Nonisospectral scattering problems and similarity reductions

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1. Introduction

The realisation that there exists a connection between completely integrable partial differential equations and Painlevé equations dates back to 1977, when Ablowitz and Segur [1] noted, amongst other examples, that the scaling similarity reduction of the modified Korteweg–de Vries (mKdV) equation gives rise to the second Painlevé equation (PII). This then allowed them to use the Inverse Scattering Transform of the mKdV equation to provide a linear integral equation for a special case of PII. Shortly afterwards Airault [2] defined a whole hierarchy of ordinary differential equations (ODEs), a P2 hierarchy, by similarity reduction of the Korteweg–de Vries (KdV) and mKdV hierarchies. However, interest in Painlevé hierarchies did not really take off until nearly twenty years later, when Kudryashov [3] derived both a first and (Airault’s) second Painlevé hierarchy. Since then, there has been a great deal of interest in Painlevé hierarchies and their properties, e.g., [4–27].

An alternative derivation of Painlevé equations, along with their underlying linear problems, was proposed in [28], with the important observation that Painlevé equations arise as stationary reductions of partial differential equations (PDEs) having nonisospectral scattering problems. This idea, and extensions thereof, were used in [5] in order to derive Painlevé hierarchies and their underlying linear problems from hierarchies of PDEs having nonisospectral scattering problems. This approach to deriving Painlevé hierarchies seems more readily applicable than the extension to completely integrable hierarchies of similarity reductions of their first members, a task that is not always so straightforward. Further applications of this approach to deriving Painlevé hierarchies, including discrete Painlevé hierarchies, can be found in [7,14,15,18–20].

In the present paper we show how certain 1 + 1-dimensional nonisospectral hierarchies and their scattering problems can be mapped onto standard PDE hierarchies and their isospectral scattering problems. Special cases of these mappings involve similarity variables, and similarity solutions of a standard hierarchy are seen to correspond to time-independent solutions of an equivalent nonisospectral hierarchy. This then explains why the use of nonisospectral scattering problems and similarity reductions yield the same Painlevé hierarchies. As examples we consider in Section 2 the KdV hierarchy and in Section 3 the dispersive water wave (DWW) hierarchy. Our final section is devoted to conclusions.

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2. The Korteweg–de Vries case

2.1. The Korteweg–de Vries hierarchy

We recall that the KdV hierarchy is a hierarchy of completely integrable PDEs given by

\[ U_t = R^n[U]U_x, \quad R^n[U] = \partial_x^n + 4U + 2U_x\partial_x^{-1}, \]  

where for ease of notation we suppress the usual labelling of the flow times. The recursion operator \( R^n[U] \) is the quotient \( R^n[U] = B_n[U]/B_{n-1}^{-1}[U] \) of the two Hamiltonian operators

\[ B_n[U] = \partial_x^n + 4U\partial_x + 2U_x, \quad B_0[U] = \partial_x. \]

By integrable, we mean that the KdV hierarchy can be solved by the Inverse Scattering Transform (IST), having the corresponding scattering problem

\[ \psi_{ss} + (U - \lambda)\psi = 0, \]

where \( \lambda \) is the (constant) spectral parameter.

Let us also recall for future use the following Lemma [29].

**Lemma 1.** The change of variables \( \tilde{U} = U + C \), where \( C \) is an arbitrary constant, in \( R^n[\tilde{U}]\tilde{U}_x \), yields

\[ R^n[U]U_x = \sum_{j=0}^{n} \alpha_{nj} C^{n-j} R^n[U] U_x, \]

where the coefficients \( \alpha_{nj} \) are determined recursively by

\[ \alpha_{n,n} = 1, \]

\[ \alpha_{nj} = 4\alpha_{n-1,j} + \alpha_{n-1,j-1}, \quad j = 1, \ldots, n-1, \]

\[ \alpha_{n,0} = \frac{4n+2}{n} \alpha_{n-1,0} \]

and where \( \alpha_{0,0} = 1 \).

We now consider the relationship between a nonisospectral KdV hierarchy, i.e., having a scattering problem where the spectral parameter is no longer constant, and a standard (isospectral) KdV hierarchy.

