# G12RAN: Real Analysis

## 1. Properties of the real numbers

### I. Review of notation, definitions and results from earlier modules

You should make sure that you read this section to remind yourself of this material.

#### Sets of real numbers

We will often work with the sets of natural numbers, integers, rational numbers or real numbers. These are, respectively, $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, $\mathbb{Q} = \{m/n : m \in \mathbb{Z}, n \in \mathbb{N}\}$ and $\mathbb{R}$ (the set of all real numbers).

Note that we could also write, for example, $\mathbb{Q} = \{x \in \mathbb{R} : x \text{ is rational}\}$.

**Warning!** Some authors include 0 in $\mathbb{N}$, but in this module 0 will *not* be an element of $\mathbb{N}$.

An *interval* in $\mathbb{R}$ is any of the following types of subset of $\mathbb{R}$ (here $a$, $b$ are in $\mathbb{R}$ with $a \leq b$): $\emptyset$, $\mathbb{R}$, $(-\infty, a)$, $(-\infty, a]$, $(a, \infty)$, $[a, \infty)$, $[a, b]$ and (when $a < b$) $(a, b)$, $[a, b)$ or $(a, b]$. Here $(a, b) = \{x \in \mathbb{R} : a < x < b\}$, $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$, $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ and the other intervals are defined similarly.

These intervals are often described as *open*, *closed* or *half-open* as appropriate. We can clearly distinguish between the *bounded* intervals and the *unbounded* intervals. The empty set $\emptyset$ is, by convention, a bounded interval. It is possible, though not common, to write $\mathbb{R} = (-\infty, \infty)$.

**Please note** that there are many subsets of $\mathbb{R}$ which are not intervals and which *can not* be described as open, closed or half-open (e.g $\mathbb{Q}$). In the case where $a = b$, then the interval $[a, b]$ has only one element.

Throughout this module, the notation $\infty$ is interchangeable with $+\infty$, and so we could also write $\mathbb{R} = (-\infty, +\infty)$. Of course, $+\infty$ and $-\infty$ are *not* elements of $\mathbb{R}$.

#### Non-negative elements of sets

We denote by $\mathbb{R}^+$ the set of non-negative real numbers, i.e. $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\} = [0, \infty)$. Similarly $\mathbb{Q}^+ = \{x \in \mathbb{Q} : x \geq 0\}$, and $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$.

**Warning!** Some authors exclude 0 from these sets, and consider strictly positive elements of these sets instead.

## Set operations

Given two sets $A$ and $B$, you should be familiar with the notions of *intersection*, *union*, and *set difference*. These are, respectively, $A \cap B = \{x : x \in A \text{ and } x \in B\}$, $A \cup B = \{x : x \in A \text{ or } x \in B(\text{or both})\}$ and $A \setminus B = \{x \in A : x \notin B\}$. For example, we have $\mathbb{Q}^+ = \mathbb{Q} \cap [0, \infty)$, while the set of all irrational (but real) numbers is $\mathbb{R} \setminus \mathbb{Q}$.

## Sequences

**Notation** Let $X$ be a set. We use the notation $(x_n)_{n=1}^{\infty} \subseteq X$ to mean that $x_1, x_2, x_3, \ldots$ is a sequence of elements of $X$. If there is no danger of ambiguity we will often shorten this to

$$(x_n) \subseteq X.$$

**Definition** Let $(x_n) \subseteq \mathbb{R}$, and let $x \in \mathbb{R}$. Then we say $(x_n)$ *converges* to $x$ in $\mathbb{R}$ if, for every positive real number $\epsilon$, from some term onwards the sequence $x_n$ stays within the interval $(x - \epsilon, x + \epsilon)$.

Equivalently, we may state this definition (more formally) in terms of $\epsilon$ and $N$: $(x_n)$ converges to $x$ if, for all $\epsilon > 0$ there exists an $N(\epsilon) \in \mathbb{N}$ such that, for all $n \geq N(\epsilon)$, $|x_n - x| < \epsilon$. If $(x_n)$ converges to $x$, we write $x_n \to x$ *as* $n \to \infty$, or $\lim_{n\to\infty}(x_n) = x$. We also say that the *limit as* $n \to \infty$ of the sequence $(x_n)$ is $x$.

If the sequence $(x_n)$ does not converge, then we say that $(x_n)$ *diverges.* In this case the notation $\lim_{n\to\infty}(x_n)$ does not mean anything.

If we ever write

$$\lim_{n\to\infty}(x_n) = x$$

we always mean that the sequence $(x_n)$ converges and the limit of the sequence is $x$.

**Exercise** Is it possible for a sequence of rational numbers to converge to an irrational number? Is it possible for a sequence of irrational numbers to converge to a rational number?

There is a notion of *divergence to* $+\infty$ (or $-\infty$). We say that a sequence of real numbers $(x_n)$ diverges to $+\infty$ if, for all $M > 0$, there is an $N(M) \in \mathbb{N}$ such that for all $n \geq N(M)$ we have $x_n > M$. This says that, from some term on, all terms are greater than $M$. How far you need to go will depend on how big $M$ is.

