{p_i\}$.

Then for each t $(1 \leqslant t \leqslant d)$, the number b_t of t-blocks is $\geqslant v$.

Proof. Fix t and let the t-blocks be $B_1, \ldots, B_{b_t}$. Let A_t be the $v \times b_t$ incidence matrix of points against t-blocks, and let $\hat{A}_t$ be the $(v+1) \times (b_t+1)$ matrix obtained by prepending a row and a column of 1's to A_t. We shall prove by induction on t that the rows of $\hat{A}_t$ are linearly independent, from which it will follow immediately that $b_t + 1 \geqslant v + 1$, or $b_t \geqslant v$, as required.

It is easy to see that the rows of $\hat{A}_1$ are linearly independent, since $\hat{A}_1 = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{bmatrix}$. So suppose $t \geqslant 2$. Suppose the rows of $\hat{A}_t$ are linearly dependent, i.e., $\exists\, a_0, a_1, \ldots, a_v$, not all zero, such that

$$\sum_{i=0}^{v} a_i = 0 \tag{1}$$

and

$$a_0 + \sum_{p_i \in B_j} a_i = 0 \quad (j = 1, \ldots, b_t). \tag{2}$$

Let B be a $(t-1)$-block that is contained in r t-blocks, say. Clearly $r > \lambda_t$ (otherwise, every point outside B would belong to all the t-blocks containing B, which would not then be *proper* subsets of S). Summing (2) over the t-blocks B_j containing B we have

$$ra_0 + \sum_{p_i \in B} ra_i + \sum_{p_i \notin B} \lambda_t a_i = 0.$$

By (1), and since $r > \lambda_t$,

$$a_0 + \sum_{p_i \in B} a_i = 0.$$

This holds for every $(t-1)$-block B, and so, since the rows of $\hat{A}_{t-1}$ are linearly independent by the induction hypothesis, $a_i = 0$ $(i = 0, 1, \ldots, v)$. This contradiction completes the proof. //