G1BCOF Complex Functions

These slides are heavily based on the lecture notes by Professor J. K. Langley for his module G12CAN (2002-3).

Dr Feinstein has made minor alterations.

The full version of these notes is available from the module web page.

SUMMARY: in this module we concentrate on functions which can be regarded as functions of a complex variable, and are differentiable with respect to that complex variable.

These "good" functions include exp, sine, cosine etc. (but log will be a bit tricky).

These are important in applied maths, and they turn out to satisfy some very useful and quite surprising and interesting formulas.

For example, one technique we learn in this module is how to calculate integrals like

$$\int_{-\infty}^{+\infty} \frac{\cos x}{x^2+1} \, dx$$

WITHOUT actually integrating.

1.1 Basic Facts on Complex Numbers

This revision section on complex numbers is in the lecture notes available from the module web page.

You should look through this to check that you are confident with this material.

1.2 Introduction to complex integrals

Suppose first of all that $[a,b]$ is a closed interval in $\mathbb{R}$ and that $g:[a,b] \to \mathbb{C}$ is continuous (this means simply that $u = \mathrm{Re}(g)$ and $v = \mathrm{Im}(g)$ are both continuous).

We can just define

$$\int_a^b g(t)\; dt$$

to be

$$\int_a^b \mathrm{Re}(g(t))\; dt + i\int_a^b \mathrm{Im}(g(t))\; dt.$$

Example Determine $\int_0^2 e^{2it} dt$.

A useful estimate on how big such an integral can be comes from the fact that

$$\left|\int_a^b g(t) \ dt\right| \leqslant \int_a^b |g(t)| \, dt.$$

Example: for $n \in \mathbb{N}$ set

$$I_n = \int_1^2 e^{it^3} (t + in)^{-1} dt$$

Show that $I_n \to 0$ as $n \to +\infty$.

1.3 Paths and contours

Suppose that f_1, f_2 are continuous real-valued functions on a closed interval $[a,b]$.

As the "time" t increases from a to b, the point $\gamma(t) = f_1(t) + i f_2(t)$ traces out a curve (or path, we make no distinction between these words in this module) in $\mathbb{C}$.

A path in $\mathbb{C}$ is then just a continuous function γ from a closed interval $[a,b]$ to $\mathbb{C}$, in which we agree that γ will be called continuous iff its real and imaginary parts are continuous.

Paths are not always as you might expect.

There is a path $\gamma:[0,2] \to \mathbb{C}$ such that γ passes through every point in the rectangle $w = u + iv$, $u,v \in [0,1]$.

(You can find this on p.224 of Math. Analysis by T. Apostol).

There also exist paths which never have a tangent although (it's possible to prove that) you can't draw one.

Because of this awkward fact, we define a special type of path with good properties:

A smooth contour is a path $\gamma:[a, b] \to \mathbb{C}$ such that the derivative γ' exists and is continuous and never 0 on $[a, b]$.

Notice that if we write $\mathrm{Re}(\gamma) = \sigma$, $\mathrm{Im}(\gamma) = \tau$ then $(\sigma'(t), \tau'(t))$ is the tangent vector to the curve, and we are assuming that this varies continuously and is never the zero vector.

For $a < t < b$ let $s(t)$ be the length of the part of the contour γ between "time" a and "time" t. Then if δt is small and positive, $s(t+\delta t)-s(t)$ is approximately equal to $|\gamma(t+\delta t)-\gamma(t)|$ and so

$$\frac{ds}{dt} = \lim_{\delta t \to 0+} \frac{|\gamma(t+\delta t)-\gamma(t)|}{\delta t} = |\gamma'(t)|.$$

Hence the length of the whole contour γ is $\int_a^b |\gamma'(t)|\, dt$, and is sometimes denoted by $|\gamma|$.

Examples

(i) A circle of centre a and radius r described once counter-clockwise.

The formula is $z = a + re^{it}, 0 \leqslant t \leqslant 2\pi$.

(ii) The straight line segment from a to b.

This is given by $z = a + t(b - a), 0 \leqslant t \leqslant 1$.

1.4 Introduction to contour integrals

Suppose that $\gamma:[a, b] \to \mathbb{C}$ is a smooth contour.

If f is a function such that $f(\gamma(t))$ is continuous on $[a, b]$ we set

$$\int_\gamma f(z)\ dz = \int_a^b f(\gamma(t))\ \gamma'(t)\ dt.$$

1.5 a very important example!

