# G13MTS Metric and Topological Spaces: Question Sheet 1

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Answers to questions 4 and 6 to be handed in to Dr Feinstein at the end of the Thursday lecture in the second week of teaching.

**Always justify your answers.**

1. Which of the following sequences of real numbers are convergent and which are divergent? For the convergent sequences, find the limit.

(i)

$$x_n = \frac{2n^2 + n + 3}{3n^2 + 1}.$$

(ii)

$$x_n = \left(1 + \frac{1}{n}\right)^{2n}.$$

(iii)

$$x_n = \sin(n).$$

2. Let $(x_n)$, $(y_n)$ be bounded sequences of real numbers. Prove that

$$\limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty}(x_n) + \limsup_{n\to\infty}(y_n)$$

and that

$$\liminf_{n\to\infty}(x_n + y_n) \geq \liminf_{n\to\infty}(x_n) + \liminf_{n\to\infty}(y_n).$$

Give examples to show that both of these inequalities may be strict.

3. Let $X$ be a set and let $d : X \times X \longrightarrow \mathbb{R}$ be a function which satisfies the three metric space axioms:

(i) $d(x, y) = 0 \iff x = y$

(ii) $d(x, y) = d(y, x)$ for all $x$, $y$ in $X$

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x$, $y$, $z$ in $X$.

Prove that $d(x, y) \geq 0$ for all $x$, $y$ in $X$, and so $d$ is a metric.

4. Prove carefully that the following three examples from the module really are metrics:

(a) On $\mathbb{R}^n$,

$$d_\infty(\mathbf{x}, \mathbf{y}) = \max\{|x_k - y_k| : 1 \leq k \leq n\}.$$

(b) On $C[0, 1]$,

$$d_1(f, g) = \int_0^1 |f(x) - g(x)| dx \ .$$

[Hint: the first axiom is the hard one here! One method involves using the Fundamental Theorem of Calculus, and differentiating the function $\int_0^x |f(t) - g(t)| \, \mathrm{d}t$.]

(c) The discrete metric: here $X$ is any set, and

$$d(x, y) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{otherwise.} \end{cases}$$

5. Consider the following functions in $C[0, 1]$: $f(x) = x$, $g(x) = \frac{5}{4}x^2$. Calculate $d_1(f, g)$ and $d_\infty(f, g)$.

6. Let $(X, d)$ be a metric space, and let $(x_n)$, $(y_n)$ be convergent sequences in $X$, with $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Prove that

$$\lim_{n\to\infty} d(x_n, y_n) = d(x, y).$$

Note: you **must not assume** that the last limit exists. This is part of the proof!

Note also that the first two sequences are contained in the general metric space $X$. But the last sequence, $(d(x_n, y_n))$, is a sequence of real numbers, whose convergence must be investigated using the usual notion of convergence in $\mathbb{R}$. (When working with sequences in $\mathbb{R}$, we will use the usual notion of convergence unless otherwise specified).

7. Write down three different metrics on $\mathbb{N}$, no two of which are multiples of each other.

8. Recall that a **pseudometric** is a function which satisfies all the conditions of a metric, except that possibly two different points may be distance zero apart (so that every metric is a pseudometric). Which of the following are metrics? Pseudometrics? Neither?

(i) on $\mathbb{R}^2$, $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2|$,

(ii) on $\mathbb{C}$, define $d(z, w)$ depending on whether $0, z, w$ lie on a straight line. If they do, define $d(z, w) = |z - w|$. Otherwise define $d(z, w) = |z| + |w|$.

(iii) on $\mathbb{R}^2$, $d((x_1, y_1), (x_2, y_2)) = \min\{|x_1 - x_2|, |y_1 - y_2|\}$.

(iv) on $\mathbb{R}^2$, $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2|^2 + |y_1 - y_2|^2$.

9. Let $X$ be the set of all bounded sequences of real numbers. We denote a typical element of $X$ by $\mathbf{x} = (x_n)$ or $(x_n)_{n=1}^\infty$. In this question you may assume that $X$ is a vector space over $\mathbb{R}$ with the usual componentwise operations: for $\mathbf{x}$ and $\mathbf{y}$ in $X$ and $\alpha \in \mathbb{R}$, with $\mathbf{x} = (x_n)$ and $\mathbf{y} = (y_n)$, we have $\mathbf{x} + \mathbf{y} = (x_n + y_n)_{n=1}^\infty$ and $\alpha\mathbf{x} = (\alpha x_n)_{n=1}^\infty$.

For $\mathbf{x} = (x_n) \in X$, define $\|\mathbf{x}\|_\infty = \sup\{|x_n| : n \in \mathbb{N}\}$. Prove that $\|\cdot\|_\infty$ is a norm on $X$.

(The space $X$ in this question is one of the basic examples in the theory of Functional Analysis. It is usually denoted by $\ell^\infty$.)