G12RAN Real Analysis: Exercises 2

You should attempt questions 1 to 3 in the second problem class. If you finish early then you can try questions 4 and 5. Try to solve the questions in advance if possible! At the very least, **make sure that you are familiar with the definitions of the concepts mentioned in these questions**.

Answers to questions 7 and 8 should be handed in at the end of the lecture on Friday 1st November.

Note that quality of exposition is very important: poor exposition/expression in the examination will cost marks. You must always justify your answers.

1. Use the definitions from lectures and the usual algebra of limits for convergent sequences to prove the algebra of limits for functions in the following form (the algebra of limits also works for the other types of one-sided or two-sided limits): let f, g be real valued functions on (a, b). Suppose that $\lim_{x\to a+} f(x) = L_1$ and $\lim_{x\to a+} g(x) = L_2$, where L_1 and L_2 are in $\mathbb{R}$.
(i) Prove that $\lim_{x\to a+}(f(x) + g(x)) = L_1 + L_2$ and that $\lim_{x\to a+}(f(x)g(x)) = L_1L_2$.
(ii) Now suppose that $L_2 \neq 0$ and that g does not take the value 0 on (a, b). Prove that $\lim_{x\to a+}(f(x)/g(x)) = L_1/L_2$.

2. For $x \in \mathbb{R}$, denote by $[x]$ the greatest integer which is less than or equal to x (often called the *integer part* of x). Define a function f from $\mathbb{R}$ to $\mathbb{R}$ by $f(x) = x$ if x is rational, while $f(x) = [x]$ if x is irrational.
(a) At which, if any, points a of $\mathbb{R}$ does $\lim_{x\to a-} f(x)$ exist? [See hint below.]
(b) At which, if any, points a of $\mathbb{R}$ does $\lim_{x\to a+} f(x)$ exist?
(c) At points where either of these one-sided limits exist, is the one-sided limit equal to $f(a)$?
[HINT: consider sequences of (i) rationals (ii) irrationals approaching a from either side. Consider separately the case where a is an integer and when a is not an integer.]

3. Prove the Sandwich Theorem (or squeeze rule) for function limits in the following form (as before, this result is true for the other forms of limit too): let f, g, h be real-valued functions on (a, b), and suppose that $f(x) \leq g(x) \leq h(x)$ for all $x \in (a, b)$. Let $L \in \mathbb{R}$. Suppose that $\lim_{x\to b-} f(x) = \lim_{x\to b-} h(x) = L$. Then $\lim_{x\to b-} g(x)$ also exists, and equals L.

4. Define f from $\mathbb{R}$ to $\mathbb{R}$ by $f(x) = x$ if x is rational, $f(x) = 1 - x$ if x is irrational. Determine all those points of $\mathbb{R}$ (if any) for which both $\lim_{x\to a-} f(x)$ and $\lim_{x\to a+} f(x)$ exist and such that

$$\lim_{x\to a-} f(x) = \lim_{x\to a+} f(x) = f(a).$$

(We say that f is *continuous at* a when this happens.)
[HINT: consider the same types of sequences as in the hint for question 2.]

5. (Harder) Does there exist a nondecreasing function f from $\mathbb{R}$ to $\mathbb{R}$ (i.e. $x \leq y \Rightarrow f(x) \leq f(y)$) such that $f(\mathbb{R}) = \mathbb{R} \setminus \{0\}$? (Here,as usual, $f(\mathbb{R})$ means $\{f(x) : x \in \mathbb{R}\}$.)

6. At which points of $\mathbb{R}$, if any, is the function defined in question 2 continuous? (See also question 4 above.)

7. Find a, b in $\mathbb{R}$ such that the following function f is continuous from $\mathbb{R}$ to $\mathbb{R}$:

$$f(x) = \begin{cases} ax + b, & \text{if } x < -1; \\ x^2 + 1, & \text{if } -1 \leq x \leq 1; \\ -ax + 2b, & \text{if } x > 1. \end{cases}$$

[Justify your answer briefly.]

8. Give an example of a surjective (onto) function f from $\mathbb{R}$ to $\mathbb{R}$ which is discontinuous at all points of $\mathbb{R}$ [as usual, you must justify your answer: make sure you check that your function really does satisfy both of the conditions asked for!]

9. Let $p(x)$ be a polynomial function of odd degree and with real coefficients. Prove that there must be at least one real number c such that $p(c) = 0$. [Hint: apply the Intermediate Value Theorem with b large and positive, and a negative and large in modulus.]

10. (Harder) The function f is defined on $(0, 1)$ as follows. If x is irrational, then $f(x) = 0$. If $x = p/q$ where p, q are positive integers with no common factor, then $f(x) = 1/q$.

Let $a \in \mathbb{R}$. Prove that f is continuous at a if and only if a is irrational.

11. Let f be a continuous function from $[0, 1]$ to $[0, 1]$. Prove that there is at least one point $c \in [0, 1]$ with $f(c) = c$ [hint: look at the function $f(x) - x$].

12. Let f be an injective, continuous, real-valued function on an open interval (a, b). Prove that f is monotone on (a, b).