Let $a \in \mathbb{C}$, let $m \in \mathbb{N}$ and $r > 0$, and set $\gamma(t) = a + re^{it}$, $0 \leqslant t \leqslant 2m\pi$.

As t increases from 0 to $2m\pi$, the point $\gamma(t)$ describes the circle $|z-a| = r$ counter-clockwise m times.

Now let $n \in \mathbb{Z}$, and consider the integral

$$\int_\gamma (z-a)^n \, dz = \int_0^{2m\pi} r^n e^{int} \, ire^{it} \, dt.$$

We have

$$\int_\gamma (z-a)^n \, dz = \int_0^{2m\pi} r^n e^{int} \, ire^{it} \, dt$$

$$= \int_0^{2m\pi} i \, r^{n+1} \, e^{(n+1)it} \, dt.$$

If $n \neq -1$ this is 0, by periodicity of $\cos((n+1)t)$ and $\sin((n+1)t)$.

If $n = -1$ then we get $2m\pi i$.

1.6 Properties of contour integrals

(a) If $\gamma:[a,b] \to \mathbb{C}$ is a smooth contour and λ is given by $\lambda(t) = \gamma(b+a-t)$ (so that λ is like γ "backwards") then

$$\int_\lambda f(z)\ dz$$

is given by

$$\int_a^b f(\gamma(b+a-t))\ (-\gamma'(b+a-t))\ dt$$

$$= -\int_\gamma f(z)\ dz.$$

(b) A smooth contour is called SIMPLE if it never passes through the same point twice (i.e. it is a one-one function).

Suppose that λ and γ are simple, smooth contours which describe the same set of points in the same direction.

Suppose λ is defined on $[a,b]$ and γ on $[c,d]$.

It is easy to see that there is a strictly increasing function $\phi:[a,b] \to [c,d]$ such that $\lambda(t) = \gamma(\phi(t))$ for $a \leqslant t \leqslant b$.

Further, it is quite easy to prove that $\phi(t)$ has continuous non-zero derivative on $[a,b]$ and we can write

$$\int_\lambda f(z)\, dz = \int_a^b f(\lambda(t))\, \lambda'(t)\, dt$$

$$= \int_a^b f(\gamma(\phi(t)))\, \gamma'(\phi(t))\, \phi'(t)\, dt$$

$$= \int_c^d f(\gamma(s))\gamma'(s)\, ds = \int_\gamma f(z)\, dz.$$

Thus the contour integral is "independent of parametrization".

(c) This is called the FUNDAMENTAL ESTIMATE; suppose that $|f(z)| \leqslant M$ on γ and the length of γ is L.

Then we have

$$\left|\int_\gamma f(z)\, dz\right| \leqslant \int_a^b |f(\gamma(t))|\, |\gamma'(t)|\, dt$$

$$\leqslant M\int_a^b |\gamma'(t)| dt = ML.$$

Example: let γ be the straight line from 2 to $3+i$, and let $I_n = \int_\gamma dz/(z^n+\bar{z})$, with n a positive integer.

Show that $I_n \to 0$ as $n \to \infty$.

Some more definitions

By a PIECEWISE SMOOTH contour γ we mean finitely many smooth contours γ_k joined end to end, in which case we define

$$\int_\gamma f(z)\ dz = \sum_k \int_{\gamma_k} f(z)\ dz.$$

The standard example is a STEPWISE CURVE: a path made up of finitely many straight line segments, each parallel to either the real or imaginary axis, joined end to end.

For example, to go from 0 to $1+i$ via 1 we can use $\gamma_1(t) = t$, $0 \leqslant t \leqslant 1$ followed by $\gamma_2(t) = 1+(1+i-1)t$, $0 \leqslant t \leqslant 1$.

Note that by 1.6(b) if you need $\int_\gamma f(z)\,dz$ it doesn't generally matter how you do the parametrization.

Suppose γ is a PSC made up of the smooth contours $\gamma_1,.....,\gamma_n$, in order.

It is sometimes convenient to combine these n formulas into one.

Assuming each γ_j is defined on [0,1] (if not we can easily modify them) we can put

$$\gamma(t) = \gamma_j(t-j+1),\ j-1 \leqslant t \leqslant j. \quad (1)$$

The formula (1) then defines γ as a continuous function on $[0,n]$.

A piecewise smooth contour is SIMPLE if it never passes through the same point twice (i.e. γ as in (1) is one-one), CLOSED if it finishes where it started (i.e. $\gamma(n) = \gamma(0)$) and SIMPLE CLOSED if it finishes where it started but otherwise does not pass through any point twice (i.e. γ is one-one except that $\gamma(n) = \gamma(0)$).

