A Proportionality Problem

(P. C. Fishburn, F. K. Hwang and H. Lee, 1986; D. R. Woodall, 1992.) Let $\mathcal{R} = (v_1, \ldots, v_n, v_1)$ be a ring of green and blue vertices. If $0 \leqslant l \leqslant r$, let

$$N_{l,r}(v_i) := \{v_{i-l},\ v_{i-l+1},\ \ldots,\ v_{i-1},\ v_{i+1},\ \ldots,\ v_{i+r}\}$$

(subscripts reduced modulo n, repetition allowed). For any set (or multiset) S, let $G(S)$ denote the number of green vertices in S, and $P(S) := G(S)/|S|$. $\mathcal{R}$ has *property* (l, r, p) if $G(\mathcal{R}) > 0$ and $P(N_{l,r}(v)) \geqslant p$ for each green vertex v in $\mathcal{R}$. What does this tell us about $P(\mathcal{R})$?

Example 1. If $\mathcal{R}$ is periodic, l and r are multiples of the period, and $P(\mathcal{R}) = p$, then $P(N_{l,r}(v)) = p$ for *every* vertex in $\mathcal{R}$. Thus property (l, r, p) does not imply $P(\mathcal{R}) > p$ in general.

Example 2. A *simple ring* $\mathcal{R}(a, b)$ consists of a green vertices followed by b blue vertices. If $p \leqslant \frac{l}{r+l}$ then $\mathcal{R}(l+1, n-l-1)$ has property (l, r, p), but $P(\mathcal{R}) = \frac{l+1}{n}$ can be arbitrarily small, and so no conclusion can be drawn (unless $p = 0$).

Example 3. If $\frac{2l}{r+l} < p < \frac{1}{2}$ ($\Longrightarrow\ l < \frac{1}{3}r$) and $p(r+l) \in \mathbb{Z}$, define $a := p(r+l) - l$ and $b := r - l - a$, so that $l < a < \frac{1}{2}(r-l) < b$. In $\mathcal{R}(a, b)$, since $r - l = a + b =$ the period, $N_{l,r}(v) = N_{l,l}(v) \cup (1 \text{ period})$, so that $P(N_{l,r}(v)) \geqslant \frac{l+a}{r+l} = p$ if v is green. Thus $\mathcal{R}(a, b)$ has property (l, r, p), and

$$P(\mathcal{R}(a, b)) = \frac{a}{a+b} = \frac{p(r+l) - l}{r - l}.$$

Conjectures. Suppose $l \leqslant r$ and $\mathcal{R}$ has property (l, r, p).

1. $P(\mathcal{R}) \geqslant p$ if $p > \frac{1}{2}$, or if $p = \frac{1}{2}$ and $l \neq r$.
2. $P(\mathcal{R}) \geqslant \frac{p(r+l)-l}{r-l}$ if $\frac{l}{r+l} < p < \frac{1}{2}$.
3. $P(\mathcal{R}) \geqslant \frac{r-\sqrt{r^2-2p(r^2-l^2)}}{2(r-l)}$ if $\frac{l}{r+l} < p < \frac{1}{2}$.

Note that all bounds $= \frac{1}{2}$ when $p = \frac{1}{2}$, and (2) > (3) iff $\frac{2l}{r+l} < p < \frac{1}{2}$. All three conjectures hold for simple rings.

Theorem 1. Conjecture 1 holds for simple rings.

Proof. Let $\mathcal{R} = \mathcal{R}(a,b)$ be a simple ring with property (l,r,p), where $l \leqslant r$, $p \geqslant \frac{1}{2}$, and $p > \frac{1}{2}$ if $l = r$. Let $v_1, \dots, v_b$ be blue and $v_{b+1}, \dots, v_{b+a} = v_0$ be green. Let

$$r = n(a+b) + r', \qquad l = m(a+b) + l',$$

where $n \geqslant 0$, $m \geqslant 0$, $0 \leqslant r' < a+b$, $0 \leqslant l' < a+b$. If $b \leqslant l' < a+b$ then $v = v_{l'+1}$ is green and $N_{l,r}(v)$ starts with b consecutive blue vertices, so that

$$P(\mathcal{R}) = \frac{a}{a+b} > P(N_{l,r}(v)) \geqslant p.$$

Thus we may suppose that $0 \leqslant l' < b$, and similarly $0 \leqslant r' < b$. Now at least one $v \in \{v_0, v_{b+1}\}$ has the property that $G(N_{l',r'}(v)) \leqslant \min\{r', l'\}$, so that

$$\begin{aligned}\tfrac{1}{2}(r+l) \leqslant p(r+l) \leqslant G(N_{l,r}(v)) &\leqslant (n+m)a + \min\{r', l'\} \\ &\leqslant (n+m)a + \tfrac{1}{2}(r'+l').\end{aligned}$$

If $n = m = 0$, then $r = r'$, $l = l'$, and we have a contradiction since $\min\{r, l\} < \frac{1}{2}(r+l)$ unless $r = l$, when $\frac{1}{2} < p$. So $n + m > 0$ and

$$\begin{aligned}P(\mathcal{R}) &= \frac{a}{a+b} = \frac{(n+m)a}{n(a+b)+m(a+b)} \\ &\geqslant \frac{p(r+l) - \frac{1}{2}(r'+l')}{(r-r')+(l-l')} \geqslant \frac{p(r+l-r'-l')}{r+l-r'-l'} = p. \quad //\end{aligned}$$

Theorem 2. Conjecture 1 holds in the following cases.

