G12RAN: Real Analysis

5. Differentiability

Definition Let f be a real-valued function defined on an open interval (b, c) and let $a \in (b, c)$. Then f is *differentiable at* a if $\lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)$ exists and is a real number (note that this quotient is not defined at a, but is defined on a punctured neighbourhood of a). In this case we also define the *derivative* of f at the point a, $f'(a)$, by $f'(a) = \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\right)$.

The function f is *differentiable* (on (b, c)) if it is differentiable at all points of (b, c).

Similarly, a function f from $\mathbb{R}$ to $\mathbb{R}$ is *differentiable* if it is differentiable at all points of $\mathbb{R}$. (You can also define differentiability on closed intervals by looking at one-sided limits at the endpoints).

Note that the function f' is defined at all points where f is differentiable, and is undefined elsewhere. For example, if $f(x) = |x|$, then f is differentiable at all points except 0. So f is defined at all points of $\mathbb{R}$ but f' is only defined on $\mathbb{R} \setminus \{0\}$.

Most standard functions met are differentiable at those points where they are defined: for example, this is true for $\exp(x)$ and for polynomial functions with real coefficients (defined and differentiable everywhere), rational functions $p(x)/q(x)$ for polynomials p and q (defined and differentiable where $q(x) \neq 0$), trigonometric functions such as $\sin(x)$, $\cos(x)$, $\tan(x)$ *etc.* (differentiable at all points where they are defined) and $\log(x)$ (defined and differentiable for $x > 0$).

The following are the basic results about differentiability and differentiation. (There will be more results and applications in the next Chapter).

Theorem (Differentiability implies continuity) *Let f be a real-valued function defined on an open interval (b, c) and let $a \in (b, c)$. If f is differentiable at a then f is also continuous at a. If f is differentiable on (b, c) then f is continuous from (b, c) to $\mathbb{R}$.*

So every differentiable function is continuous. But not every continuous function is differentiable, as the example $f(x) = |x|$ shows. In fact it is possible for a continuous function to fail differentiability at *all* points! One example will be described in lectures, but those interested might want to read about Brownian motion.

Theorem (The algebra of differentiable functions) *Let f and g be real-valued functions defined on an open interval (b, c) and let $a \in (b, c)$. Suppose that f and g are both differentiable at a. Then:*

(i) $f + g$ *is differentiable at a and* $(f + g)'(a) = f'(a) + g'(a)$;

(ii) fg *is differentiable at a and* $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$;

(iii) if $g(a) \neq 0$ then f/g is defined on some open interval centred on a and f/g is differentiable at a, with

$$\left(\frac{f}{g}\right)'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{(g(a))^2}.$$

Similarly, if f and g are differentiable throughout the interval (a, b) then so are f, g and (apart from where $g(x) = 0$) f/g.

Theorem (The chain rule) *Let f, g be real-valued functions defined on (b, c), (d, e) respectively. Let $a \in (b, c)$ and suppose that $f(a) \in (d, e)$. If f is differentiable at a and g is differentiable at $f(a)$, then $g \circ f$ (the composite function) is defined on an open interval containing a, is differentiable at a, and $(g \circ f)'(a) = g'(f(a))f'(a)$.*

Theorem (Rolle's theorem). *Let a, b be real numbers with $a < b$. Let f be a continuous, real-valued function defined on $[a, b]$ such that f is differentiable on (a, b). Suppose that $f(a) = f(b)$. Then there is at least one point $c \in (a, b)$ such that $f'(c) = 0$.*

One related result is that if you want to find the greatest and least values taken by a differentiable function f on an interval $[a, b]$, you only need to check the values of f at the endpoints a and b, and at any 'stationary points' in (a, b) (i.e. points $x \in (a, b)$ where $f'(x) = 0$).

Theorem (The mean value theorem, or MVT) *Let a, b be real numbers with $a < b$. Let f be a continuous, real-valued function defined on $[a, b]$ such that f is differentiable on (a, b). Then there is at least one point $c \in (a, b)$ such that $f'(c) = (f(b) - f(a))/(b - a)$.*

See lectures for geometrical interpretations of these results.

The MVT has many important applications, and here are a few.

Proposition *Let a, b be real numbers with $a < b$. Let f be a real-valued function which is differentiable on (a, b). Then:*

(i) if $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing on (a, b);

(ii) if $f'(x) \geq 0$ for all $x \in (a, b)$, then f is nondecreasing on (a, b);

(iii) if $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b);

(iv) if $f'(x) \leq 0$ for all $x \in (a, b)$, then f is nonincreasing on (a, b);

(v) if $f'(x) < 0$ for all $x \in (a, b)$, then f is strictly decreasing on (a, b).

However, it is possible to be strictly increasing on an interval and yet still have some points where the derivative is 0, as the function $f(x) = x^3$ shows.

Proposition *Let f be a differentiable, real-valued function defined on an open interval (a, b). Let $A > 0$. Then the following are equivalent (TFAE):*

(i) $|f(x) - f(y)| \leq A|x - y|$ for all x, y in (a, b);

(ii) $|f'(x)| \leq A$ for all x in (a, b).

The same result applies if you work on $\mathbb{R}$ rather than on (a, b). A function (differentiable or not) for which such an $A > 0$ exists is said to satisfy a *Lipschitz condition of order* 1, but we shall call such functions *Lipschitz continuous* for short. Every Lipschitz continuous function is continuous, but not every continuous function is Lipschitz continuous.