G13MTS Metric and Topological Spaces: Question Sheet 2

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Answers to questions 1 and 6 to be handed in to Dr Feinstein at the end of the Thursday lecture in the fourth week of teaching.

If X is a subset of $\mathbb{R}$ or $\mathbb{C}$ then, **unless otherwise specified**, the metric we will use on X will be the 'usual' metric: $d_X(x, y) = |x - y|$ for x, y in X.

Always justify your answers.

1.(i) In $\mathbb{R}^2$ with the metric d_∞ (as defined in the notes), sketch the ball $B((0, 0), 1)$.

(ii) In $\mathbb{R}^2$ with the jungle river metric (defined in the notes), sketch the ball $B((2, 1), 3/2)$.

2. Which, if any, of the following functions from $(0, \infty)$ to $\mathbb{R}$ are (a) uniformly continuous? (b) Lipschitz continuous?

(i) $f_1(x) = \sqrt{x}$ (ii) $f_2(x) = \sin(x)$ (iii) $f_3(x) = x^2$ (iv) $f_4(x) = 1/\sqrt{x}$.

3. Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose that d_X is the discrete metric. Prove that every function $f : X \longrightarrow Y$ is continuous.

4. Let (X, d) be a metric space, let (x_n) be a sequence in X and let $x \in X$. Prove that the following statements are equivalent.

(i) $x_n \to x$ as $n \to \infty$.

(ii) For every subsequence (y_n) of (x_n) there is at least one subsequence (z_n) of (y_n) such that $z_n \to x$ as $n \to \infty$.

(Another way to write (ii) is to say that every subsequence of (x_n) has at least one subsequence which converges to x. Here you will be dealing with subsequences of subsequences, which are sometimes called subsubsequences. However, you should be able to convince yourself that every subsubsequence of a sequence (x_n) is also a subsequence of (x_n).

Warning: remember that, in general, a divergent sequence may possibly have some convergent subsequences!)

5. Let X and Y be metric spaces, let f be a function from X to Y and let $x \in X$. Prove that the following three statements are equivalent. (This is slightly stronger than one of the results stated in the notes.)

(a) The function f is continuous at x.

(b) For every sequence $(x_n) \subseteq X$ which converges to x, we have $f(x_n) \to f(x)$ in Y as $n \to \infty$.

(c) For every sequence $(x_n) \subseteq X$ which converges to x, there is at least one subsequence (z_n) of (x_n) such that $f(z_n) \to f(x)$ in Y as $n \to \infty$.

6. Let X be $C[0,1]$. Define $F : X \longrightarrow \mathbb{R}$ by $F(f) = f(0) \quad (f \in C[0,1])$. Is F continuous when X is given (i) the metric d_1 (ii) the metric d_∞? (These metrics were defined in the notes).

7. Let $X = [0,2) \cup [3,5]$ with the usual metric (which is, of course, the same as the subspace metric induced by the usual metric on $\mathbb{R}$). Which of the following sets are open (a) in $\mathbb{R}$ (b) in X?

(i) $X \cap (1,3)$ (ii) $X \cap [3,6]$ (iii) $X \cap [4,6]$ (iv) X.

8. Let X, Y be metric spaces, let $Z \subseteq Y$, and give Z the subspace metric induced by the metric on Y. Let $f : X \longrightarrow Z$ be a function, and note that f may also be regarded as a function from X to Y. Thinking of f in these two ways, show that f is continuous from X to Z if and only if f is continuous from X to Y.