Statement and proof of Lemma 5.5

Lemma 5.5 Let X be a metric space, let $(x_n) \subseteq X$ and let $x \in X$. For each $n \in \mathbb{N}$, set

$$A_n = \{x_n, x_{n+1}, x_{n+2}, \ldots\} = \{x_k : k \geq n\}.$$

Then the following statements are equivalent:

(a) (x_n) has a subsequence which converges to x;

(b) $x \in \bigcap_{n=1}^{\infty} \operatorname{clos} A_n$.

Thus (x_n) has a convergent subsequence if and only if

$$\bigcap_{n=1}^{\infty} \operatorname{clos} A_n \neq \emptyset.$$

Proof It is clear that the last statement follows once we have proved the equivalence of (a) and (b).

((a)⇒(b)) Given (a), there are $n_1 < n_2 < \cdots$ in $\mathbb{N}$ such that $x_{n_k} \to x$ as $k \to \infty$. Let $N \in \mathbb{N}$. Then, for $k \geq N$, we have $x_{n_k} \in A_N$, and so we have a sequence in A_N which converges to x. Thus $x \in \operatorname{clos} A_N$. As $N \in \mathbb{N}$ was arbitrary, (b) follows.

((b)⇒(a)) Given (b), we proceed as follows. Since $x \in \operatorname{clos} A_1$, we may choose $n_1 \in \mathbb{N}$ with $x_{n_1} \in B_X(x, 1)$. Then $x \in \operatorname{clos} A_{n_1+1}$, so there must be an $n_2 > n_1$ with $x_{n_2} \in B_X(x, 1/2)$. Inductively, having chosen $n_1 < n_2 < \cdots < n_k$ such that $x_{n_j} \in B_X(x, 1/j)$ for $j = 1, 2, \ldots, k$, we choose n_{k+1} as follows: we know that $x \in \operatorname{clos} A_{n_k+1}$. Thus we may choose $n_{k+1} > n_k$ such that $x_{n_{k+1}} \in B_X(x, 1/(k+1))$.

This produces a subsequence (x_{n_k}) of (x_n), and clearly this subsequence converges to x, as required.

An alternative proof involves proving the equivalence of (a) and (b) to each of the following conditions:
(c) for all $\varepsilon > 0$ and all $N \in \mathbb{N}$, there exists $n \geq N$ with $x_n \in B_X(x, \varepsilon)$;
(d) for all $\varepsilon > 0$, the set $\{n \in \mathbb{N} : x_n \in B_X(x, \varepsilon)\}$ is an infinite subset of $\mathbb{N}$.

Exercise: check the details of this method.