5 Gelfand's problem revisited, mathematical curiosities and some open problems

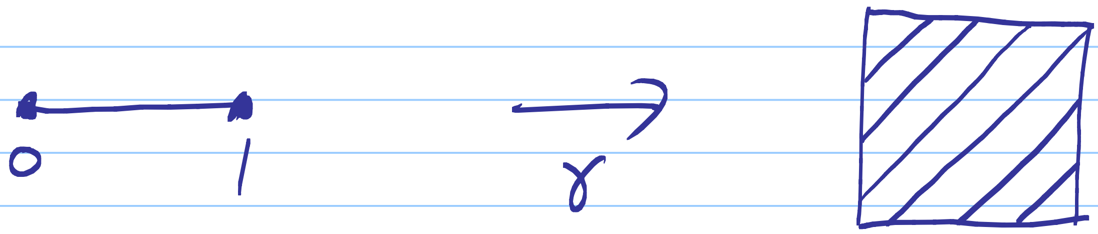
5.1 Space-filling curves and Jordan arcs with positive area

In order to give examples of non-trivial uniform algebras on the interval, we shall need the fact that there are Jordan arcs in $\mathbb{C}$ which have positive area.

The first such arcs were constructed by Osgood in 1903.

These arcs are not, of course, space-filling curves, but the construction is similar.

$\exists$ cts γ from $[0,1]$ onto closed square in $\mathbb{C}$.

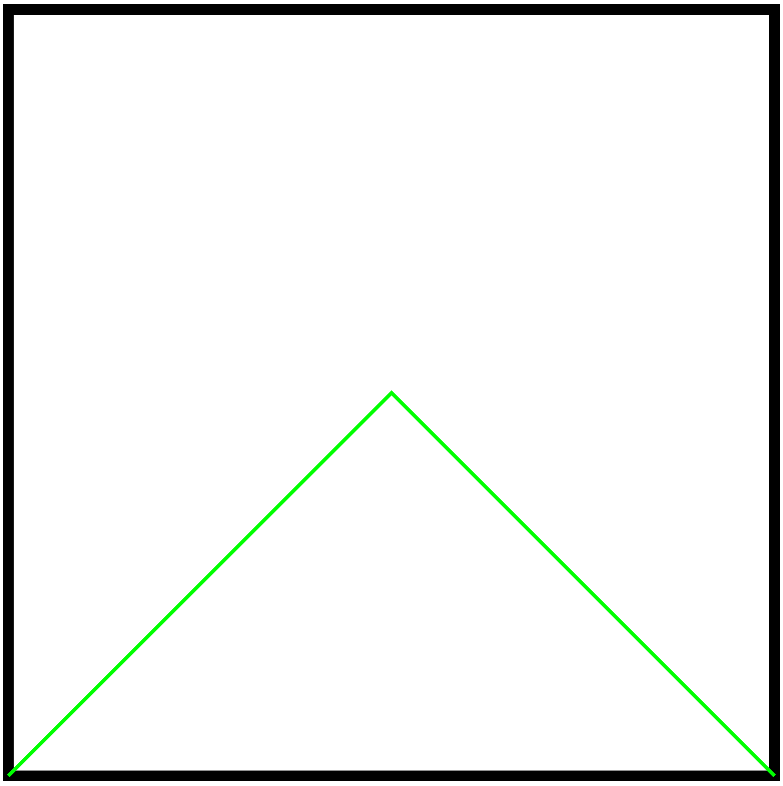


Space-filling, "Peano curve".