**Example** The sequence $x_n = n^2$ diverges to $+\infty$.

**Exercise** Write down a formal definition for the notion of divergence to $-\infty$.

**Proposition 1.1.** (The algebra of limits). If $(x_n)$, $(y_n)$ are convergent sequences of real numbers, with $\lim_{n\to\infty}(x_n) = x$ and $\lim_{n\to\infty}(y_n) = y$, then

(i) $x_n + y_n \to x + y$ as $n \to \infty$,

(ii) $x_n y_n \to xy$ as $n \to \infty$,

(iii) if $y \neq 0$ then $x_n/y_n \to x/y$ as $n \to \infty$. ∎

**Remark:** strictly speaking, in 1.1 (iii) we should be worried about division by zero. Just because $y \neq 0$ does not mean that none of the $y_n$ are zero. However, since $\lim_{n\to\infty}(y_n)$ is not zero, it follows that from some term onwards we will have $y_n \neq 0$. From this term onwards, $x_n/y_n$ makes sense, and gives a sequence in $\mathbb{R}$ which converges to $x/y$.

The following well known false statement and proof illustrates the danger of assuming that sequences are convergent.

**False theorem 1.2.** Let $(x_n) \subseteq \mathbb{R}$ and suppose that $\lim_{n\to\infty}(x_n^2) = 1$. Then

$$\lim_{n\to\infty}(x_n) = 1 \text{ or } -1.$$

**False proof.** By the algebra of limits,

$$\lim_{n\to\infty}(x_n^2) = (\lim_{n\to\infty}(x_n))^2,$$

i.e. $(\lim_{n\to\infty}(x_n))^2 = 1$. The result follows. ∎

The fact that this statement is false is shown by using, for example, the example $x_n = (-1)^n$. This sequence satisfies the conditions of the false theorem, but it does not converge at all, and this is where the false proof breaks down.

However, there is some standard theory available that helps to guarantee convergence under certain conditions.

**Theorem 1.3.** (Squeeze rule, or Sandwich Theorem). Let $(a_n)$, $(b_n)$, $(c_n)$ be sequences of real numbers such that, for all $n \in \mathbb{N}$,

$$a_n \leq b_n \leq c_n.$$

Suppose that the sequences $(a_n)$ and $(c_n)$ both converge, and that they have the same limit. Then the sequence $(b_n)$ also converges, and

$$\lim_{n\to\infty}(a_n) = \lim_{n\to\infty}(b_n) = \lim_{n\to\infty}(c_n).$$

∎

## Bounded sets in $\mathbb{R}$

You should be familiar with the following fact about $\mathbb{R}$ (known as the *completeness axiom* for $\mathbb{R}$): if $E$ is a non-empty subset of $\mathbb{R}$ which is bounded above, then $E$ has a *least upper bound* (also called the *supremum* of $E$). Similarly, if $E$ is non-empty and bounded below, then $E$ has a *greatest lower bound* (or *infimum*). We will use the terms infimum, supremum throughout this module, and use the following notation.

**Notation.** Let $E$ be a non-empty subset of $\mathbb{R}$. If $E$ is bounded above, then we denote the supremum of $E$ (the least upper bound of $E$) by $\sup(E)$. If $E$ is bounded below then we denote the infimum of $E$ (the greatest lower bound of $E$) by $\inf(E)$.

Note that if $E$ is the open interval $(0,1)$ and $F$ is the closed interval $[0,1]$, then $\inf(E) = \inf(F) = 0$ and $\sup(E) = \sup(F) = 1$. This shows that (for non-empty, bounded sets) the least upper bound and greatest lower bound may or may not be elements of the set in question. In other words, *non-empty bounded sets may or may not have maximum and/or minimum elements.*

**Definition** Let $(x_n)$ be a sequence of real numbers. We say that $(x_n)$ is a *monotone increasing* (or *nondecreasing*) sequence if we have $x_1 \leq x_2 \leq x_3 \leq \ldots$, while $(x_n)$ is *monotone decreasing* (or *nonincreasing*) if we have $x_1 \geq x_2 \geq x_3 \leq \ldots$. In either of these two cases we say that $(x_n)$ is a *monotone sequence.*

**Warning!** Most sequences are not monotone! For example, the sequence $x_n = (-1)^n/n$ is far from monotone, even though it converges to 0.

**Theorem 1.4** (Monotone Sequence Theorem or MST) Let $(x_n)$ be a monotone sequence of real numbers. Then either $(x_n)$ converges to some real number $x$, or else the sequence $(x_n)$ diverges to either $+\infty$ or $-\infty$. ∎

In fact, if $(x_n)$ is monotone increasing and is bounded above in $\mathbb{R}$, then $(x_n)$ converges to $\sup(\{x_1, x_2, x_3, \ldots\})$. Similarly, if $(x_n)$ is monotone decreasing and is bounded below in $\mathbb{R}$, then $(x_n)$ converges to $\inf(\{x_1, x_2, x_3, \ldots\})$.

**Exercise.** Let $(x_n)$ be a monotone decreasing sequence of non-negative real numbers. Is it necessarily the case that $(x_n)$ converges to 0?

## Density of the rationals and the irrationals

**Proposition 1.5** Let $a$, $b$ be real numbers with $a < b$. Then there are infinitely many rational numbers in the interval $(a, b)$ and there are also infinitely many irrational numbers in the interval $(a, b)$.

Because of this, we say that the rational numbers are *dense* in $\mathbb{R}$, and so are the irrationals numbers.