2.2. A nonisospectral Korteweg–de Vries hierarchy

We consider the nonisospectral hierarchy

\[ U_t = R^n[u]u_x + \sum_{j=0}^{n-1} B_j[U] R^n[u] u_x + g_{n-1}(t)(4u + 2\xi u_x) + g_n(t), \]  

where \( R^n[u] \) is given by \( R^n[U] \) in (2.1) with \( U \) replaced by \( u \) and \( \partial_t \) by \( \partial_t \). This hierarchy is a special case of a 2 + 1-dimensional hierarchy considered in [5], and can also be found in [30]. This equation is of interest from the point of view of Painlevé hierarchies since its stationary reduction, where \( g_{n-1}(t), g_n(t) \) and all \( B_j(t) \) are now constant, yields the equation

\[ R^n[u]u_x + \sum_{j=0}^{n-1} B_j R^n[u] u_x + g_{n-1}(t)(4u + 2\xi u_x) + g_n(t) = 0. \]

This last equation is equivalent to Eq. (3.29) in [5] and gives rise to a generalized thirty-fourth Painlevé \((P_{34})\) hierarchy and to a generalized first Painlevé \((P_1)\) hierarchy. These hierarchies arise in the cases \( g_{n-1} \neq 0 \) (when we can also assume \( g_n = 0 \)) and \( g_{n-1} = 0 \) respectively. Generalized and non-generalized \( P_{34} \) and \( P_1 \) hierarchies can be found in [3–5,9,11].

We now show that Eq. (2.8) can in fact be mapped to an isospectral KdV hierarchy generalized by the inclusion of lower-order flows.

**Proposition 1.** The nonisospectral hierarchy (2.8) is equivalent to the isospectral KdV hierarchy

\[ U_t = R^n[U]U_x + \sum_{i=1}^{n-1} \beta_i(t) R^i[U] U_x \]

under a change of variables of the form

\[ U = g(t)^2 u(\xi, \tau) + m(t), \quad \xi = g(t)x + h(t), \quad \tau = k(t). \]
Using Lemma 1 we see that substituting (2.11) in (2.10) gives
\[
\sum_{p=0}^{n} \gamma_p(t) \partial^2_p[u - g(t)g'(t)(2u + \zeta u)] - m'(t) - g(t)^2 \partial^2_0[u - g(t)g'(t)(2u + \zeta u)] = 0.
\]
We recall that each \(a_i = 1\), and so in particular \(\gamma_n(t) = g(t)^{2n+3}\).
We solve the equations
\[
\gamma_p(t) = B_p(k(t))g(t)^{2n+3}, \quad p = n - 1, \ldots, 1
\]
recursively for the coefficients \(\beta_p(t)\) and the equation
\[
\gamma_0(t) = B_0(k(t))g(t)^{2n+3}
\]
for \(h(t)\), where \(g(t), m(t)\) and \(k(t)\) are determined by the equations
\[
g'(t) = -2g_{n-1}(k(t))g(t)^{2n+2},
\]
\[
m'(t) = -g_n(k(t))g(t)^{2n+3},
\]
\[
k'(t) = g(t)^{2n+1}.
\]
Cancelling an overall factor of \(g(t)^{2n+3}\) in (2.12) and writing the coefficients as functions of \(\tau\) then gives (2.8).

The mapping between the scattering problems is given by the following:

**Proposition 2.** Extending the change of variables (2.11) with
\[
\psi(x, t) = \varphi(\xi, \tau), \quad \mu(k(t)) = \frac{\lambda - m(t)}{g(t)^2}
\]
the scattering problem (2.3) is transformed into the nonisospectral scattering problem
\[
\varphi_{\xi\xi} + [u - \mu(\tau)]\varphi = 0
\]
where \(\mu(\tau)\) satisfies the nonisospectral condition
\[
\frac{d\mu}{d\tau} = 4g_{n-1}(\tau)\mu(\tau) + g_n(\tau).
\]

**Proof.** Eq. (2.19) follows easily from the substitution of (2.11) and (2.18) in (2.3). Eq. (2.20) follows from differentiating the second equation of (2.18) with respect to \(t\) to obtain
\[
\left[\frac{d\mu}{d\tau}(k(t))\right]k'(t) = -2\left[\frac{\lambda - m(t)g'(t)}{g(t)^2}\right]m'(t) \quad g(t)^2
\]
and then using Eqs. (2.15)–(2.17) and the second equation of (2.18), and writing the result in terms of \(\tau\).

