G13MTS METRIC AND TOPOLOGICAL SPACES

Model Solutions to the 1997–1998 Examination Paper

1 (a) (i) A *metric* on X is a function $d\colon X \times X \to \mathbb{R}^+$ satisfying, for all x, y, z in X,

(A) $d(x, y) = 0 \Leftrightarrow x = y$,

(B) $d(x, y) = d(y, x)$,

(C) $d(x, z) \leq d(x, y) + d(y, z)$.

(ii) A *pseudometric* on X is a function $d\colon X \times X \to \mathbb{R}^+$ satisfying (B), (C) as above and such that $d(x, x) = 0$ for all x in X.

(b) (i) This is a metric, and hence also a pseudometric. Clearly $d\colon X \times X \to \mathbb{R}^+$. Let $x, y, z \in X$.

(A) If $x = y$ then $\frac{1}{x} = \frac{1}{y}$ so $\left|\frac{1}{x} - \frac{1}{y}\right| = 0$.
Conversely, if $d(x, y) = 0$ then $\left|\frac{1}{x} - \frac{1}{y}\right| = 0$ so $\frac{1}{x} = \frac{1}{y}$ and hence $x = y$.
[**Aliter**: "This is clear from the fact that $x \mapsto \frac{1}{x}$ is $1 - 1$ from X to $\mathbb{R}$."]

(B) This is immediate from properties of $|\cdot|$.

(C) $\left|\frac{1}{x} - \frac{1}{z}\right| \leq \left|\frac{1}{x} - \frac{1}{y}\right| + \left|\frac{1}{y} + \frac{1}{z}\right|$ by the triangle law in $\mathbb{R}$, so $d(x, z) \leq d(x, y) + d(y, z)$.

(ii) This is also a metric. Clearly $d\colon X \times X \to \mathbb{R}^+$. Let $x, y, z \in X$.

(A) If $x = y$ then

$$\begin{aligned} d(x, y) &= \min\{|x - y|, 1\} \\ &= \min\{0, 1\} = 0. \end{aligned}$$

Conversely, if $\min\{|x - y|, 1\} = 0$, then we must have $|x - y| = 0$ and so $x = y$.

(B) This is clear, since $|x - y| = |y - x|$.

(C) This is trickier! However, we do know that $|x - z| \leq |x - y| + |y - z|$ and that $d(x, z) \leq 1$.

If either $d(x, y)$ or $d(y, z) = 1$, then clearly $d(x, z) \leq d(x, y) + d(y, z)$. So we may assume that $d(x, y) = |x - y|$ and $d(y, z) = |y - z|$. But then

$$\begin{aligned} d(x, z) \leq |x - z| &\leq |x - y| + |y - z| \\ &= d(x, y) + d(y, z). \end{aligned}$$

(iii) This is neither a metric nor a pseudometric since, for example, $d(1, 1) = 1 \neq 0$.

(iv) This is a pseudometric but not a metric (mainly because $x \mapsto x^2$ is not $1-1$). We have $d(x,x) = |x^2 - x^2| = 0$, but $d(-1,1) = |1-1| = 0$, and $-1 \neq 1$. The proofs of (B), (C) are the same as in part (i).

2 (a) Let X be a metric space, and suppose that E, F are disjoint, non-empty, closed subsets of X. Then there is a continuous function $f\colon X \to [0,1]$ such that $f(x) = 0$ for all x in E, and $f(x) = 1$ for all x in F.

Proof We use properties of the distance function, $\text{dist}(x,A)$. If A is a *closed* set, then $\text{dist}(x,A) = 0 \Leftrightarrow x \in A$. Otherwise $\text{dist}(x,A) > 0$. [Here A is assumed non-empty!] The function $x \mapsto \text{dist}(x,A)$ is continuous.

Set

$$f(x) = \frac{\text{dist}(x,E)}{\text{dist}(x,E) + \text{dist}(x,F)}.$$

This makes sense, because $E \cap F = \emptyset$, so $x \notin E$ or $x \notin F$ (or both), and so (by above) $\text{dist}(x,E) + \text{dist}(x,F) > 0$ for all x in X.