These are equivalent to:

γ is CLOSED if it finishes where it starts i.e. the last point of γ_n is the first point of γ_1.

γ is SIMPLE if it never passes through the same point twice (apart from the fact that γ_{k+1} starts where γ_k finishes).

γ is SIMPLE CLOSED if it finishes where it starts but otherwise doesn't pass through any point twice (apart again from the fact that γ_{k+1} starts where γ_k finishes).

Example

Let σ be the straight line from i to 1, and let γ be the stepwise curve from i to 1 via 0. Show that

$$\int_\gamma \bar{z}\, dz \neq \int_\sigma \bar{z}\, dz.$$

Thus the contour integral is not always independent of path (we will return to this important theme later).

1.7 Open Sets and Domains

Let $z \in \mathbb{C}$ and let $r > 0$.

We define $B(z,r) = \{w \in \mathbb{C}: |w - z| < r\}$.

This is called the open disc of centre z and radius r.

It consists of all points lying inside the circle of centre z and radius r, the circle itself being excluded.

Now let $U \subseteq \mathbb{C}$.

We say that U is OPEN if it has the following property:

for each $z \in U$ there exists $r_z > 0$ such that $B(z,r_z) \subseteq U$.

Note that r_z will usually depend on z.

Examples

(i) An open disc $B(z,r)$ is itself an open set.

(ii) Let $H = \{z : \mathrm{Re}(z) > 0\}$. Then H is open.

(iii) Let $K = \{z = x + iy : x, y \in \mathbb{R} \setminus \mathbb{Q}\}$. Then K is not open.

The point $u = \sqrt{2} + i\sqrt{2}$ is in K, but any open disc centred at u will contain a point with rational coordinates.

A DOMAIN is an open subset D of $\mathbb{C}$ which has the following additional property:

any two points in D can be joined by a stepwise curve which does not leave D.

An open disc is a domain, as is the half-plane $\mathrm{Re}(z) > 0$, but the set $\{z : \mathrm{Re}(z) \neq 0\}$ is not a domain, as any stepwise curve from -1 to 1 would have to pass through $\mathrm{Re}(z) = 0$.

We will say that a set E in $\mathbb{R}^2$ is open/a domain if the set in $\mathbb{C}$ corresponding to E, that is, $\{x + iy : (x, y) \in E\}$, is open/a domain.

A useful fact about domains

Let D be a domain in $\mathbb{R}^2$, and let u be a real-valued function such that $u_x \equiv 0$ and $u_y \equiv 0$ on D. Then u is constant on D.

Here the partials $u_x = \partial u/\partial x$, $u_y = \partial u/\partial y$, are defined by

$$u_x(a,b) = \lim_{x \to a} \frac{u(x,b)-u(a,b)}{x-a},$$

$$u_y(a,b) = \lim_{y \to b} \frac{u(a,y)-u(a,b)}{y-b}.$$

Why is this fact true?

Take any straight line segment S in D, parallel to the x axis, on which $y = y_0$, say.

On S we can write $u(x, y) = u(x, y_0) = g(x)$, and we have $g'(x) = u_x(x, y_0) = 0$.

This shows that u is constant on S, and similarly constant on any line segment in D parallel to the y axis.

Since any two points in D can be linked by finitely many such line segments joined end to end, u is constant on D.

2. Functions

2.1 Limits

If (z_n) is a sequence (i.e. non-terminating list) of complex numbers, we say that $z_n \to a \in \mathbb{C}$ if $|z_n - a| \to 0$ (i.e. the distance from z_n to a tends to 0).

As usual, if $E \subseteq \mathbb{C}$ a function f from E to $\mathbb{C}$ is a rule assigning to each $z \in E$ a unique value $f(z) \in \mathbb{C}$.

Such functions can usually be expressed either in terms of Re(z) and Im(z) or in terms of z and $\bar{z}$.

For example, consider $f(z) = \bar{z}z^2$. If we put $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$ then we have

$$f(z) = z(\bar{z}z) = z(x^2+y^2)$$

$$= u(x,y)+iv(x,y),$$

where

$$u(x,y) = x(x^2+y^2)$$

and

$$v(x,y) = y(x^2+y^2).$$

It is standard to write

$$f(x+iy) = u(x,y)+iv(x,y) \,, \qquad (1)$$

with x,y real and u,v real-valued functions (of x and y).

For any non-trivial study of functions you need limits.