### 2.3. Similarity reductions of the Korteweg–de Vries hierarchy

As mentioned in the Introduction, one method of deriving Painlevé hierarchies is as similarity reductions of completely integrable PDE hierarchies. However this process is not always so straightforward. For example, it is only recently that the accelerating-wave reduction and generalized scaling reduction of the KdV equation have been extended to the KdV hierarchy; see [29] and [31] respectively. Here we consider the relationship between the derivation of Painlevé hierarchies using nonisospectral scattering problems and using similarity reductions; it turns out that the key to understanding this relationship lies in the transformation (2.11). This then allows us to explain why the same Painlevé hierarchies are found using these two techniques.

We consider two special cases of (2.8) such that the stationary reduction (2.9) gives rise to a generalized \(P_1\) hierarchy or a generalized \(P_n\) hierarchy. The two cases that we consider have all \(B_0(\tau)\) constant, and in addition \(g_{n-1}(\tau) = 0\) and \(g_n(\tau)\) constant, and \(g_n(\tau) = 0\) and \(g_{n-1}(\tau)\) constant, respectively, and are both nonisospectral equations.
Corollary 1. For the special case of Eq. (2.10) and transformation (2.11) having \( g(t) = 1 \), \( m(t) = -g_nt \) (\( g_n \) constant) \( k(t) = t \) and \( h(t) \) and all \( \beta_k(t) \) such that all \( B_k(t) \), \( k = 0, 1, \ldots n - 1 \) are constant, the corresponding equivalent nonisospectral equation is

\[
\frac{u_t}{R^n[u]} = \frac{n-1}{\sum_{j=0}^{n-1} B_j R^j[u] u_t} + g_n.
\]

Proof. A simple substitution in Proposition 1. We note in particular that since \( g(t) \) is constant, \( g_{n-1}(t) = 0 \) (see (2.15)).

Remark 1. The stationary reduction of (2.22) yields (2.9) with \( g_{n-1} = 0 \),

\[
\frac{u_t}{R^n[u]} + \frac{n-1}{\sum_{j=0}^{n-1} B_j R^j[u] u_t} = 0
\]

and the substitution used to obtain this last from the corresponding case of (2.10), that is, the special case of the transformation (2.11) given in Corollary 1 but now with \( u \) a function of \( \xi \) only, then corresponds precisely to the similarity reduction of this case of (2.10) given in [29] which yields the generalized \( P_1 \) hierarchy, this last being obtained by integrating (2.23) (to give for example (2.13) in [29]). That is, the change of variables of Corollary 1 explains why the stationary reduction of the nonisospectral hierarchy (2.22) can also be obtained as a similarity reduction of the corresponding case of the standard hierarchy (2.10). Both techniques then yield the same Painlevé hierarchy. It is interesting to note that these similarity solutions of this case of the hierarchy (2.10) correspond to time-independent solutions of the equivalent hierarchy (2.22). These results represent a generalization to the case of hierarchies—a generalization which relies on Lemma 1 published recently in [29]—of the special case \( n = 1 \) considered in [28], where it was shown that the accelerating-wave reduction of the KdV equation, which gives \( P_1 \), could be extended to give a nonisospectral equation having \( P_1 \) as stationary reduction. Similarly, Proposition 1 represents an extension to the case of hierarchies of the results in [32–34], where generalizations and special cases of (2.8) for \( n = 1 \) were considered along with the question of mappings to the KdV equation; again, this generalization relies on Lemma 1 in order to precisely formulate Eqs. (2.13) and (2.14). We recall that from the point of view of Painlevé hierarchies, the principal interest of the hierarchy (2.8) is its reduction to (2.9), and that for this last there is no mapping from the case \( g_{n-1} = g_n \) arbitrary to the case \( g_{n-1} = g_n = 0 \).