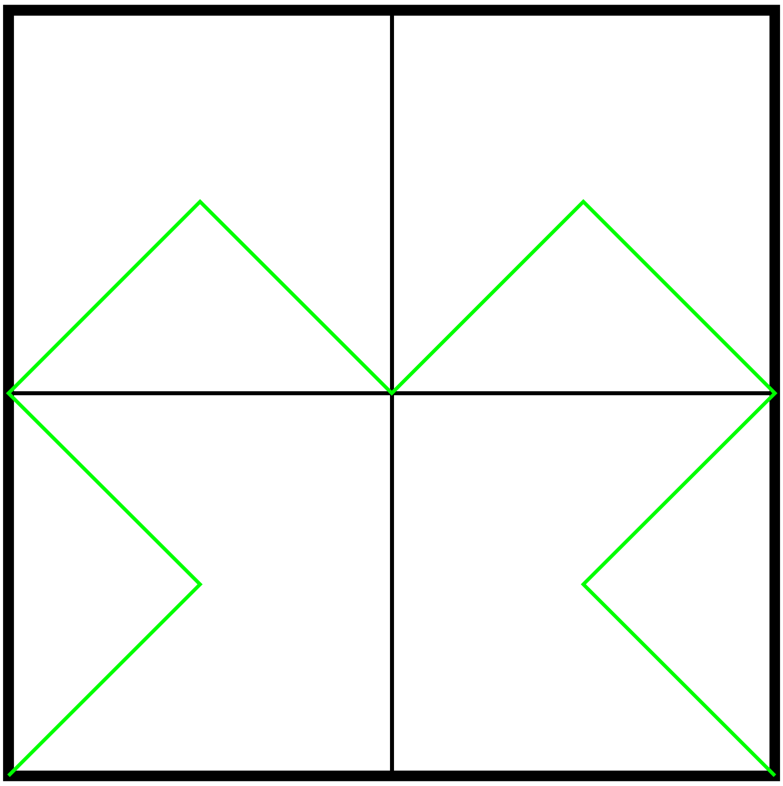
Can't do this if γ is 1-1, because then γ is a homeomorphism. Jordan arcs in $\mathbb{C}$ are homeomorphic to $[0,1]$, so NO INTERIOR.

First let us recall how we can obtain space-filling curves, using the following pictures.

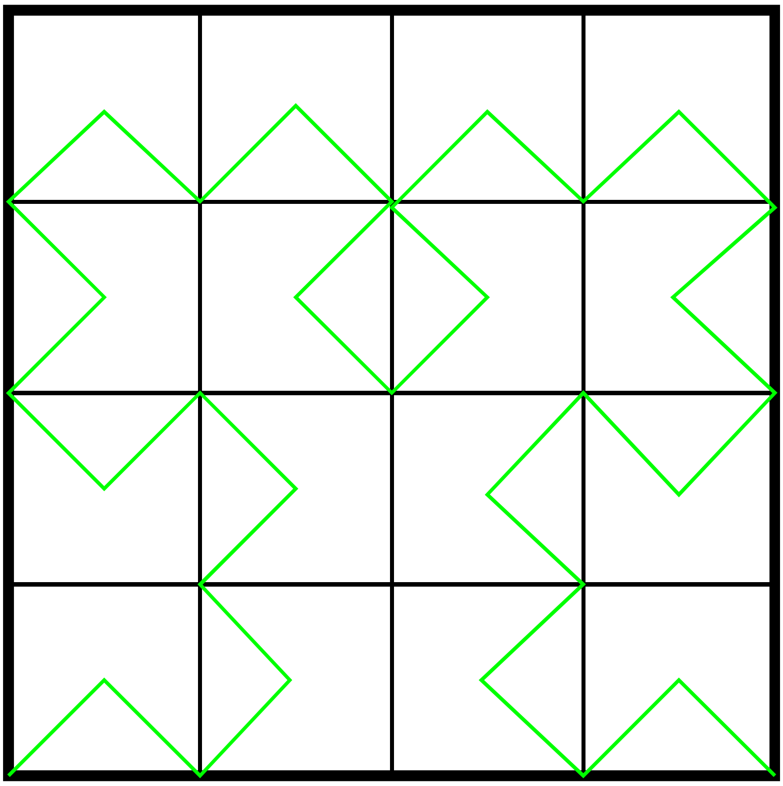
γ_1



γ_2



γ_3



Gap to fill in

γ_n defined this way

$[0,1] \to$ square, "constant speed"

entering bottom left, leaving bottom right.