It is now clear that $f\colon x \to [0,1]$. Since sums and quotients of continuous, real-valued functions are continuous, f is continuous.

Finally, for $x \in E$, $\text{dist}(x,E) = 0$ so $f(x) = 0$. For $x \in F$, $\text{dist}(x,F) = 0$, so

$$f(x) = \frac{\text{dist}(x,E)}{\text{dist}(x,E)} = 1.$$

(b) A topological space X is *normal* if whenever E, F are disjoint, closed subsets of X, then there are disjoint open subsets of X, U, V satisfying $E \subseteq U$ and $F \subseteq V$.

(c) Let (X,d) be a metric space. Let E, F be disjoint closed subsets of X.
If either E or F is empty we can take as open sets X and $\emptyset$ to satisfy the above condition. So we may assume that E, F are non-empty. Now define f as in (a).
Set $U = \{x \in X\colon f(x) < \frac{1}{3}\}$ and set $V = \{x \in X\colon f(x) > \frac{2}{3}\}$.
Since f is continuous, it follows that U and V are open, being the pre-images of open subsets of $[0,1]$ (or $\mathbb{R}$, if they prefer).
Clearly $U \cap V = \emptyset$ and $E \subseteq U$, $F \subseteq V$ as required.

(d) With our definition, the indiscrete topology *does* give a normal space. So they must be careful! There is an example with 3 points.

$$X = \{1,2,3\}, \quad \tau = \big\{\phi, X, \{1\}, \{1,2\}, \{1,3\}\big\}.$$

[It is easy to see that τ is a topology.]

Then $\{2\}, \{3\}$ are closed, but any open set containing $\{2\}$ must also contain $\{1\}$, and similarly for $\{3\}$, so (X, τ) is not normal.

3 (a) $\mathcal{B}$ is a *base* for τ if $\mathcal{B} \subseteq \tau$ and every $U \in \tau$ is a union of some or all of the sets in $\mathcal{B}$.

($U = \bigcup_{\gamma \in \Gamma} V_\gamma$ for some sets $V_\gamma \in \mathcal{B}$.)

(Here the 'empty union' is allowed.)

(b) Let U be an open subset of X. For all $x \in U \; \exists \, r(x) > 0$ such that $B_X\big(x, r(x)\big) \subseteq U$. But then it is clear that

$$U = \bigcup_{x \in U} B_X\big(x, r(x)\big).$$

So every open subset of X is a union of sets in $\mathcal{B}$.

Since 'open balls' *are* open, we have that $\mathcal{B} \subseteq \tau$, and so $\mathcal{B}$ *is* a base for τ.

(c) Yes, $\mathcal{B}$ *is* a base for the usual topology. Clearly all the sets in $\mathcal{B}$ are open in $\mathbb{R}$. Let U be an open subset of $\mathbb{R}$. For all $x \in U$, there is an $\varepsilon > 0$ with $(x - \varepsilon, x + \varepsilon) \subseteq U$. Choose $a_x < b_x < c_x < d_x$ in $\mathbb{R}$ with

$$x - \varepsilon < a_x < x < b_x < c_x < d_x < x + \varepsilon.$$

Then $x \in (a_x, b_x) \cup (c_x, d_x) \subseteq U$. We clearly then have

$$U = \bigcup_{x \in U} \big((a_x, b_x) \cup (c_x, d_x)\big),$$

a union of sets in $\mathcal{B}$. So $\mathcal{B}$ *is* a base, as required.

(d) You can take $\mathcal{B} = \{(-\infty, 1), (0, \infty)\}$. Then $(-\infty, 1) \cup (0, \infty) = \mathbb{R}$, but $\mathcal{B}$ cannot be a base for a topology, since if $\mathcal{B}$ *is* a base for a topology τ we would have $(-\infty, 1), (0, \infty) \in \tau$ and so $(-\infty, 1) \cap (0, \infty) = (0, 1) \in \tau$.

But $(0, 1)$ is *not* a union of elements of $\mathcal{B}$, so this is not possible.