What do we mean by

$$\lim_{z \to a} f(z) = L \in \mathbb{C}?$$

We mean that as z approaches a, in any manner whatever, the value $f(z)$ approaches L.

The only restriction here on the manner in which z approaches a is that we do not allow z to actually equal a: as usual with function limits, the value or existence of $f(a)$ makes no difference here.

Definition

Let f be a complex-valued function defined near $a \in \mathbb{C}$ (but not necessarily at a itself).

We say that $\lim_{z \to a} f(z) = L \in \mathbb{C}$ if the following is true: for every sequence z_n which converges to a with $z_n \neq a$, we have $\lim_{n \to \infty} f(z_n) = L$.

This must hold for all sequences tending to, but not equal to, a, regardless of direction: the condition that $z_n \neq a$ is there because the existence or value of $f(a)$ makes no difference to the limit.

Using the decomposition (1) (with x, y, u, v real) it is easy to see that

$$\lim_{z \to a} f(z) = L \in \mathbb{C}$$

if and only if both

$$\lim_{(x,y) \to (\mathrm{Re}(a), \mathrm{Im}(a))} u(x,y) = \mathrm{Re}(L)$$

and

$$\lim_{(x,y) \to (\mathrm{Re}(a), \mathrm{Im}(a))} v(x,y) = \mathrm{Im}(L).$$

This is because

$$|u - \mathrm{Re}(L)| + |v - \mathrm{Im}(L)| \leqslant 2|f - L|$$

$$\leqslant 2(|u - \mathrm{Re}(L)| + |v - \mathrm{Im}(L)|).$$

The usual Algebra of Limits results are also true, as in the case of real functions.

Examples

(a) Let $g(x,y) = (x^3 + y^2 x^2)/(x^2 + 4y^2)$.
Then $\lim_{(x,y) \to (0,0)} g(x,y) = 0$.

(b) Let $f(z) = |z|/(\pi + \operatorname{Arg} z)$ for $z \neq 0$.

Does $\lim_{z \to 0} f(z)$ exist?

If we let $z \to 0$ along some ray $\arg z = t$ with t in $(-\pi, \pi]$, then the denominator is constant and $f(z) \to 0$.

However, the limit does not in fact exist, as we see by considering

$$z = se^{i(-\pi + s^2)}$$

for small positive s.

Continuity

This is easy to handle.

We say f is continuous at a if $\lim_{z \to a} f(z)$ exists and is $f(a)$.

Thus $f(z)$ is as close as desired to $f(a)$, for all z sufficiently close to a.

Note that $\operatorname{Arg} z$ is discontinuous on the negative real axis.

2.2 Complex differentiability

Now we can define our "good" functions.

Let f be a complex-valued function defined on some open disc $B(a,r)$ and taking values in $\mathbb{C}$.

We say that f is complex differentiable at a if there is a complex number $f'(a)$ such that

$$f'(a) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Examples

1. Try $f(z) = \bar{z}$.

Then we look at

$$\lim_{z \to a} \frac{\bar{z} - \bar{a}}{z - a} = \lim_{h \to 0} \frac{\bar{h}}{h}.$$

For $f'(a)$ to exist, the limit must be the same regardless of in what manner h approaches 0.

If we let $h \to 0$ through real values, we see that $\bar{h}/h = 1$. But,

If we let $h \to 0$ through imaginary values, say $h = ik$ with k real, we see that $\bar{h}/h = -ik/ik = -1$.

So $\bar{z}$ is not complex differentiable anywhere.

This is rather surprising, as $\bar{z}$ is a very well behaved function.

It doesn't blow up anywhere and is in fact everywhere continuous.

If you write it in the form $u(x,y) + iv(x,y)$ you get $u = x$ and $v = -y$, and these have partial derivatives everywhere.

We'll see in a moment why $\bar{z}$ fails to be complex differentiable.

2. Try $f(z) = z^2$.

Then, for any a,

$$\lim_{z \to a} \frac{z^2 - a^2}{z - a} = \lim_{z \to a} (z + a) = 2a,$$

and so the function z^2 *is* complex differentiable at every point, and $(d/dz)(z^2) = 2z$ as you'd expect.

In fact, the chain rule, product rule and quotient rules all apply just as in the real case.

So, for example, $(z^3 - 4)/(z^2 + 1)$ is complex differentiable at every point where $z^2 + 1 \neq 0$, and so everywhere except i and $-i$.

Meaning of the derivative

In real analysis we think of $f'(x_0)$ as the slope of the graph of f at x_0.

