G13MTS: Metric and Topological Spaces

Question Sheet 5.

Answers to Questions 1, 2 and 10 to be handed in at the end of the Thursday lecture in the tenth week of teaching.

Unless otherwise specified, the topology on any subset of $\mathbb{R}$ is assumed to be the usual topology (induced by the usual metric on the subset, $d(x,y) = |x-y|$), and the topology on $\mathbb{R}^2$ is the one induced by any of the following three metrics on $\mathbb{R}^2$: d_1, d_2 or d_∞ (your choice!).

Always justify your answers.

Recall that when (X,d) is a metric space, E is a non-empty subset of X and $x \in X$, then $\mathrm{dist}(x,E) = \mathrm{dist}(E,x) = \inf\{d(x,y) : y \in E\}$.

1. Let (X,d) be a metric space.
(i) Let E be a non-empty, compact subset of X, and let $x \in X$. Show that there exists $y \in E$ such that $d(x,y) = \mathrm{dist}(x,E)$.
(ii) Let E, F be two disjoint, non-empty subsets of X with E compact and F closed. Show that

$$\inf\{d(x,y) : x \in E, y \in F\} > 0.$$

[Warning! Part (iii) shows some care is needed here.]
(iii) Give an example of two disjoint closed subsets of $\mathbb{R}^2$ such that

$$\inf\{d(\mathbf{x},\mathbf{x}') : \mathbf{x} \in E, \mathbf{x}' \in F\} = 0.$$

2. Let (X,d) be a totally bounded metric space, and let Y be a subset of X. Give Y the subspace metric $\widetilde{d}$ induced by d. Prove that $(Y,\widetilde{d})$ is also a totally bounded metric space.

3.(a) Prove that every compact, Hausdorff topological space is regular.

(b) Prove that every compact, Hausdorff topological space is normal. (Hint: use part (a).)

(c) A topological space X is said to be **locally compact** if every point x of X has a neighbourhood base consisting of compact neighbourhoods.

Prove that every compact, Hausdorff topological space is locally compact. (Hint: use part (a).)

(d) Show that $\mathbb{Q}$ is not locally compact.

4. Prove that all of the following subsets A, B, C of $\mathbb{R}^2$ are homeomorphic to each other. (you may assume that they are compact, and you can save some time by using Corollary 5.27).

$$A = \{(x,y) \in \mathbb{R}^2 : |x| + |y| = 1\}$$

$$B = \{(x,y) \in \mathbb{R}^2 : \max\{|x|,|y|\} = 1\}$$

$$C = ([0,1] \times \{0\}) \cup (\{0\} \times [0,1]) \cup \{(x,y) : x,y \in \mathbb{R}^+, x+y = 1\}.$$

5. (i) Is $\mathbb{R}$ homeomorphic to $[-1,1)$? (ii) (harder) Is $\mathbb{R}$ homeomorphic to $\mathbb{R}^2$?

[Question 5 (ii) is made more interesting by the fact that there are some continuous functions which map $\mathbb{R}$ onto $\mathbb{R}^2$. Those interested can read up about space-filling curves.]

6. For each type of non-empty interval $I \subseteq \mathbb{R}$, give an example of a continuous surjection from $\mathbb{R}$ onto I.

7. Let X and Y be non-empty, connected topological spaces. Prove that $X \times Y$, with the (standard) product topology, is connected. [Hint: assume for a contradiction that there is a continuous, non-constant, continuous function from $X \times Y$ to the discrete topological space $\{0, 1\}$.]

8. Find (a) the connected components and (b) all the connected subsets of the following metric spaces:
(i) $\mathbb{R} \setminus \mathbb{Q}$.
(ii) $\mathbb{Q} \times \mathbb{Q}$ (product topology).
(iii) $[0, 1] \cup [2, 3) \cup (4, 5)$.

9. Choose one of the sets A, B, C from Question 4 above. Prove that your choice is a connected subset of $\mathbb{R}^2$.

10. Let E be a connected subset of a topological space X, and let F be a subset of X satisfying $E \subseteq F \subseteq \overline{E}$. Prove that F is connected. (Hint: use the closed sets version of connectedness).

11. Let X be the subset of $\mathbb{R}^2$ defined by

$$X = \{(0, y) : -1 \leq y \leq 1\} \cup \{(x, \sin(1/x)) : x > 0\}.$$

Prove that X is connected.

12. Which of the following subsets of $\mathbb{R}$ are (i) compact (ii) complete?
(a) $[0, 1)$ (b) $[0, \infty)$ (c) $\{x \in \mathbb{R} : x^4 \leq 3x^2 - 1\}$.

13. Suppose that X and Y are metric spaces which are isometric to each other, and that X is complete. Prove that Y is complete. [Warning: it is not enough to say that X and Y are homeomorphic, because completeness is not always preserved by homeomorphisms: for example $\mathbb{R}$ is homeomorphic to $(-1, 1)$, but with the usual metrics only one of these is complete].

14. (i) Show that $\mathbb{Q}$ is not complete.
(ii) Find an infinite subset A of $\mathbb{Q}$ such that A is complete with the usual metric, but such that A is not homeomorphic to $\mathbb{N}$. [Warning: make sure that the example you give really is complete!]

15. Let X be $C[0, 1]$ with the metric d_∞. Set

$$E = \{f \in X : |f(t)| \leq 1 \text{ for all } t \in [0, 1]\}.$$

(i) Is E complete with the subspace metric induced by d_∞?
(ii) Is E a compact subset of X?