(a) $l = 0$ (even if $p < \frac{1}{2}$).
(b) $l = r$ (and $p > \frac{1}{2}$).
(c) $l = r - 1$ (and $p \geqslant \frac{1}{2}$).

Proof. (a) is obvious if $p = 0$. In *all* other cases, $p > \frac{l}{r+l}$, so $G(N_{0,r}(v)) > 0$ for every green vertex v. Let g_1 be any green vertex. Given g_i, let g_{i+1} be the green vertex in $N_{0,r}(g_i)$ furthest from g_i. Continue until the first repetition: $g_k = g_j$ for some $j < k$.

(a) The intervals $(g_i, g_{i+1}]$ ($j \leqslant i \leqslant k-1$) cover $\mathcal{R}$ uniformly, and $P((g_i, g_{i+1}]) \geqslant P(N_{0,r}(g_i)) \geqslant p$ for each i, so that $P(\mathcal{R}) \geqslant p$.

(b) and (c): Define multisets

$$A_i := (g_i, g_{i+1}] + [g_i, g_{i+1}),$$
$$B_i := N_{0,r}(g_i) + N_{l,0}(g_{i+1}),$$
$$A := \sum_{i=j}^{k-1} A_i, \qquad B := \sum_{i=j}^{k-1} B_i = \sum_{i=j}^{k-1} N_{l,r}(g_i)$$

since $g_k = g_j$. A covers $\mathcal{R}$ uniformly, so that $P(A) = P(\mathcal{R})$, and $P(B) \geqslant p$ by hypothesis. It remains to prove that $P(A) \geqslant P(B)$. Let $v_i^{+t} := v_{i+t}$.

(b) Here $l = r$. For any i ($j \leqslant i \leqslant k-1$), suppose that $g_{i+1} = g_i^{+t}$ and let $s := r - t \geqslant 0$, so that $g_{i+1}^{+s} = g_i^{+r}$. A_i is formed from B_i by deleting s blue vertices (in $[g_{i+1}^{+1}, g_{i+1}^{+s}]$) and s vertices of unknown colour (in $[g_i^{-s}, g_i^{-1}]$). This holds for every i. Since at most half the deleted vertices are green, and since $P(B) \geqslant p > \frac{1}{2}$, it follows that $P(A) \geqslant P(B)$ as required.

(c) Now $l = r - 1$. If s (as in (b)) $= 0$, then A_i is formed from B_i by adding one green vertex (g_i). Otherwise, A_i is formed from B_i by deleting s blue vertices and $s - 1$ vertices of unknown colour. Since $s - 1 < s$, the required conclusion follows as before. //

Exercise. $P(\mathcal{R}) \geqslant p$ if and only if, for each green vertex v in $\mathcal{R}$, $\exists\, r(v) > 0$ s.t. $P(N_{0,r(v)}(v)) \geqslant p$.

Theorem 3. (B. J. Tarlow, 1998.) Conjecture 1 holds if $l \leqslant 3$.

Proof. W.l.o.g. $p = \frac{g}{r+l}$ for some $g \in \mathbb{N}$, $g > l$. It suffices to prove that, for each green vertex v in $\mathcal{R}$, $\exists\, l(v) > 0$ s.t. $P(N_{l(v),0}(v)) \geqslant p$. Given v, relabel $\mathcal{R}$ so that $v = v_{r+1}$.

If v_0 is green, then $P([v_{-l}, v_r]) > p$; so assume v_0 is blue.

If v_1 is green, then $P([v_{-l+1}, v_r]) \geqslant p$; so assume v_1 is blue.

Choose $s, t > 0$ minimal such that v_s, v_{-t} are green. Let there be n_{G} green vertices in $[v_2, v_r]$ (or, equivalently, in $[v_s, v_r]$). Then $g \leqslant G(N_{l,r}(v_{-t})) \leqslant l + n_{\mathrm{G}}$, and so $n_{\mathrm{G}} \geqslant g - l$.

Suppose $n_{\mathrm{G}} = g - l$. If $s \leqslant l$ then $N_{l,r}(v_s)$ contains s blue vertices $v_0, \ldots, v_{s-1}$ and the s vertices $v_{r+1}, \ldots, v_{r+s}$, and so $G(N_{l,r}(v_s)) \leqslant (l - s) + (g - l - 1) + s = g - 1$, a contradiction. Thus $s \geqslant l + 1$, and so $P([v_{l+1}, v_r]) = \frac{g-l}{r-l} \geqslant \frac{g}{r+l} = p$ since $\frac{1}{2} \leqslant p$.

Thus we may assume that $n_{\mathrm{G}} \geqslant g - l + 1 \geqslant g - \frac{1}{2}l - \frac{1}{2}$, since $l \leqslant 3$. But then $P([v_2, v_r]) \geqslant \frac{g - \frac{1}{2}l - \frac{1}{2}}{r-1} \geqslant \frac{g}{r+l} = p$. //

Conjecture 1 has been proved also when $l = 4$ and $l = r - 2$, and (by Ben Tarlow) for ‘rings of order 4’.

Exercise. If $p > \frac{l}{r+l}$ and $\mathcal{R}$ has property (l, r, p), then $P(\mathcal{R}) \geqslant \frac{p(r+l)-l}{r}$.