G13MTS METRIC AND TOPOLOGICAL SPACES

Model Solutions to the 2003–2004 Examination Paper

1 (a) (i) A **metric** on X is a function $d\colon X \times X \to \mathbb{R}^+$ satisfying three conditions for x, y, z in X:

I $d(x,y) = 0 \Leftrightarrow x = y$;

II $d(x,y) = d(y,x)$;

III $d(x,z) \leq d(x,y) + d(y,z)$.

(ii) A **pseudometric** on X is a function $d\colon X \times X \to \mathbb{R}^+$ satisfying II, III above and also

I' $d(x,x) = 0$ for all x in X.

(b) (i) This is neither a metric nor a pseudometric. Although both I and II hold, III fails. Indeed, with $x = 1$ and $z = 2$, $d(x,z) = \sin\left(\frac{\pi}{2}\right) = 1$, but $\lim_{n\to\infty} \left(\sin\left(\frac{\pi}{n}\right) + \sin\left(\frac{\pi}{n+1}\right)\right) = 0$ so (in fact for $n \geq 6$) taking $y = n+1$ for large enough n we obtain $d(x,z) > d(x,y) + d(y,z)$.

(ii) This is both a metric and a pseudometric. The proof is similar to that for the discrete metric.
The function $d(x,y)$ takes only values in

$$\left\{\cos\left(\frac{\pi}{n}\right) : n = 2, 3, 4, \ldots\right\}.$$

These values are all in $\mathbb{R}^+$, and we have $\cos\left(\frac{\pi}{n}\right) = 0$ when $n = 2$ but $\cos\left(\frac{\pi}{n}\right) \geq \frac{1}{2}$ for $n \geq 3$. Thus when $x = y$ we have $d(x,y) = 0$ while when $x \neq y$ we have $d(x,y) \in [\frac{1}{2}, 1]$.
Conditions I and II are now clear. For III, suppose for contradiction that

$$d(x,z) > d(x,y) + d(y,z).$$

Then it is clear that $x \neq y$ and $y \neq z$ (else equality holds). But then $d(x,y) \geq \frac{1}{2}$ and $d(y,z) \geq \frac{1}{2}$ giving $d(x,z) > \frac{1}{2} + \frac{1}{2} = 1$. This contradiction proves that III holds.

(iii) An easy one to finish. This is neither a metric nor a pseudometric since III fails. For example, with $x = 1$, $y = 2$, $z = 3$ we have

$$d(x,z) = 8 > 2 = d(x,y) + d(y,z).$$

2 (a) A **topology** on a set X is a collection $\tau \subseteq \mathcal{P}(X)$ satisfying

I ϕ and $X \in \tau$,

II whenever $U, V \in \tau$ then $U \cap V \in \tau$,

III whenever U_γ $(\gamma \in \Gamma)$ are in τ then $\bigcup_{\gamma \in \Gamma} U_\gamma$ is also in τ.

A **topological space** is a pair (X, τ) where X is a set and τ is a topology on X.

(b) Let $(X, \tau_1), (Y, \tau_2)$ be topological spaces. A function $f\colon X \to Y$ is **continuous** if, for all $U \in \tau_2$, we have $f^{-1}(U) \in \tau_1$. The map $f\colon X \to Y$ is a **homeomorphism** if f is a bijection from X to Y and both f and f^{-1} are continuous. The topological space X is **homeomorphic** to Y if there exists at least one homeomorphism $f\colon X \to Y$.

(c) Let $(X, d_X), (Y, d_Y)$ be metric spaces. A function $f\colon X \to Y$ is an **isometry** if, for all x_1, x_2 in X, $d_Y\big(f(x_1), f(x_2)\big) = d_X(x_1, x_2)$. The metric spaces X, Y are isometric to each other if there is at least one SURJECTIVE isometry $f\colon X \to Y$.

(d) We can define a homeomorphism $f\colon \mathbb{R} \to (0, \infty)$ by $f(x) = \exp(x)$. This is clearly continuous and is a bijection $\mathbb{R} \to (0, \infty)$. The inverse function is $\log(y)$ $(y \in (0, \infty))$ (or $\ln(y)$) and this is also continuous. Thus $\mathbb{R}$ is homeomorphic to $(0, \infty)$.