Corollary 2. For the special case of Eq. (2.10) and transformation (2.11) having \( g(t) = 1/|2(2n + 1)g_{n-1}|^{1/(2n-1)} \) (\( g_{n-1} \) constant), \( m(t) = d \) (constant), \( k(t) = \log(t)/|2(2n + 1)g_{n-1}| \) and \( h(t) \) and all \( \beta_k(t) \) such that all \( B_k(\tau) \), \( k = 0, 1, \ldots n - 1 \) are constant, the corresponding equivalent nonisospectral equation is

\[
\frac{u_t}{R^n[u]} = \frac{n-1}{\sum_{j=0}^{n-1} B_j R^j[u] u_t} + g_{n-1}(4u + 2\zeta u_t).
\]

Proof. A simple substitution in Proposition 1. We note in particular that since \( m(t) \) is constant, \( g_n(\tau) = 0 \) (see (2.16)).

Remark 2. The stationary reduction of (2.24) yields (2.9) with \( g_n = 0 \),

\[
\frac{u_t}{R^n[u]} + \frac{n-1}{\sum_{j=0}^{n-1} B_j R^j[u] u_t} = 0
\]

and the substitution used to obtain this last from the corresponding case of (2.10), that is, the special case of the transformation (2.11) given in Corollary 2 but now with \( u \) a function of \( \xi \) only, then corresponds precisely to the generalized scaling reduction of this case of (2.10) given in [31] which yields the generalized \( P_{14} \) hierarchy, this last being obtained as a first integral of (2.25) (and given for example as (2.10) in [31]). That is, the change of variables of Corollary 2, which involves scaling similarity variables, explains why the stationary reduction of the nonisospectral hierarchy (2.24) can also be obtained as a generalized scaling reduction of the corresponding case of the standard hierarchy (2.10). Both techniques then yield the same Painlevé hierarchy. It is interesting to note that scaling similarity solutions of this case of the hierarchy (2.10) correspond to time-independent solutions of the equivalent hierarchy (2.24).

3. The dispersive water wave case

3.1. The dispersive water wave hierarchy

The DWW hierarchy is a two-component completely integrable hierarchy in \( U = (U, V)^T \) given by [35]
where once again for ease of notation we suppress the usual labelling of the flow times. The recursion operator $\mathcal{R}$ is the quotient $\mathcal{R}[\mathbf{U}] = \mathcal{B}_2[\mathbf{U}]\mathcal{B}_1^{-1}[\mathbf{U}]$ of the two Hamiltonian operators

$$
\mathcal{B}_2[\mathbf{U}] = \frac{1}{2} \left( \frac{2\partial_x}{U\partial_x + \partial_x^2} \frac{\partial_x U - \partial_x^2}{V\partial_x + \partial_x V} \right),
$$

(3.2)

and

$$
\mathcal{B}_1[\mathbf{U}] = \left( \begin{array}{c} 0 \\ \partial_x \end{array} \right).
$$

(3.3)

As is well-known, the DWW hierarchy has an energy-dependent scattering problem,

$$
\psi_{xx} = \left[ \left( \lambda - \frac{1}{2} U \right)^2 + \frac{1}{2} U_x - V \right] \psi,
$$

(3.4)

where $\lambda$ is the spectral parameter.

We recall for future use the following Lemma [29].

**Lemma 2.** The change of variables $\mathbf{U} = (U + C, V)^T$, where $C$ is an arbitrary constant, in $\mathcal{R}^n[\mathbf{U}]\mathbf{U}_x$, yields

$$
\mathcal{R}^n[\mathbf{U}]\mathbf{U}_x = \sum_{j=0}^n \mathcal{B}_j(t)\mathcal{R}^j[\mathbf{U}]\mathbf{U}_x,
$$

(3.5)

where the coefficients $\mathcal{B}_j$ are determined recursively by

$$
\mathcal{B}_0 = 1,
$$

(3.6)

$$
\mathcal{B}_j = \frac{1}{2} \mathcal{B}_{j-1} + \mathcal{B}_{n-j-1}, \quad j = 1, \ldots, n-1,
$$

(3.7)

$$
\mathcal{B}_n = \frac{1}{2} \left( \frac{n+1}{n} \right) \mathcal{B}_{n-1}
$$

(3.8)

and where $\mathcal{B}_0 = 1$.

We now turn to the relationship between a nonisospectral DWW hierarchy and a standard DWW hierarchy.

