G13MTS: Metric and Topological Spaces, Prize Problems

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Please note that Dr Feinstein did not invent these prize problems (they are well-known). For each of the prize problems, a prize will be awarded by Dr Feinstein for the first correct solution (including justification sufficient to convince Dr Feinstein!) handed in (by the deadline below) by a student registered for G13MTS: Metric and Topological Spaces in the academic year 2004-5. Solutions should be handed directly to Dr Feinstein on or before January 7 2005. Note that the solutions to these prize problems need not necessarily have anything to do with G13MTS: Metric and Topological Spaces. The amount of the prize is intended to give a rough indication of the difficulty of each problem.

Prize problem 1. [Prize: 15 pounds] Give (with proof, of course!) an explicit example of a positive real number t such that the sequence $\cos(n!t)$ does not converge.

Prize problem 2. [Prize: 15 pounds] Suppose that a rectangle is formed from finitely many smaller rectangles, and that each of the smaller rectangles has the property that at least one of its sides has integer length. Show that the original rectangle must also have at least one side of integer length.

Prize problem 3. [Prize: 12 pounds] Show that there is no infinite sequence of pairwise disjoint, non-empty closed intervals $[a_n, b_n] \subseteq [0, 1]$ whose union is all of $[0, 1]$.

Prize problem 4. [Prize: 15 pounds] Does there exist an uncountable collection of subsets of $\mathbb{N}$ with the property that, for each pair of sets A, B in the collection, either A is a subset of B or B is a subset of A?

Prize problem 5. [Prize: 20 pounds] Let f be a real-valued function defined on the unit square $[0, 1]^2$. Suppose that, for each fixed $x \in [0, 1]$, $f(x, y)$ is a polynomial in y, and that, for each fixed $y \in [0, 1]$, the function $f(x, y)$ is a polynomial in x. Prove that f must be a polynomial in the two variables x and y.

Prize problem 6. [Prize: 10 pounds] Show that there DOES exist a function f from $\mathbb{Q} \times \mathbb{Q}$ to $\mathbb{Q}$ such that $f(x, y)$ is NOT a polynomial in x and y, and yet for each fixed $x \in \mathbb{Q}$, $f(x, y)$ is a polynomial in y, and, for each fixed $y \in \mathbb{Q}$, the function $f(x, y)$ is a polynomial in x.

Prize problem 7. [Prize: 20 pounds] Does there exist a pair of sequences (λ_n), (a_n) of non-zero complex numbers such that

(i) no two of the a_n are equal,

(ii) $\sum_{n=1}^{\infty} |\lambda_n| < \infty$,

(iii) $|a_n| < 2$ for all $n \in \mathbb{N}$, and yet,

(iv) for all $z \in \mathbb{C}$,

$$\sum_{n=1}^{\infty} \lambda_n \exp(a_n z) = 0?$$

Prize problem 8. [Prize: 5 pounds] Is there a continuous function f from $[0,1]$ to $\mathbb{R}$ with the following property: for all $x \in [0,1]$, $f(x)$ is rational if and only if x is irrational?

Prize problem 9. [Prize: 15 pounds] Let X be a metric space, and let A be a subset of X. Consider the collection of all subsets of X which can be obtained from A by taking successively either the complement in X or the closure (e.g. A, $X \backslash A$, clos$(X \backslash A)$, clos(A), etc). Show that no more than 14 of these sets may be different from each other. (This result holds for general topological spaces also.) Show also that it is possible to obtain 14 sets when $X = \mathbb{R}$ (with the usual metric) if A is a suitable subset of $\mathbb{R}$.

Prize problem 10. [Prize: 15 pounds] Let f be a function from $\mathbb{R}$ to $\mathbb{R}$ which is infinitely differentiable, i.e. f can be differentiated as many times as you like at all points of $\mathbb{R}$. Denote the nth derivative of f by $f^{(n)}$ (by convention, $f^{(0)}$ just means f).
Suppose that for all $a \in \mathbb{R}$ there exists at least one integer (possibly depending on the point a) $n \in \{0, 1, 2, \ldots\}$ such that $f^{(n)}(a) = 0$ (i.e. at every point of $\mathbb{R}$ there is a derivative of some order of f which is 0 at that point). Prove that f must be a polynomial function.

Prize problem 11. [Prize: 5 pounds] Let A and B be two convex polygons in $\mathbb{R}^2$ with $A \subseteq B$ (these polygons need not be regular, but do have only finitely many sides). Prove that the length of the boundary of A is no greater than the length of the boundary of B, and that equality holds if and only if $A = B$.