(e) We can take $X = \mathbb{N}$ with the usual metric and $A = \{2, 3, 4, \dots\}$. Then the map $f\colon X \to A$ defined by $f(n) = n + 1$ is clearly a surjective isometry, so A is isometric to $\mathbb{N}$, $A \subseteq \mathbb{N}$ and $A \neq \mathbb{N}$.

(f) Given $A \subseteq \mathbb{Q}$ and A isometric to $\mathbb{Q}$, there exists a surjective isometry $f\colon \mathbb{Q} \to A$. Let $q \in \mathbb{Q}$. We show that $q \in A$ as follows: choose any $a \in A$ (clearly A is not empty!). If $q = a$ there is nothing to prove. Otherwise set $r = |q - a|$. Then $f\big(f^{-1}(a) - r\big)$ and $f\big(f^{-1}(a) + r\big)$ are two different rational numbers distance r from a, so one of these must be q. Thus $q \in A$ and the result follows.

3 (a) Let (X, τ) be a topological space and let $Y \subseteq X$. Then the **subspace topology** τ_Y on Y is defined by

$$\tau_Y = \{U \cap Y : U \in \tau\}.$$

(b) Let (X, τ) be a topological space. An **open cover** of X is a non-empty collection of open subsets of X

$$U_\gamma \quad (\gamma \in \Gamma)$$

such that $X = \bigcup_{\gamma \in \Gamma} U_\gamma$.

Such an open cover has a **finite subcover** if there are $n \in \mathbb{N}$ and $\gamma_1, \gamma_2, \ldots, \gamma_n$ in Γ with $X = \bigcup_{k=1}^{n} U\gamma_k$.

The topological space X is **compact** if **every** open cover of X has a finite subcover.

(c) Given that a topological space (X, τ) is compact, let E be a closed subset of X. We show that E is compact.

Let U_γ $(\gamma \in \Gamma)$ be a non-empty collection of open subsets of X with $E \subseteq \bigcup_{\gamma \in \Gamma} U_\gamma$.

We must show that some finite number of the sets U_γ cover E. We have $X = (X \backslash E) \cup \bigcup_{\gamma \in \Gamma} U_\gamma$, and since $X \backslash E$ is open in X this is an open cover of X.

This open cover must have a finite subcover (since X is compact) so there must be $n \in \mathbb{N}$ and $\gamma_1, \gamma_2, \ldots, \gamma_n$ in Γ such that

$$X = (X \backslash E) \cup \bigcup_{k=1}^{n} U_{\gamma_k}.$$

We then have $E \subseteq \bigcup_{k=1}^{n} U_{\gamma_k}$, as required.

(d) Suppose that U_γ $(\gamma \in \Gamma)$ is a non-empty collection of open subsets of X with $X = \bigcup_{\gamma \in \Gamma} U_\gamma$. We show that this open cover must have a finite subcover. Clearly we may assume $X \neq \emptyset$, so there is at least one $\gamma_0 \in \Gamma$ with $U_{\gamma_0} \neq \emptyset$. Set $E = X \backslash U_{\gamma_0}$. Then E is closed in X and $E \neq X$. Thus E is compact.

We have $E \subseteq \bigcup_{\gamma \in \Gamma} U_\gamma$, so there are $n \in \mathbb{N}$ and $\gamma_1, \ldots, \gamma_n \in \Gamma$ with

$$E \subseteq \bigcup_{k=1}^{n} U_{\gamma_k}.$$

But then $X = \bigcup_{k=0}^{n} U_{\gamma_k}$, and so the open cover of X has a finite subcover, as claimed.

(e) The Heine–Borel theorem tells us that a subset of $\mathbb{R}$ is compact if and only if it is closed and bounded. Thus we need an infinite subset of $\mathbb{Q}$ which is bounded in $\mathbb{R}$ and closed in $\mathbb{R}$. There are many examples, e.g. $E = \{0\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$.

