FAQ for G13MTS: Metric and Topological Spaces

J. F. Feinstein

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1 Why is the pre-image of the codomain equal to the domain?

When $f : X \to Y$, why is $f^{-1}(Y)$ equal to the whole of X?

Answer For $E \subseteq Y$, the definition of the notation $f^{-1}(E)$ is that $f^{-1}(E) = \{x \in X : f(x) \in E\}$. In particular, $f^{-1}(Y) = \{x \in X : f(x) \in Y\}$. However, f is a function from X to Y and so, for all $x \in X$, we have $f(x) \in Y$. Thus $f^{-1}(Y) = X$.

2 What makes the sets in a topology open?

If U is in a topology τ, does this mean that U is open in the conventional sense, or is U effectively open by definition because it is in τ? In other words, does U need to be conventionally open to be in τ, or does its being in τ force U to be open?

Answer This is one of the most frequently asked questions about topologies. In general, when you take a set X and want to put a topology on it, **there is no conventional notion of open available.** Until you define the topology, there are no open sets. Once you have set up your topology τ, the sets are open **because they are in tau**. This is the topological space notion of open set.
However, in the special case of a metric space, you DO have a conventional notion of open. You usually use this to define your topology, and then the sets in the topology are open in both the conventional AND the topological space sense.

3 How can the identity map be discontinuous?

Answer If (X, τ) is a topological space, then the identity map Id_X (defined by $x \mapsto x$) is clearly a homeomorphism from X to itself. The problem comes when we have two different topologies on X. So now suppose that τ_1 and τ_2 are two topologies on X. Let X_1 be X with the topology τ_1 and let X_2 be X with the topology τ_2. Then we may ask whether Id_X is continuous from X_1 to X_2, By

definition, this means we are asking whether, for all $U \in \tau_2$, we have $\mathsf{Id}_X^{-1}(U) \in \tau_1$. Obviously $\mathsf{Id}_X^{-1}(U) = U$ here, so we are simply asking whether or not $\tau_2 \subseteq \tau_1$. The conclusion is that Id_X is continuous from (X, τ_1) to (X, τ_2) if and only if $\tau_2 \subseteq \tau_1$, which is the same as saying that τ_2 is weaker than τ_1. Of course Id_X is a homeomorphism from (X, τ_1) to (X, τ_2) if and only if $\tau_1 = \tau_2$.
If we restict our attention to metric spaces, we obtain some familiar results mentioned in the notes. Two metrics d_1 and d_2 on a set X are equivalent if and only if they generate the same topology on X, and this is the case if and only if Id_X is a homeomorphism from (X, d_1) to (X, d_2). (Our original definition is in terms of sequences: d_1 and d_2 are equivalent if they give rise to the same convergent sequences in X and with the same limits.)

4 Why is that set closed?

Given a topological space X, a continuous function $f : X \to \mathbb{R}$ and a real number y, why is $\{x \in X : f(x) \leq y\}$ a closed subset of X?

Answer Here it may be helpful to note the following frequently forgotten fact: $(-\infty, y]$ is a closed subset of $\mathbb{R}$, since its complement is (y, ∞) (which is clearly open).
Set $E = \{x \in X : f(x) \leq y\}$. One way to answer the question is to note that $E = f^{-1}((-\infty, y])$. Since f is continuous, this pre-image of a closed set must be closed, as required.
If X is a metric space, there is another proof involving convergent sequences: if $(x_n) \subseteq E$ and $x_n \to x$ as $n \to \infty$, it is easy to show that x must also be in E. The result follows.