G13MTS Metric and Topological Spaces: Question Sheet 3

Dr J. F. Feinstein

Answers to questions 8 and 9 to be handed in to Dr Feinstein at the end of the Thursday lecture in the sixth week of teaching.

Unless otherwise specified, the topology on any subset of $\mathbb{R}$ is assumed to be the usual topology (induced by the usual metric on the subset, $d(x, y) = |x - y|$).

Always justify your answers.

1. In $\mathbb{C}$ with the French railway metric (as in Sheet 1, Q8 (ii)), sketch the open ball $B(1, 3/2)$.

2. (i) Let (X, τ) be a topological space. Prove that every finite union of closed subsets of X is again a closed subset of X.

Now suppose that E_γ $(\gamma \in \Gamma)$ is a collection of closed subsets of X, where Γ is a (non-empty, and possibly infinite) indexing set. Prove that

$$\bigcap_{\gamma \in \Gamma} E_\gamma$$

is closed in X.
(ii) Give an example of a metric space X together with an infinite family of open subsets of X whose intersection is not open in X.

Recall that a topological space X is **Hausdorff** if, whenever x, y are distinct elements of X then there are open subsets U, V of X such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Recall also that every metric space is a topological space, using the topology induced by the given metric. However, not all topological spaces are metric spaces.

3. (i) Prove that every metric space is a Hausdorff topological space.
(ii) Let (X, τ) be a Hausdorff topological space, and suppose that $x \in X$. Prove that $\{x\}$ is closed in X.

4. Find all possible topologies on the set $\{1, 2\}$, and decide for each of them whether they can be induced (i) by a pseudometric on $\{1, 2\}$, (ii) by a metric on $\{1, 2\}$.

5. Define a collection of subsets of $\mathbb{R}$, τ, by

$$\tau = \{\emptyset, \mathbb{R}\} \cup \{(-\infty, a) : a \in \mathbb{R}\}.$$

Prove that τ is a topology on $\mathbb{R}$.

6. Let (X, d) be a metric space and let $x \in X$. Is it necessarily true that the closure of the ball $B(x, 1)$ is equal to the set

$$\{y \in X : d(y, x) \leq 1\}$$

or can you find a counterexample?

7. Let X be a topological space and let $Y \subseteq X$. Prove that the following statements are equivalent.

(i) Y is dense in X;

(ii) for every $x \in X$ and every neighbourhood N of x, we have $N \cap Y \neq \emptyset$;

(iii) $\overline{Y} = X$;

(iv) $\text{int}\,(X \setminus Y) = \emptyset$.

8. (i) Let A and B be subsets of a topological space X. Prove that $\text{clos}\,(A \cup B) = \overline{A} \cup \overline{B}$.
(ii) Give an example of two subsets A, B of $\mathbb{R}$ such that $\text{clos}\,(A \cap B) \neq \overline{A} \cap \overline{B}$.
(iii) Let U, V be dense open subsets of a topological space X. Prove that $U \cap V$ is dense in X. (Warning! Don't forget what you discovered in (ii). To avoid a common error here, you are recommended to use the original definition of dense in terms of intersections with non-empty open sets).

9. Give an example of an open set U in a topological space X such that

$$U \neq \text{int}\,(\overline{U}).$$

10. Let X be a topological space and let $A \subseteq X$. Let U be the interior of A and let B be the boundary of A.
(i) Prove that $A \subseteq U \cup B$.
(ii) Prove that A is open if and only if $A \cap B = \emptyset$.
(iii) Prove that A is closed if and only if $B \subseteq A$.

11. (i) Let X be a metric space and let U be a non-empty open subset of X. Prove that there is a (possibly uncountably infinite) collection of open balls in X whose union is U.
(ii) Let X be a metric space and let Y be a subset of X. Give Y the subspace metric. Let $U \subseteq Y$. Prove that U is open in Y if and only if there exists an open subset of X, V, satisfying $U = V \cap Y$. (Hint: to prove the 'only if' part, use part (i). Prove the 'if' part directly).
(iii) (For those of you who know about countability and uncountability) Prove that every non-empty open subset of $\mathbb{R}$ is a **countable** union of open intervals. (Note this is slightly better than the result given in (i), because in (i) you may need uncountably many sets. It should help you if you use the countability of the dense subset $\mathbb{Q}$).