### 3.2. A nonisospectral dispersive water wave hierarchy

We consider the nonisospectral DWW hierarchy in $\mathbf{u} = (u, v)^T$,

$$
\mathbf{u}_t = \mathcal{R}^n[\mathbf{u}]\mathbf{u}_x + \sum_{j=0}^{n-1} \mathcal{B}_j(t)\mathcal{R}^j[\mathbf{u}]\mathbf{u}_x + \frac{1}{2} \mathcal{B}_n(t) \left( \frac{\xi u}{2v + \xi v} \right) + \mathcal{B}_{n+1}(t) \left( \begin{array}{c} 1 \\ 0 \end{array} \right),
$$

(3.9)

where $\mathcal{R}[\mathbf{u}]$ is given by $\mathcal{R}[\mathbf{U}]$ in (3.1) with $\mathbf{U}$ replaced by $\mathbf{u}$ and $\partial_x$ by $\partial_t$. This hierarchy is a special case of a hierarchy in $2 + 1$ dimensions considered in [8] and is of interest from the point of view of the derivation of Painlevé hierarchies since its stationary reduction

$$
\mathcal{R}^n[\mathbf{u}]\mathbf{u}_t + \sum_{j=0}^{n-1} \mathcal{B}_j(t)\mathcal{R}^j[\mathbf{u}]\mathbf{u}_x + \frac{1}{2} \mathcal{B}_n(t) \left( \frac{\xi u}{2v + \xi v} \right) + \mathcal{B}_{n+1}(t) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right)
$$

(3.10)

—where $\mathcal{B}_n(t)$, $\mathcal{B}_{n+1}(t)$ and all $\mathcal{B}_j(t)$ are now constant—gives rise to a $P_n$ hierarchy and to a $P_{nv}$ hierarchy. This last equation is equivalent to the special case $g_{n-1} = 0$ of (53) in [8]. The $P_n$ hierarchy and $P_{nv}$ hierarchy correspond respectively to the cases $g_n = 0$ and $g_0 \neq 0$ (when we may also assume $g_{n-1} = 0$). For $P_n$ and $P_{nv}$ hierarchies see [8,10–12,16,17].

We now show that Eq. (3.9) can be mapped to an isospectral DWW hierarchy generalized by the inclusion of lower order flows.

**Proposition 3.** The nonisospectral hierarchy (3.9) is equivalent to the isospectral DWW hierarchy

$$
\mathbf{u}_t = \mathcal{R}^n[\mathbf{U}]\mathbf{U}_x + \sum_{j=1}^{n-1} \mathcal{Q}_j(t)\mathcal{R}^j[\mathbf{U}]\mathbf{U}_x
$$

(3.11)

under a change of variables of the form


\[ U = g(t)u(\xi, \tau) + m(t), \quad V = g(t)^2 v(\xi, \tau), \quad \xi = g(t)x + h(t), \quad \tau = k(t). \quad (3.12) \]

**Proof.** Using Lemma 2 we see that substituting (3.12) in (3.11) gives

\[
\sum_{p=0}^{n} \Gamma_p(t) R^p [u, u] = - \left( g(t)^2 \frac{\partial^2}{\partial t^2} + g(t)^2 \frac{\partial}{\partial t} \right) \left( \begin{array}{c} g(t) \psi'(t) \\ g(t) \psi(t) \end{array} \right) - \left( \begin{array}{c} g(t)^2 \psi''(t) \\ g(t)^2 \psi'(t) \end{array} \right) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) - \left( \begin{array}{c} g(t)^2 \psi''(t) \\ g(t)^2 \psi'(t) \end{array} \right) \left( \begin{array}{c} g(t) \psi'(t) \\ g(t) \psi(t) \end{array} \right) \left( \begin{array}{c} 0 \\ 0 \end{array} \right),
\]

where

\[
\Gamma_p(t) = \left( \begin{array}{cc} g(t)^p & 0 \\ 0 & g(t)^{p+1} \end{array} \right). \quad (3.14)
\]

We recall that each \( \alpha_i = 1 \), and so in particular \( \Gamma_{n-1}(t) = G_{n-1}(t) \).

