G13MTS Metric and Topological Spaces: Question Sheet 1

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Answers to questions 4 and 6 to be handed in to Dr Feinstein at the end of the Thursday lecture in the second week of teaching.

Always justify your answers.

1. Which of the following sequences of real numbers are convergent and which are divergent? For the convergent sequences, find the limit.

(i)

$$x_n = \frac{2n^2 + n + 3}{3n^2 + 1}.$$

(ii)

$$x_n = \left(1 + \frac{1}{n}\right)^{2n}.$$

(iii)

$$x_n = \sin(n).$$

2. Let (x_n), (y_n) be bounded sequences of real numbers. Prove that

$$\limsup_{n\to\infty}(x_n + y_n) \le \limsup_{n\to\infty}(x_n) + \limsup_{n\to\infty}(y_n)$$

and that

$$\liminf_{n\to\infty}(x_n + y_n) \ge \liminf_{n\to\infty}(x_n) + \liminf_{n\to\infty}(y_n).$$

Give examples to show that both of these inequalities may be strict.

3. Let X be a set and let $d : X \times X \longrightarrow \mathbb{R}$ be a function which satisfies the three metric space axioms:

(i) $d(x, y) = 0 \iff x = y$

(ii) $d(x, y) = d(y, x)$ for all x, y in X

(iii) $d(x, z) \le d(x, y) + d(y, z)$ for all x, y, z in X.

Prove that $d(x, y) \ge 0$ for all x, y in X, and so d is a metric.

4. Prove carefully that the following three examples from the module really are metrics:

(a) On $\mathbb{R}^n$,

$$d_\infty(\mathbf{x}, \mathbf{y}) = \max\{|x_k - y_k| : 1 \le k \le n\}.$$

(b) On $C[0,1]$,

$$d_1(f, g) = \int_0^1 |f(x) - g(x)| dx \ .$$

[Hint: the first axiom is the hard one here! One method involves using the Fundamental Theorem of Calculus, and differentiating the function $\int_0^x |f(t) - g(t)| \,\mathrm{d}t$.]

(c) The discrete metric: here X is any set, and

$$d(x, y) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{otherwise.} \end{cases}$$

5. Consider the following functions in $C[0,1]$: $f(x) = x$, $g(x) = \frac{5}{4}x^2$. Calculate $d_1(f, g)$ and $d_\infty(f, g)$.

6. Let (X, d) be a metric space, and let (x_n), (y_n) be convergent sequences in X, with $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Prove that

$$\lim_{n\to\infty} d(x_n, y_n) = d(x, y).$$

Note: you **must not assume** that the last limit exists. This is part of the proof!

Note also that the first two sequences are contained in the general metric space X. But the last sequence, $(d(x_n, y_n))$, is a sequence of real numbers, whose convergence must be investigated using the usual notion of convergence in $\mathbb{R}$. (When working with sequences in $\mathbb{R}$, we will use the usual notion of convergence unless otherwise specified).

7. Write down three different metrics on $\mathbb{N}$, no two of which are multiples of each other.

8. Recall that a **pseudometric** is a function which satisfies all the conditions of a metric, except that possibly two different points may be distance zero apart (so that every metric is a pseudometric). Which of the following are metrics? Pseudometrics? Neither?

(i) on $\mathbb{R}^2$, $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2|$,
(ii) on $\mathbb{C}$, define $d(z, w)$ depending on whether $0, z, w$ lie on a straight line. If they do, define $d(z, w) = |z - w|$. Otherwise define $d(z, w) = |z| + |w|$.
(iii) on $\mathbb{R}^2$, $d((x_1, y_1), (x_2, y_2)) = \min\{|x_1 - x_2|, |y_1 - y_2|\}$.
(iv) on $\mathbb{R}^2$, $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2|^2 + |y_1 - y_2|^2$.

9. Let X be the set of all bounded sequences of real numbers. We denote a typical element of X by $\mathbf{x} = (x_n)$ or $(x_n)_{n=1}^\infty$. In this question you may assume that X is a vector space over $\mathbb{R}$ with the usual componentwise operations: for $\mathbf{x}$ and $\mathbf{y}$ in X and $\alpha \in \mathbb{R}$, with $\mathbf{x} = (x_n)$ and $\mathbf{y} = (y_n)$, we have $\mathbf{x} + \mathbf{y} = (x_n + y_n)_{n=1}^\infty$ and $\alpha\mathbf{x} = (\alpha x_n)_{n=1}^\infty$.

For $\mathbf{x} = (x_n) \in X$, define $\|\mathbf{x}\|_\infty = \sup\{|x_n| : n \in \mathbb{N}\}$. Prove that $\|\cdot\|_\infty$ is a norm on X.

(The space X in this question is one of the basic examples in the theory of Functional Analysis. It is usually denoted by ℓ^∞.)