Note $\gamma_n : [0,1] \to \mathbb{C}$ cts,

so $\gamma_n \in C[0,1]$.

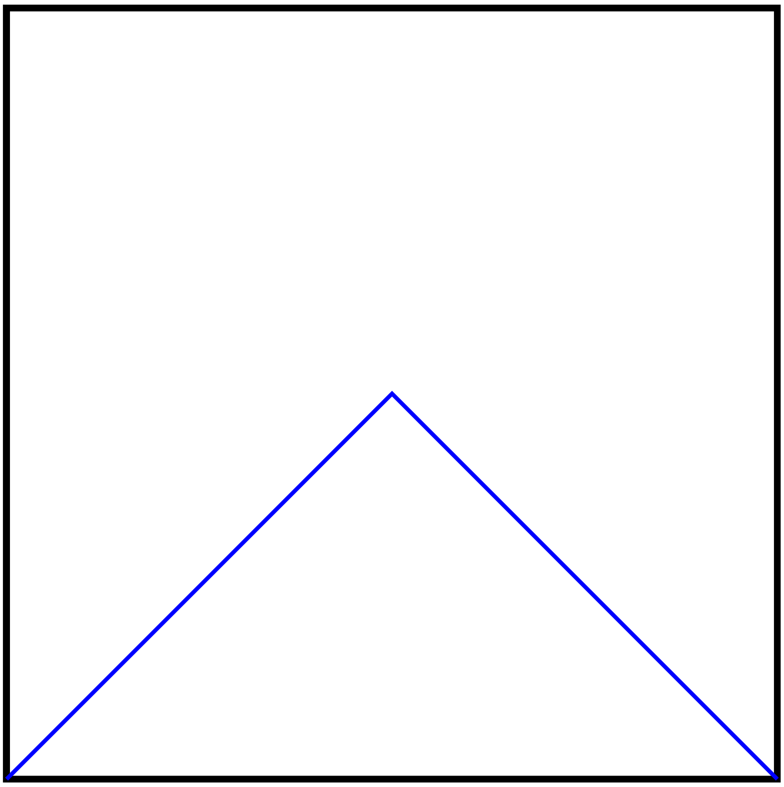
Easy exercise:

$$\|\gamma_{n+1} - \gamma_n\|_\infty \le C/2^n$$

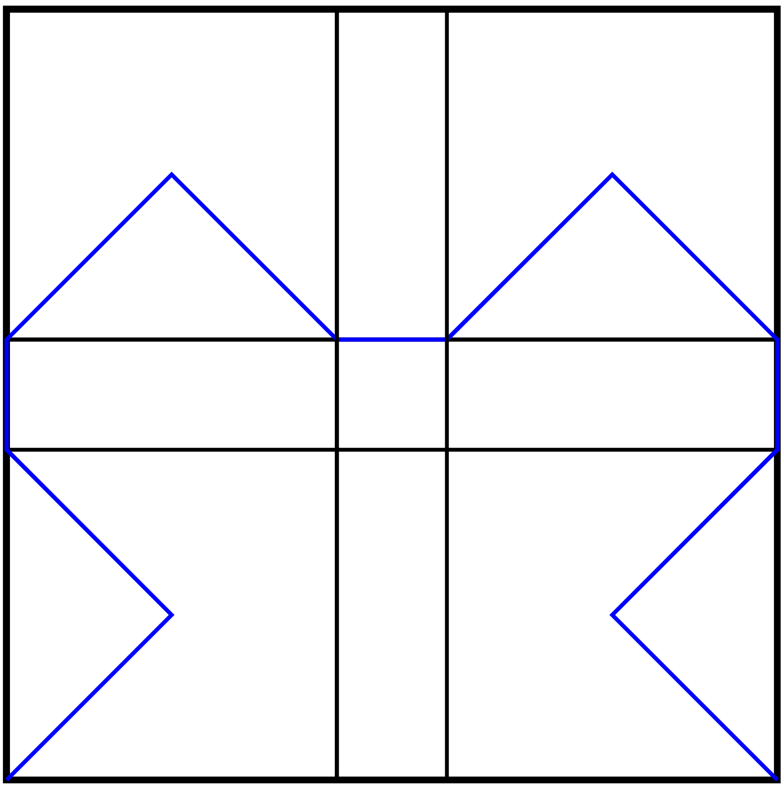
so $\sum_1 \|\gamma_{n+1} - \gamma_n\|_\infty < \infty$, and γ_n converge uniformly to some curve $\gamma : [0,1] \to \mathbb{C}$. Easy to see $\gamma([0,1])$ includes elements of <u>all</u> subsquares that turn up, so, since $\gamma([0,1])$ is compact, $\gamma([0,1])$ is the whole closed square.