In complex analysis it doesn't make sense to attempt to "draw a graph" but we can think of the derivative in terms of approximation.

If f is complex differentiable at a then as $z \to a$ we have $\dfrac{f(z)-f(a)}{z-a} \to f'(a)$ and so $\dfrac{f(z)-f(a)}{z-a} = f'(a)+\rho(z)$, where $\rho(z) \to 0$, and we can write this as $f(z)-f(a) = (z-a)(f'(a)+\rho(z))$.

In particular, f is continuous at a.

2.3 Cauchy-Riemann equations, first encounter

Assume that the complex-valued function f is complex differentiable at $a = A+iB$, and as usual write

$$f(x+iy) = u(x,y)+iv(x,y) \qquad (1)$$

with A,B,x,y,u,v all real.

Now, by assumption,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

and the limit is the same regardless of how h approaches 0.

If we let h approach 0 through real values, putting $h = t$,

$$f'(a) = \lim_{t \to 0} \frac{f(a+t) - f(a)}{t}$$

which we can easily show gives

$$f'(a) = u_x(A,B) + iv_x(A,B).$$

(In particular, the partials u_x, v_x do exist.)

Now put $h = it$ and again let $t \to 0$ through real values.

This time we get

$$f'(a) = (1/i)(u_y(A,B) + iv_y(A,B)).$$

Equating real and imaginary parts we now see that, at (A,B), we have

$$u_x = v_y \quad , \quad u_y = -v_x .$$

These are called the Cauchy-Riemann equations.

We also have (importantly) $f'(a) = u_x + iv_x$.

These relations must hold if f is complex differentiable.

Next we need a result in the other direction.

2.4 Cauchy-Riemann equations, second encounter

Theorem

Suppose that the functions f,u,v are as in (1) above, and that the following is true.

The partial derivatives u_x,u_y,v_x,v_y all exist near (A,B), and are continuous at (A,B), and the Cauchy-Riemann equations are satisfied at (A,B).

Then f is complex differentiable at $a = A+iB$, and $f'(a) = u_x+iv_x$.

Remark: the continuity of the partials won't usually be a problem in G1BCOF: e.g. this is automatic if they are polynomials in x,y and (say) functions like $e^x, \cos y$.

If there are denominators which are 0 at (A, B) some care is needed, though.

Those interested can see the proof of this theorem in the full lecture notes.

Example

Where is $x^2 + iy^2$ complex differentiable?

2.5 Analytic Functions

We say that a function f is complex differentiable on an open set U if it is complex differentiable at every point of U.

We say that f is ANALYTIC at a point a (resp. analytic on a set X) if f is complex differentiable on some open set G which contains the point a (resp. the set X).

It is NOT enough for f to be complex differentiable at a!

Obviously, if f is complex differentiable on a domain D in $\mathbb{C}$ then f is analytic on D (take $G = D$).

Other words for analytic are regular, holomorphic and uniform.

A sufficient condition for analyticity at a is that the partial derivatives of u,v are continuous and satisfy the Cauchy-Riemann equations at all points near a.

Examples

1. The exponential function. We've already defined $e^{it} = \cos t + i \sin t$ for t real.

We now define

$$\exp(x+iy) = e^{x+iy} = e^x e^{iy}$$

$$= e^x \cos y + i e^x \sin y$$

for x,y real.

We then have, using the standard decomposition,

$$u(x,y) = e^x \cos y \ , \ v(x,y) = e^x \sin y \ ,$$

and

$$u_x = e^x \cos y \, , \, u_y = -e^x \sin y \, ,$$

$$v_x = e^x \sin y \, , \, v_y = e^x \cos y \, ,$$

and so the Cauchy-Riemann equations are satisfied.

Obviously these partials are continuous.

Thus $\exp(z)$ is complex differentiable at every point in $\mathbb{C}$, and so is analytic in $\mathbb{C}$, or ENTIRE.

Further, the derivative of exp at z is $u_x + iv_x = \exp(z)$.

It is easy to check that $e^{z+w} = e^z e^w$ for all complex z,w.

Also $|e^z| = e^{\mathrm{Re}(z)} \neq 0$, so exp(z) never takes the value zero.

Since $e^0 = e^{2\pi i} = 1$ and $e^{\pi i} = -1$ this means that two famous theorems from real analysis are not true for functions of a complex variable!

2. sine and cosine. For $z \in \mathbb{C}$ we set

$$\sin z = (e^{iz} - e^{-iz})/2i$$

$$\cos z = (e^{iz} + e^{-iz})/2 \quad .$$

Exercise: put $z = x \in \mathbb{R}$ in these definitions and check that you just get $\sin x$, $\cos x$ on the RHS.

