G13MTS Metric and Topological Spaces: Question Sheet 4

Answers to questions 5, 11, 15 and 17 to be handed in to Dr Feinstein at the end of the Thursday lecture in the eighth week of teaching.

Unless otherwise specified, the topology on any subset of $\mathbb{R}$ is assumed to be the usual topology.

Recall that when (X, d) is a metric space, E is a non-empty subset of X and $x \in X$, then $\text{dist}(x, E) = \text{dist}(E, x) = \inf\{d(x, y) : y \in E\}$.

Always justify your answers.

1. In the lecture notes there are two notions of continuity at a point: one is for functions between metric spaces, the other is for functions between topological spaces (defined in terms of neighbourhoods). Prove that these two notions agree for functions between metric spaces.

2. Let X and Y be topological spaces and let f be a function from X to Y. Prove that the following statements are equivalent.

(i) The function f is continuous from X to Y.

(ii) For every closed subset E of Y, $f^{-1}(E)$ is closed in X.

(iii) (Slightly harder) For every subset A of X, we have

$$f(\text{clos}\, A) \subseteq \text{clos}\,(f(A))\,.$$

3. Is $\mathbb{R}$ isometric to $(-\pi/2, \pi/2)$ (using the usual metrics)?

4. Let X be a set, let (Y, d_Y) be a metric space, and let $f : X \to Y$. Define $\rho_f(x, x') = d_Y(f(x), f(x'))$. prove that ρ_f is a pseudometric on X, and that ρ_f is a metric on X if and only if f is injective.

The result of Question 4 is a standard fact: you may find it useful to refer to this fact in some of the questions below.

5. By giving an explicit example of a homeomorphism, show that $\mathbb{R}$ is homeomorphic to the open interval $(-\pi/2, \pi/2)$. Hence give an example of a metric on $(-\pi/2, \pi/2)$ which is equivalent to the usual metric, but is NOT uniformly equivalent to it.

6. Is $\mathbb{R}$ homeomorphic to $[-1, 1]$?

7. Give an example of two metrics on $\mathbb{N}$ which are not equivalent to each other. (Note that it is NOT enough to show that the metrics you choose are not uniformly equivalent. One of the earlier questions may help you here).

8. Let X be a metric space, and let E, F be non-empty subsets of X. Prove that

$$\inf\{\text{dist}(x, F) : x \in E\} = \inf\{\text{dist}(E, y) : y \in F\}.$$

[Remember that, in general, an infimum need not necessarily be achieved.]

9. Let X be a metric space, and let E_k $(1 \le k \le 3)$ be non-empty, closed subsets of X which are pairwise disjoint (i.e. no two of these sets have any points in common). Let $\alpha_k \in \mathbb{R}$ $(1 \le k \le 3)$. Use Urysohn's lemma to prove that there is a continuous function $f : X \longrightarrow \mathbb{R}$ such that, for $1 \le k \le 3$, $f(x) = \alpha_k$ for all $x \in E_k$.

Recall that a topological space X is **regular** if, whenever E is a closed subset of X and $x \in X \setminus E$, there are open subsets U and V of X such that $x \in U$, $E \subseteq V$ and $U \cap V = \emptyset$. The topological space X is **normal** if, whenever E and F are closed subsets of X with $E \cap F = \emptyset$, then there are open subsets U, V of X such that $E \subseteq U$, $F \subseteq V$ and $U \cap V = \emptyset$.

10. Let (X, d) be a metric space, and give X the topology induced by its metric.

(a) (Easy directly, or you can deduce this from (b)) Prove that X is regular.

(b) (Needs more care: it is easiest to use Urysohn's Lemma, but you should also see if you can find a direct proof.) Prove that X is normal.

11. Give an example of a topological space which is not normal.
(Hint: the indiscrete topology does not work here. Try to find an example of such a space which has exactly three points.)

12. Let (X, τ_1) and (Y, τ_2) be topological spaces and let $f : X \to Y$. Prove the following statements (stated in the lecture notes).

(a) Let $x \in X$, let $\mathcal{N}$ be a base of neighbourhoods at $f(x)$ and let $\mathcal{N}'$ be a base of neighbourhoods at x. Then f is continuous at x if and only if the following condition holds: for all $N \in \mathcal{N}$ there exists $N' \in \mathcal{N}'$ with $f(N') \subseteq N$.

(b) Let $\mathcal{B}$ be a base for τ_2. Then f is continuous from X to Y if and only if, for all $U \in \mathcal{B}$, $f^{-1}(U) \in \tau_1$.

(c) Let τ_3 be another topology on Y. Then τ_2 is weaker than τ_3 (i.e., $\tau_2 \subseteq \tau_3$) if and only if $\mathcal{B} \subseteq \tau_3$.

13. Let (X, τ) be a topological space, let $Y \subseteq X$ and let $Z \subseteq Y$. Let τ_Y be the subspace topology on Y induced by τ, and let τ_Z be the subspace topology on Z induced by τ. Prove that τ_Z is the same as the subspace topology induced on Z by τ_Y.

14. Let (X, τ) be a topological space, let W be an open subset of X and let F be a closed subset of X. Let $U \subseteq W$ and $E \subseteq F$. Give W and F the subspace topologies induced by τ. Show that U is open in W if and only if U is open in X, and that E is closed in F if and only if E is closed in X. How much of this remains true if U is not open in X or F is not closed in X?

15. Let (X, τ_1) be a topological space, let Y be a set and let $q : X \to Y$ be a function. Give Y the quotient topology τ_2 induced by q. Now let (Z, τ_3) be another topological space, let $f : Z \to X$ and let $g : Y \to Z$.

(a) Prove that g is continuous from Y to Z if and only if $g \circ q$ is continuous from X to Z.

(b) Is it necessarily the case that f is continuous if and only if $q \circ f$ is continuous?

16. Let $X = [-1, 1]$, and give X the usual topology.
(a) Define a function $f : X \longrightarrow [0, 1]$ by $f(x) = x^2$. Show that the quotient topology on $[0, 1]$ induced by f is the same as the usual topology on $[0, 1]$.
(b) (Harder) Give an example of a surjective function g from X onto $[0, 1]$ such that the quotient topology on $[0, 1]$ induced by g is not Hausdorff.

17. Let X, Y be non-empty topological spaces. Give $X \times Y$ the product topology. Let p_X, p_Y be the usual coordinate projections from $X \times Y$ onto X, Y respectively.
(a) Prove that $X \times Y$ is Hausdorff if and only if both X and Y are Hausdorff.
(b) Let Z be a topological space and let f be a function from Z to $X \times Y$. Prove that f is continuous if and only if both $p_X \circ f$ and $p_Y \circ f$ are continuous (from Z to X, Z to Y respectively).
(c) Let $y \in Y$. Set $X' = X \times \{y\}$, and give X' the subspace topology induced by the topology on $X \times Y$. Prove that X is homeomorphic to X'.
(d) Let $X' \subseteq X$ and let $Y' \subseteq Y$. Then there are two obvious ways to put a topology on $X' \times Y'$. One is to give X' and Y' the subspace topologies from X and Y respectively and then give $X' \times Y'$ the product topology accordingly, as a product of the topological spaces X' and Y'. The other is to regard $X' \times Y'$ as a subset of $X \times Y$, and give $X' \times Y'$ the subspace topology induced by the product topology on $X \times Y$. Prove that these two topologies on $X' \times Y'$ are the same.