A slight modification of this construction produces arcs with positive area.

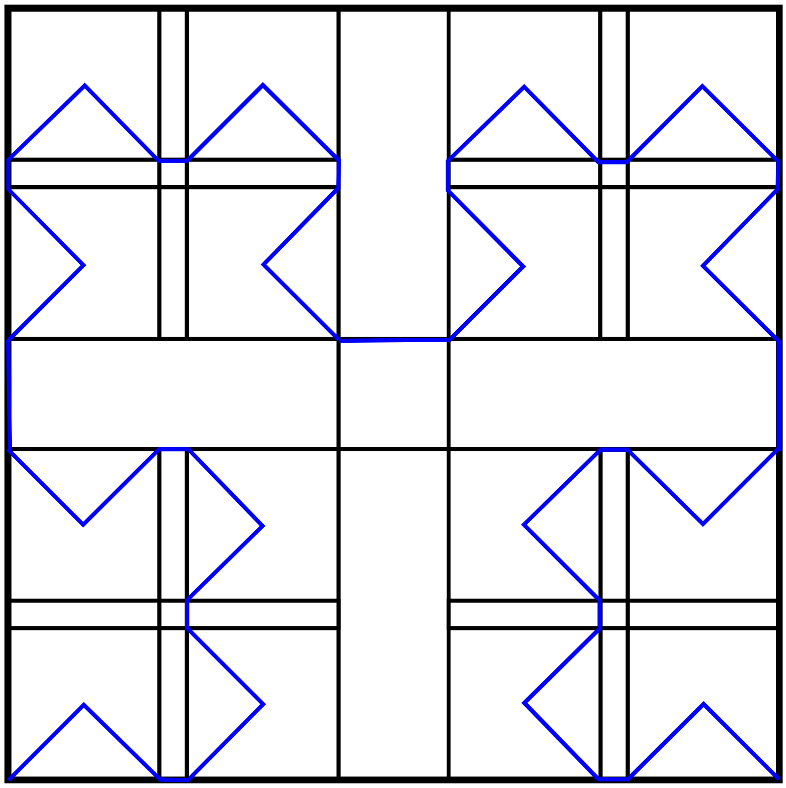
γ_1



γ_2



γ_3



Gap to fill in

Again γ_n converges uniformly to some continuous $\gamma : [0,1] \to \mathbb{C}$.

But this time γ is 1-1. Also

$$\gamma([0,1]) \supseteq C \times C$$

for (suitably chosen) Cantor set C of positive length.

We can now describe some non-trivial uniform algebras on the unit interval.

These examples are not, however, natural on $[0, 1]$

Gap to fill in

Let J be a Jordan arc in $\mathbb{C}$ with positive area.

Set $A =$

$\{ f \in C(J) \mid f$ extends to be cts on $\mathbb{C}_\infty$, and analytic off $J \}$

$= \{ f \in C(J) \mid f$ extends to be bdd, cts on $\mathbb{C}$, and analytic on $\mathbb{C} \setminus J \}$

Note. Define

$$f(z) = \int_J \frac{1}{w-z} \, dA(w) \quad \text{area}.$$

f is in A and

$z f(z) \to -$ area of J

as $z \to \infty$.