4 (a) Let X be a topological space. An *open cover* of X is a collection of open subsets of X, U_γ ($\gamma \in \Gamma$), satisfying

$$X = \bigcup_{\gamma\in\Gamma} U_\gamma.$$

Such an open cover has a *finite subcover* if there are $\gamma_1, \dots, \gamma_n$ in Γ with

$$X = \bigcup_{k=1}^{n} U_{\gamma_k}.$$

X is *compact* if *every* open cover of X has a finite subcover.

(b) Let X be a Hausdorff space, and let E be a compact subset of X. We may assume that $E \neq \emptyset$ (since $\emptyset$ *is* closed).
We show that $X\backslash E$ is open. Let $x \in X\backslash E$. We show that $x \in \operatorname{int}(X\backslash E)$ [it then follows that $X\backslash E \subseteq \operatorname{int}(X\backslash E)$, and so $X\backslash E = \operatorname{int}(X\backslash E)$ and is open].
For all $y \in E$ we can find open sets U_y, V_y with $x \in U_y$, $y \in V_y$ and $U_y \cap V_y = \emptyset$. [X is Hausdorff.]
We then have $E \subseteq \bigcup\limits_{y\in E} V_y$, and, since E is compact, there must be a finite subcover: $\exists\, y_1, \dots, y_n \in E$ with $E \subseteq \bigcup\limits_{k=1}^{n} V_{y_k}$.
Set $U = \bigcap\limits_{k=1}^{n} U_{y_k}$. Then $x \in U$, U is open, and $U \cap E \subseteq U \cap \big(\bigcup\limits_{k=1}^{n} V_{y_k}\big) = \emptyset$, hence $U \subseteq X\backslash E$ and $x \in \operatorname{int}(X\backslash E)$, as claimed. $\square$

(c) Take $X = \{1, 2\}$ with the indiscrete topology. Then $\{1\}$ is a compact subset of X, but is not closed.

(d) $x_n \to x$ as $n \to \infty$, $\quad E = \{x\} \cup \{x_1, x_2, \dots\}$.
Let U_γ ($\gamma \in \Gamma$) be a collection of open subsets of X with $E \subseteq \bigcup\limits_{\gamma\in\Gamma} U_\gamma$.
Then there is a $\gamma_0 \in \Gamma$ with $x \in U_{\gamma_0}$.
Since $x_n \to x$ as $n \to \infty$, there is an $N \in \mathbb{N}$ such that, for all $n \geq N$, $x_n \in U_{\gamma_0}$.
There are then $\gamma_1, \gamma_2, \dots, \gamma_N \in \Gamma$ with $x_k \in U_{\gamma_k}$ for $k = 1, \dots, N$. We have

$$E \subseteq U_{\gamma_0} \cup \bigcup_{k=1}^{N} U_{\gamma_k},$$

and so we have found a finite subcover, as required.

5 (a) The topological space X is connected if the *only* subsets of X which are both open and closed are $\emptyset$ and X itself.

[Or various acceptable alternatives!]

(b) Suppose that U_1, U_2 are open subsets of Y with $f(E) \subseteq U_1 \cup U_2$, and such that $U_1 \cap U_2 \cap f(E) = \emptyset$. We must show that $f(E) \subseteq U_1$ or $f(E) \subseteq U_2$.

Since f is continuous, we have that $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are open. Also

$$E \subseteq f^{-1}\big(f(E)\big) \subseteq f^{-1}(U_1) \cup f^{-1}(U_2)$$

and

$$E \cap f^{-1}(U_1) \cap f^{-1}(U_2) \subseteq f^{-1}\big(f(E) \cap U_1 \cap U_2\big) = \emptyset.$$

Since E is connected, we have $E \subseteq f^{-1}(U_1)$ or $E \subseteq f^{-1}(U_2)$ or both.

Thus $f(E) \subseteq U_1$ or $f(E) \subseteq U_2$ (or both!) as required.

(c) Here it is best to use the 'continuous function to $\{0,1\}$' version.

Quote: If Z is a topological space, then Z is connected $\Leftrightarrow$ the only continuous functions from Z to the discrete space $\{0,1\}$ are the constant functions.

Suppose, for contradiction, that $X \times Y$ is disconnected. Then there is a continuous function f from $X \times Y$ to the discrete space $\{0,1\}$ which is not constant. There are then $(x_1, y_1), (x_2, y_2)$ in $X \times Y$ with $f\big((x_1, y_1)\big) = 0$ and $f\big((x_2, y_2)\big) = 1$.