With these definitions, the usual rules for derivatives tell us that sine and cosine are also entire, but it is important to note that they are not bounded in $\mathbb{C}$.

3. Some more elementary examples. What about $\exp(1/z)$?

We’ve already observed that the chain rule holds for complex differentiability, as does the quotient rule.

So this function is complex differentiable everywhere except at 0, and so analytic everywhere except at 0.

Similarly $\sin(\exp(1/(z^4+1))$ is analytic everywhere except at the four roots of $z^4+1 = 0$.

4. At which points is

$$g(x+iy) = x^2+4y^2+ixy\ , \qquad x,y \in \mathbb{R}$$

(i) complex differentiable (ii) analytic?

5. Does there exist any function h analytic on a domain D in $\mathbb{C}$ such that Re(h) is x^2+4y^2 at each point $x+iy \in D$ (x,y real) ?

6. The logarithm. The aim is to find an analytic function $w = h(z)$ such that $\exp(h(z)) = z$. This is certainly NOT possible for $z = 0$, as $\exp(w)$ is never 0.

Further,

$$\exp(h(z)) = e^{\mathrm{Re}(h(z))} e^{i\mathrm{Im}(h(z))}$$

and

$$z = |z| e^{i \arg z}.$$

So if such an h exists on some domain it follows that $\mathrm{Re}(h(z)) = \ln|z|$ and that $\mathrm{Im}(h(z))$ is an argument of z.

Here we use $\ln x$ to denote the logarithm, base e , of a POSITIVE real number x.

A problem arises with this.

If we start at $z = -1$, and fix some choice of the argument there, and if we then continue once clockwise around the origin, we find that on returning to -1 the argument has decreased by 2π and the value of the logarithm has changed by $-2\pi i$.

Indeed, we've already seen that the argument of a complex number is discontinuous at the negative real axis.

So to make our logarithm analytic we have to restrict the domain in which z can lie.

Let D_0 be the complex plane with the origin and the negative real axis both removed, and define, for z in D_0,

$$w = \operatorname{Log} z = \ln|z| + i \operatorname{Arg} z.$$

Remember that Arg will be taking values in $(-\pi,\pi)$.

This choice for w gives

$$e^w = \exp(\operatorname{Log} z) = e^{\ln|z|}\exp(i \operatorname{Arg} z) = z$$

as required.

Now for $z \in D_0$ we have $-\infty < \ln|z| < +\infty$ and $-\pi < \operatorname{Arg} z < \pi$ and so $w = \operatorname{Log} z$ maps D_0 one-one onto the strip $W = \{w \in \mathbb{C} : |\operatorname{Im}(w)| < \pi\}$.

For $z, z_0 \in D_0$ and $w = \operatorname{Log} z$, $w_0 = \operatorname{Log} z_0$, we then have $z \to z_0$ if and only if $w \to w_0$.

Hence

$$\lim_{z \to z_0} \frac{\operatorname{Log} z - \operatorname{Log} z_0}{z - z_0} = \lim_{z \to z_0} \frac{w - w_0}{z - z_0} =$$

$$= \lim_{w \to w_0} \frac{w - w_0}{z - z_0} = \lim_{w \to w_0} \frac{w - w_0}{e^{w} - e^{w_0}}.$$

The last limit is the reciprocal of the derivative of exp at w_0 and so is $1/\exp(w_0) = 1/z_0$.

We conclude:

The PRINCIPAL LOGARITHM defined by $\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$ is analytic on the domain D_0 obtained by deleting from $\mathbb{C}$ the origin and the negative real axis, and its derivative is $1/z$.

It satisfies $\exp(\operatorname{Log} z) = z$ for all $z \in D_0$.

Warning: It is not always true that $\operatorname{Log}(\exp(z)) = z$, nor that $\operatorname{Log}(zw) = \operatorname{Log} z + \operatorname{Log} w$. e.g. try $z = w = -1 + i$.

Powers of *z*

Suppose we want to define a complex square root $z^{1/2}$.

A natural choice is

$$w = \sqrt{|z|}\, e^{\frac{1}{2} i \arg z},$$

because this gives $w^2 = |z| e^{i \arg z} = z$.

If we do this, however, we encounter the same problem as with the logarithm.

If we start at -1 and go once around the origin clockwise the argument decreases by 2π and the value we obtain on returning to -1 is the original value multiplied by $e^{-i\pi} = -1$.