So f and zf are in A.

(etc.)

Then $A|_J$ turns out to be a uniform algebra on J, extensions are unique, and for all $z \in \mathbb{C}_\infty \setminus J$ and $f \in A$,

$f(z) \in f(J)$.

For full details, see Stout's book. [Or Gamelin, "Uniform algebras".]

Φ_A turns out to be $\mathbb{C}_\infty$ instead of J.

Since J is homeomorphic to $[0,1]$, this gives us non-trivial uniform algebras on $[0,1]$.

[See also John Wermer's paper in Advances in Mathematics, Volume 1.]

5.2 Progress on Gelfand's problem

We now discuss some of the results related to Gelfand's problem.

Proposition 5.2.1 (D. Wilken, 1969) The only strongly regular uniform algebra on $[0, 1]$ is $C[0, 1]$.

This result was strengthened somewhat by Feinstein and Somerset ('Strong regularity for uniform algebras', 1998), but it is an open question whether or not every (natural) regular uniform algebra on $[0, 1]$ is trivial.

Note that Wilken showed in 1965 that every natural uniform algebra on $[0, 1]$ is 'approximately normal'.

Gap to fill in

See also Gamelin's book on uniform algebras, Rossi's local maximum modulus theorem.

A rather different attack on the problem may be found in the paper of Dawson and Feinstein (2003).

Proposition 5.2.2 Let A be a natural uniform algebra on $[0,1]$.

Suppose that $\mathrm{Inv}\, A$ is dense in A. Then $A = C[0,1]$.

Here the condition that $\mathrm{Inv}\, A$ is dense in A says that A has **topological stable rank** equal to 1.

Gap to fill in

Čirka's Theorem:

Let A be uniform algebra on a locally connected, compact space X. Suppose that $\{f^2 \mid f \in A\}$ is dense in A. Then $A = C(X)$.

Proof of 5.2.2

Given A as in the statement, we show $\{f^2 \mid f \in A\}$ is dense in A. The result then follows from Čirka's Theorem.

By Arens-Royden theorem, $\mathrm{Inv}(A) = \exp(A)$.

Thus $\{f^2 \mid f \in A\} \supseteq \mathrm{Inv}(A)$.

$\square$

In the case where A is finitely generated as a Banach algebra, Gelfand's problem comes down to a question about $P(J)$ for polynomially convex arcs J in C^N.

Under some fairly mild conditions on the polynomially convex arc you can see that $\mathrm{Inv}\, P(J)$ must be dense in $P(J)$, and so $P(J) = C(J)$.

However, even in this setting, the solution to Gelfand's problem remains elusive.

Gap to fill in

5.3 Open problems

Here we collect together some of the open problems in this area.

Many of these have already been discussed in this course.

Question 5.3.1 Is there a non-trivial, natural uniform algebra on $[0,1]$?

Question 5.3.2 Is there a non-trivial, natural, regular uniform algebra on $[0,1]$?

Question 5.3.3 Is there a non-trivial uniform algebra which has spectral synthesis?

Question 5.3.4 Let X be a compact plane set such that $R(X) \neq C(X)$. Can $R(X)$ be strongly regular? Can $R(X)$ have spectral synthesis?

Question 5.3.5 Let A be a regular Banach function algebra on Φ_A, and let $x \in \Phi_A$.

Suppose that M_x has a b.a.i.

Does it follow that A is strongly regular at x?

(This question is also open for uniform algebras.)

Question 5.3.6 Let A be a Banach function algebra on Φ_A.

Suppose that the only closed ideals in A are the kernels $I(E)$ for closed sets $E \subseteq \Phi_A$.

Does it follow that A is regular, and hence that A has spectral synthesis?

NOTE ADDED

For some discussion of these questions, listen to the audio file (which will be made available from the web page) of the final Discussion Session.