We solve the equations

\[
\Gamma_p(t) = B_p(k(t)) G_{n-2}(t), \quad p = n-2, \ldots, 1 \quad (3.15)
\]

recursively for the coefficients \( \gamma_p(t) \) and the equation

\[
\Gamma_0(t) = B_0(k(t)) G_{n-2}(t) \quad (3.16)
\]

for \( h(t) \), where \( g(t), m(t) \) and \( k(t) \) are determined by the equations

\[
g'(t) = -\frac{1}{2} g_n(k(t)) g(t)^{n-2}, \quad (3.17)
\]

\[
m'(t) = -g_{n-1}(k(t)) g(t)^{n-2}, \quad (3.18)
\]

\[
k'(t) = g(t)^{n-1}. \quad (3.19)
\]

Multiplying by the inverse of \( G_{n-2}(t) \) in (3.13) and writing the coefficients as functions of \( \tau \) then gives (3.9).

As in the KdV case, we can give the mapping between the corresponding isospectral and nonisospectral scattering problems:

**Proposition 4.** Extending the change of variables (3.12) with

\[
\psi(x, t) = \varphi(\xi, \tau), \quad \mu(k(t)) = \frac{2\dot{\varphi} - m(t)}{2g(t)}, \quad (3.20)
\]

the scattering problem (3.4) is transformed into the nonisospectral scattering problem

\[
\varphi_{\xi \xi} = \left( \mu(\tau) - \frac{1}{2} \dot{\mu} \right)^2 + \frac{1}{2} u^2 - v \varphi \quad (3.21)
\]

where \( \mu(\tau) \) satisfies the nonisospectral condition

\[
\frac{d\mu}{d\tau} = \frac{1}{2} g_n(\tau) \mu(\tau) + \frac{1}{2} g_{n+1}(\tau). \quad (3.22)
\]

**Proof.** Eq. (3.21) follows easily from the substitution of (3.12) and (3.20) in (3.4). Eq. (3.22) follows from differentiating the second equation of (3.20) with respect to \( t \) to obtain

\[
\frac{d\mu}{d\tau} \left( k(t) \right) = -\frac{2\dot{\mu} - m(t)}{2g(t)} g'(t) - \frac{1}{2} \frac{m'(t)}{g(t)} \quad (3.23)
\]

and then using Eqs. (3.17)–(3.19) and the second equation of (3.20), and writing the result in terms of \( \tau \).

### 3.3. Similarity reductions of the dispersive water wave hierarchy

We now consider once again the relationship between the derivation of Painlevé hierarchies using nonisospectral scattering problems and using similarity reductions; we take as examples the extensions to the DWW hierarchy of the
accelerating-wave reduction and generalized scaling reduction of the DWW equation, given in [29] and [31] respectively. Similarly to the KdV case, it is the change of variables (3.12) that provides an understanding of this relationship, and allows us to explain why the same Painlevé hierarchies are found using these two techniques.

We consider two special cases of (3.9) such that the stationary reduction (3.10) gives rise to a $P_k$ hierarchy or a $P_m$ hierarchy. The two cases that we consider have all $B_k(\tau)$ constant, and in addition $g_n(\tau) = 0$ and $g_{n+1}(\tau)$ constant, and $g_{n+2}(\tau)$ constant, respectively, and are both nonisospectral equations.

Corollary 3. For the special case of Eq. (3.11) and transformation (3.12) having $g(t) = 1, m(t) = -g_{n+1}t$ ($g_{n+1}$ constant), $k(t) = t$ and $h(t)$ and all $\gamma_k(t)$ such that all $B_k(\tau), k = 0, 1, \ldots n - 1$ are constant, the corresponding equivalent nonisospectral equation is

$$u_\tau = R^n[u]u_\tau + \sum_{j=0}^{n-1} B_j R^j[u]u_\tau + g_{n+1} \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right).$$

(3.24)

Proof. A simple substitution in Proposition 3. We note in particular that since $g(t)$ is constant, $g_n(0) = 0$ (see (3.17)).