So we again have to restrict our domain of definition.

We first note that if n is a positive integer, then, on D_0,

$$\exp(n \operatorname{Log} z) = (\exp(\operatorname{Log} z))^n = z^n,$$

$$\exp(-n \operatorname{Log} z) = (\exp(n \operatorname{Log} z))^{-1}$$

$$= (\exp(\operatorname{Log} z))^{-n} = z^{-n}.$$

So, on D_0 , we can define, for each complex number α,

$$z^\alpha = \exp(\alpha \operatorname{Log} z).$$

With this definition and properties of exp,

$$z^{\alpha} z^{\beta} = \exp(\alpha \operatorname{Log} z)\exp(\beta \operatorname{Log} z)$$

$$= \exp((\alpha+\beta) \operatorname{Log} z) = z^{\alpha+\beta}.$$

However, it isn't always true that with this definition, $(z^{\alpha})^{\beta} = z^{\alpha\beta}$.

For example, take $z = i$, $\alpha = 3$, $\beta = 1/2$.

Then

$$z^{\alpha\beta} = i^{3/2} = \exp((3/2)\ \mathrm{Log}\, i)$$

$$= \exp((3/2) i\pi/2) = \exp(3\pi i/4).$$

But

$$z^{\alpha} = i^{3} = \exp(3\ \mathrm{Log}\, i) = \exp(3\pi i/2),$$

and this has principal logarithm equal to $-\pi i/2$, so that

$$(z^{\alpha})^{\beta} = \exp((1/2)(-\pi i/2))$$

$$= \exp(-\pi i/4) \neq \exp(3\pi i/4).$$

Section 3

Integrals involving analytic functions

Theorem 3.1

Suppose that $\gamma:[a, b] \to D$ is a smooth contour in a domain $D \subseteq \mathbb{C}$, and that $F:D \to \mathbb{C}$ is analytic with continuous derivative f.

Then $\int_\gamma f(z)\ dz = F(\gamma(b)) - F(\gamma(a))$ and so is 0 if γ is closed.

One way to prove this is to look at the derivative of the function $F(\gamma(t))$.

If we do the same for a PSC γ, we find that the integral of f is the value of F at the finishing point of γ minus the value of F at the starting point of γ, and again if γ is closed we get 0.

Now we state a very important theorem (for proof see full lecture notes).

Theorem 3.2 (Cauchy-Goursat)

Let $D \subseteq \mathbb{C}$ be a domain and let T be a contour which describes once counter-clockwise the perimeter of a triangle whose perimeter and interior are contained in D. Let $f{:}D \to \mathbb{C}$ be analytic. Then $\int_T f(z)\, dz = 0$.

Example

Let T describe once counter-clockwise the triangle with vertices at 0, 1, i, and let $a=(1+i)/4$ (a point inside T).

We have already seen that $\int_T \bar{z}\,dz \neq 0$.

Which of the following functions have integral around T equal to 0?

(i) $(z-a)^{-2}$, (ii) $\exp(1/(z-a))$,

(iii) $\exp(1/(z-10))$.

3.3 A special type of domain

A star domain D is a domain (in $\mathbb{C}$) which has a star centre, α say, with the following property.

For every z in D the straight line segment from α to z is contained in D.

Examples include an open disc, the interior of a rectangle or triangle, a half-plane.

On star domains we can prove a more general theorem about contour integrals than Theorem 3.2.

Useful fact: if γ is a simple closed PSC in a star domain D, and w is a point inside γ, then w is in D.

Theorem 3.4

Suppose that $f{:}D \to \mathbb{C}$ is continuous on the star domain $D \subseteq \mathbb{C}$ and is such that $\int_T f(z)\,dz = 0$ whenever T is a contour describing once counter-clockwise the boundary of a triangle which, together with its interior, is contained in D.

Then the function F defined by $F(z) = \int_\alpha^z f(u)\,du$, in which the integration is along the straight line from α to z, is analytic on D and is such that $F'(z) = f(z)$ for all z in D.

Remark

An analytic function is continuous (see Section 2) and so Theorem 3.4 applies in particular when f is analytic in D.

This leads at once to the following very important theorem.

Theorem 3.5 (stronger than 3.2)

Let $D \subseteq \mathbb{C}$ be a star domain and let $f{:}D \to \mathbb{C}$ be analytic.

Let γ be any closed piecewise smooth contour in D.

Then $\int_\gamma f(z)\, dz = 0.$

To see this, Theorem 3.4 gives us a function F such that $F'(z) = f(z)$ in D, and so the integral of f is just F evaluated at the final point of γ minus F evaluated at the initial point of γ. But these points are the same!