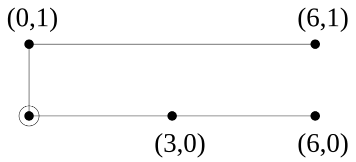
4 (a) Any of the equivalent standard definitions accepted. For example, X is **connected** if the only clopen subsets of X are $\emptyset$ and X.

(b) The connected subsets of $\mathbb{R}$ are the intervals (bounded, unbounded, open, half-open, closed, possibly empty). (Full marks for saying that they are the intervals.)

(c) Set $Z = f(E)$ and set $g = f|_E$. Then g is continuous and surjective from E to Z (here E, Z are given the subspace topologies). Let A be a clopen subset of Z. Then $g^{-1}(A)$ is a clopen subset of E. Since E is connected, we have $g^{-1}(A) = \emptyset$ or $g^{-1}(A) = E$. Thus $A = \emptyset$ or $A = f(E)$ as required.

(d) The function $(x, y) \mapsto x + y - xy$ is continuous from $\mathbb{R}^2$ to $\mathbb{R}$, so by (c) $\{x + y - xy \colon (x, y) \in E\}$ is a connected subset of $\mathbb{R}$, and so is an interval, I. Considering the points $(1, 5)$ and $(2, 4)$ in E we see that 1 and -2 are in I. Hence $0 \in I$ and the result follows.

(e) We know that path connected spaces are connected. In fact we can construct an example in $\mathbb{R}^2$ as follows.



Set

$$X = \{(x, 0) \colon 0 \le x \le 6\} \cup \{(x, 1) \colon 0 \le x \le 6\} \cup \{(0, y) \colon 0 \le y \le 1\}.$$

Give X the usual Euclidean metric. Clearly X is path connected, hence connected. But

$$B_X\big((3, 0), 2\big)$$

is clearly disconnected, as it consists of two non-empty line-segments distance 1 apart.

5 (a) A metric space (X,d) is **sequentially compact** if **every** sequence in X has at least one convergent subsequence.

A **Cauchy sequence** in a metric space (X,d) is a sequence $(x_n) \subseteq X$ such that

for all $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t. for all m and $n \geq N$ we have $d(x_n, x_m) < \varepsilon$.

The metric space (X,d) is **complete** if **every** Cauchy sequence in X converges in X.

(b) We quote the standard fact that any Cauchy sequence which has a convergent subsequence must itself converge.

Let (X,d) be a sequentially compact metric space and let (x_n) be a Cauchy sequence in X. Since X is sequentially compact, (x_n) has a convergent subsequence and so, by the fact stated above, (x_n) must itself converge.

(c) (i) A **topological property** is a property which is preserved by homeomorphism. Thus if X and Y are topological spaces which are homeomorphic to each other than any topological property which X has is shared by Y and vice versa.

(ii) We know that $\mathbb{R}$ is homeomorphic to $(0,\infty)$ (from lectures or 2(d)). However, with the usual metrics, $\mathbb{R}$ is complete and $(0,\infty)$ is not; a subset of $\mathbb{R}$ is complete with the usual metric if and only if it is closed in $\mathbb{R}$. Thus completeness is not presented by homeomorphism.

(iii) Let $(X,d_X),(Y,d_Y)$ be metric spaces. A function $f\colon X \to Y$ is **Lipschitz continuous** if there is a constant $A > 0$ such that for all x_1, x_2 in X we have

$$d_Y\big(f(x_1), f(x_2)\big) \leq A\ d_X(x_1, x_2).$$

(iv) By symmetry it is enough to show that if (X,d_X) is complete then so is (Y,d_Y).

Given that (X,d_X) is complete, let (y_n) be a Cauchy sequence in Y.
Since f^{-1} is Lipschitz continuous, $f^{-1}(y_n)$ is clearly a Cauchy sequence in X and so converges in X, say $f^{-1}(y_n) \to x$ as $n \to \infty$. But then f is continuous, so

$$y_n = f\big(f^{-1}(y_n)\big) \to f(x)$$

as $n \to \infty$, and so (y_n) converges in Y. This shows that (Y,d_Y) is complete, as required.