Remark 3. We make similar remarks to those made in the KdV case. The stationary reduction of (3.24) yields (3.10) with $g_n = 0,

$$R^n[u]u_\tau + \sum_{j=0}^{n-1} B_j R^j[u]u_\tau + g_{n+1} \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right)$$

(3.25)

and the substitution used to obtain this last from the corresponding case of (3.11), that is, the special case of the transformation (3.12) given in Corollary 3 but now with $u$ and $v$ functions of $\xi$ only, then corresponds precisely to the similarity reduction of this case of (3.11) given in [29] which yields a $P_0$ hierarchy (this last being obtained by integrating each equation of (3.25) to obtain for example (3.7) in [29]). Thus the change of variables of Corollary 3 explains why the stationary reduction of the nonisospectral hierarchy (3.24) can also be obtained as a similarity reduction of the corresponding case of the standard hierarchy (3.11). Both techniques then yield the same Painlevé hierarchy. Again we note that these similarity solutions of this case of the hierarchy (3.11) correspond to time-independent solutions of the equivalent hierarchy (3.24).

Corollary 4. For the special case of Eq. (3.11) and transformation (3.12) having $g(t) = 1/\sqrt{(n+1)g_n t}^{1/(n+1)}$ ($g_n$ constant), $m(t) = d$ (constant), $k(t) = \log(t)/(2(n+1)g_n)$ and $h(t)$ and all $\gamma_k(t)$ such that all $B_k(\tau), k = 0, 1, \ldots n - 1$ are constant, the corresponding equivalent nonisospectral equation is

$$u_\tau = R^n[u]u_\tau + \sum_{j=0}^{n-1} B_j R^j[u]u_\tau + g_{n+1} \left( \begin{array}{c} (\xi u)_\xi \\ 2 v + \xi \nu \xi \end{array} \right).$$

(3.26)

Proof. A simple substitution in Proposition 3. We note in particular that since $m(t)$ is constant, $g_{n+1}(\tau) = 0$ (see (3.18)).

Remark 4. Again we make remarks similar to those made in the KdV case. The stationary reduction of (3.26) yields (3.10) with $g_{n+1} = 0,

$$R^n[u]u_\tau + \sum_{j=0}^{n-1} B_j R^j[u]u_\tau + g_{n+1} \left( \begin{array}{c} (\xi u)_\xi \\ 2 v + \xi \nu \xi \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right)$$

(3.27)

and the substitution used to obtain this last from the corresponding case of (3.11), that is, the special case of the transformation (3.12) given in Corollary 4 but now with $u$ and $v$ functions of $\xi$ only, then corresponds precisely to the generalized scaling reduction of this case of (3.11) given in [31] which yields a $P_m$ hierarchy, this last being obtained as a first integral of (3.27) (and given for example as (3.12) and (3.13) in [31]). That is, the change of variables of Corollary 4, which involves scaling similarity variables, explains why the stationary reduction of the nonisospectral hierarchy (3.26) can also be obtained as a generalized scaling reduction of the corresponding case of the standard hierarchy (3.11). Both techniques then yield the same Painlevé hierarchy. It is interesting to note that scaling similarity solutions of this case of the hierarchy (3.11) correspond to time-independent solutions of the equivalent hierarchy (3.26).

4. Conclusions

We have shown that certain 1 + 1-dimensional nonisospectral hierarchies and their scattering problems can be mapped onto standard PDE hierarchies and their isospectral scattering problems. Special cases of these transformations involve similarity variables, and time-independent solutions of corresponding nonisospectral hierarchies are then seen to correspond to
similarity reductions of standard hierarchies. This explains why the use of nonisospectral scattering problems and similarity reductions yield the same Painlevé hierarchies. For example, in the KdV case, we have seen why the ODE hierarchies (2.23) and (2.25), which may be derived as stationary reductions of nonisospectral hierarchies and which yield a generalized P1 hierarchy and a generalized P3 hierarchy respectively, are precisely those obtained in [29] and [31] by respectively extending the accelerating-wave and generalized scaling reductions of the KdV equation to corresponding standard (isospectral) hierarchies. Similarly, in the DWW case, we have seen why the ODE hierarchies (3.25) and (3.27), which may be derived as stationary reductions of nonisospectral hierarchies and which yield a P3 hierarchy and a P5 hierarchy respectively, are precisely those obtained in [29] and [31] by respectively extending the accelerating-wave and generalized scaling reductions of the DWW equation to corresponding standard (isospectral) hierarchies. Further examples of mappings between nonisospectral and isospectral scattering problems, and of special cases of such transformations related to similarity reduction, will be considered in future papers.

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