# G13MTS Metric and Topological Spaces: Question Sheet 2

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Answers to questions 1 and 6 to be handed in to Dr Feinstein at the end of the Thursday lecture in the fourth week of teaching.

If $X$ is a subset of $\mathbb{R}$ or $\mathbb{C}$ then, **unless otherwise specified**, the metric we will use on $X$ will be the 'usual' metric: $d_X(x, y) = |x - y|$ for $x$, $y$ in $X$.

**Always justify your answers.**

1.(i) In $\mathbb{R}^2$ with the metric $d_\infty$ (as defined in the notes), sketch the ball $B((0, 0), 1)$.

(ii) In $\mathbb{R}^2$ with the jungle river metric (defined in the notes), sketch the ball $B((2, 1), 3/2)$.

2. Which, if any, of the following functions from $(0, \infty)$ to $\mathbb{R}$ are (a) uniformly continuous? (b) Lipschitz continuous?

(i) $f_1(x) = \sqrt{x}$ (ii) $f_2(x) = \sin(x)$ (iii) $f_3(x) = x^2$ (iv) $f_4(x) = 1/\sqrt{x}$ .

3. Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Suppose that $d_X$ is the discrete metric. Prove that every function $f : X \longrightarrow Y$ is continuous.

4. Let $(X, d)$ be a metric space, let $(x_n)$ be a sequence in $X$ and let $x \in X$. Prove that the following statements are equivalent.

(i) $x_n \to x$ as $n \to \infty$.

(ii) For every subsequence $(y_n)$ of $(x_n)$ there is at least one subsequence $(z_n)$ of $(y_n)$ such that $z_n \to x$ as $n \to \infty$.

(Another way to write (ii) is to say that every subsequence of $(x_n)$ has at least one subsequence which converges to $x$. Here you will be dealing with subsequences of subsequences, which are sometimes called subsubsequences. However, you should be able to convince yourself that every subsubsequence of a sequence $(x_n)$ is also a subsequence of $(x_n)$.

Warning: remember that, in general, a divergent sequence may possibly have some convergent subsequences!)

5. Let $X$ and $Y$ be metric spaces, let $f$ be a function from $X$ to $Y$ and let $x \in X$. Prove that the following three statements are equivalent. (This is slightly stronger than one of the results stated in the notes.)

(a) The function $f$ is continuous at $x$.

(b) For every sequence $(x_n) \subseteq X$ which converges to $x$, we have $f(x_n) \to f(x)$ in $Y$ as $n \to \infty$.

(c) For every sequence $(x_n) \subseteq X$ which converges to $x$, there is at least one subsequence $(z_n)$ of $(x_n)$ such that $f(z_n) \to f(x)$ in $Y$ as $n \to \infty$.

6. Let $X$ be $C[0,1]$. Define $F : X \longrightarrow \mathbb{R}$ by $F(f) = f(0) \quad (f \in C[0,1])$. Is $F$ continuous when $X$ is given (i) the metric $d_1$ (ii) the metric $d_\infty$? (These metrics were defined in the notes).

7. Let $X = [0,2) \cup [3,5]$ with the usual metric (which is, of course, the same as the subspace metric induced by the usual metric on $\mathbb{R}$). Which of the following sets are open (a) in $\mathbb{R}$ (b) in $X$?

(i) $X \cap (1,3)$ (ii) $X \cap [3,6]$ (iii) $X \cap [4,6]$ (iv) $X$.

8. Let $X$, $Y$ be metric spaces, let $Z \subseteq Y$, and give $Z$ the subspace metric induced by the metric on $Y$. Let $f : X \longrightarrow Z$ be a function, and note that $f$ may also be regarded as a function from $X$ to $Y$. Thinking of $f$ in these two ways, show that $f$ is continuous from $X$ to $Z$ if and only if $f$ is continuous from $X$ to $Y$.