Define $f_1 \colon X \to \{0,1\}$ and $f_2 \colon Y \to \{0,1\}$ by

$$f_1(x) = f\big((x, y_1)\big) \quad \text{and}$$

$$f_2(y) = f\big((x_2, y)\big).$$

Then f_1, f_2 are continuous, and so must be constant (since X, Y are connected).

But $f_1(x_1) = 0$, so $f_1(x_2) = 0$ i.e. $f(x_2, y_1) = 0$, while $f_2(y_2) = 1$, so $f_2(y_1) = 1$, i.e. $f(x_2, y_1) = 1$.

This contradiction shows that $X \times Y$ must be connected. $\square$

(d) We know that $E_1 \times E_2$ is a connected subset of $X \times X$, by (c).

But now E is just the continuous image of $E_1 \times E_2$ under the continuous map $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\big((x_1, y_1), (x_2, y_2)\big) \mapsto (x_1 + x_2, y_1 + y_2).$$

Thus E is connected, by (b).

6 (a) Consider the sequence $x_n = \dfrac{1}{2n}$.

Then $(x_n) \subseteq (0,1)$, and $x_n \to 0$ in $\mathbb{R}$ with usual metric, so (x_n) is Cauchy with the usual metric, and so

$$\lim_{m,n\to\infty} |x_m - x_n| = 0.$$

Thus (x_n) is Cauchy in $(0,1)$. But there is no point of $(0,1)$ to which (X_n) converges, so (X,d) is not complete.

Aliter: $(0,1)$ is not closed in $\mathbb{R}$. Any subset of $\mathbb{R}$ which is complete with the subspace metric must be closed in $\mathbb{R}$, so $(0,1)$ is not complete with the usual (subspace) metric.

(b) (i) (A) Clearly $d' \colon X \times X \to \mathbb{R}^+$. If $x = y$ then $d'(x,y) = |x-y| + |\ln(x) - \ln(y)| = 0$.
Conversely, if $d'(x,y) = 0$ then $0 \le |x-y| \le d'(x,y) = 0$, so $|x-y| = 0$ and $x = y$.

(B) This is clear from properties of $|\cdot|$.

(C) For $x,y,z \in (0,1)$

$$\begin{aligned} d'(x,z) &= |x-z| + |\ln(x) - \ln(z)| \\ &\le |x-y| + |y-z| + |\ln(x) - \ln(y)| + |\ln(y) - \ln(z)| \\ &\qquad\qquad\qquad\qquad \text{(by triangle law for } |\cdot|\text{)} \\ &= d'(x,y) + d'(y,z). \end{aligned}$$

(ii) We show that d, d' have the same convergent sequences with the same limits. Let $(x_n) \subseteq X$ and $x \in X$.
If $x_n \to x$ with respect to d, then, since $\ln$ is continuous from $(0,\infty)$ to $\mathbb{R}$ (with respect to the usual metrics), we have $\ln(x_n) \to \ln(x)$ as $n \to \infty$ (usual metric on $\mathbb{R}$) and so $|\ln(x_n) - \ln(x)| \to 0$ as $n \to \infty$. Thus $d'(x_n, x) = |x_n - x| + |\ln(x_n) - \ln(x)| \to 0$ as $n \to \infty$. (Usual metric on $\mathbb{R}$ here!)
Conversely, if $x_n \to x$ with respect to d', then,

$$0 \le d(x_n, x) \le d'(x_n, x) \to 0 \quad \text{as} \quad n \to \infty,$$

(where this is usual convergence in $\mathbb{R}$)

and so $d(x_n, x) \to 0$ as $n \to \infty$ by the sandwich rule.

(iii) d and d' cannot be uniformly equivalent, because d is bounded and d' is not. [$d(x,y) \le 1$ for x,y in X, but $d'\left(\frac{1}{2}, \frac{1}{2n}\right)$ diverges to ∞ as $n \to \infty$, since $|\ln(\frac{1}{2}) - \ln(\frac{1}{2n})| = \ln(n)$.]