In fact, even more than this is true.

We state without proof:

The general Cauchy theorem: let γ be a simple closed piecewise smooth contour, and let D be a domain containing γ and its interior. Let $f : D \to \mathbb{C}$ be analytic. Then $\int_{\gamma} f(z)\,dz = 0$.

The general case is surprisingly difficult to prove, and is beyond the scope of G1BCOF.

We will, however, use the result, since the contours encountered in this module will be fairly simple geometrically.

Example 3.6

Suppose that f is analytic in the disc $|z| < S$.

Suppose that $0 < s < S$, and that $w \in \mathbb{C}$, $|w| \neq s$. We compute

$$\frac{1}{2\pi i}\int_{|z| = s} \frac{f(z)}{z-w}\, dz,$$

in which the integral is taken once counter-clockwise.

First, if $|w| > s$ then $g(z) = f(z)/(z-w)$ is analytic on and inside $|z| = s$, so the integral is 0.

Next, assume that $|w| < s$ and let δ be small and positive.

Consider the domain D_1 given by

$$|z| < s,\ |z-w| > \delta.$$

Then $g(z) = f(z)/(z-w)$ is analytic on D_1.

By cross-cuts, we see that the integral of $g(z)$ around the boundary of D_1, the direction of integration keeping D_1 to the left, is 0.

This gives

$$\int_{|z| = s} g(z)\,dz = \int_{|z-w| = \delta} g(z)\,dz$$

$$= \int_0^{2\pi} f(w+\delta e^{i\theta})\,i\,d\theta$$

$$\to \int_0^{2\pi} f(w)\,i\,d\theta = 2\pi i f(w)$$

as $\delta \to 0$. Hence

$$\int_{|z| = s} f(z)/(z-w)\,dz = 2\pi i f(w)$$

when $|w| < s$.

3.7 Cauchy's integral formula

Suppose that f is analytic on a domain containing the simple closed piecewise smooth contour γ and its interior.

Let $w \in \mathbb{C}$, with w not on γ.

Then, integrating once counter-clockwise,

$$\frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{z-w}\, dz$$

is $f(w)$ if w lies inside γ, and is 0 if w lies outside γ.

Note that we have to exclude the case where w lies on γ, as in this case the integral may fail to exist.

3.8 Liouville's theorem

Suppose that f is entire (i.e. analytic in $\mathbb{C}$) and bounded as $|z|$ tends to ∞, i.e. there exist $M > 0$ and $R_0 > 0$ such that $|f(z)| \leqslant M$ for all z with $|z| \geqslant R_0$. Then f is constant.

Proof We take any u and v in $\mathbb{C}$ and estimate $f(u) - f(v)$ using Cauchy's integral formula. Take R very large.

By Cauchy's integral formula, we have, integrating once counter-clockwise,

$$f(u) - f(v) = \frac{1}{2\pi i} \int_{|z| = R} \frac{f(z)(u-v)}{(z-u)(z-v)}\, dz.$$

Using the fundamental estimate we can show that this tends to 0 as $R \to \infty$, and so we must have $f(u) = f(v)$.

A corollary to this is the *fundamental theorem of algebra*: if $P(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial in z of positive degree n (i.e. $a_n \neq 0$) then there is at least one z in $\mathbb{C}$ with $P(z) = 0$.

For otherwise $1/P$ is entire, and $1/P(z) \to 0$ as $|z| \to \infty$.

By Liouville's theorem, it would follow that $1/P(z)$ is constant on $\mathbb{C}$, which is clearly a contradiction.

3.9 (Non-examinable) An application, and a physical interpretation of Cauchy's theorem

Analytic functions can be used to model fluid flow. For details, see the full lecture notes. There is a question on this in the set for Problem Class 3 but you need not attempt this unless you are interested.

4.1 Complex Series

Let $a_p, a_{p+1}, a_{p+2}, \ldots.$ be a sequence (i.e. a non-terminating list) of complex numbers.

As with real numbers, for $n \geqslant p$, define

$$s_n = \sum_{k=p}^{n} a_k,$$

the *sequence of partial sums*.

If the sequence s_n converges (i.e. tends to a finite limit as $n \to \infty$, with $S = \lim_{n \to \infty} s_n$, then we say that the series $\sum_{k=p}^{\infty} a_k = \lim_{n \to \infty} \sum_{k=p}^{n} a_k$ converges, with sum S.

Many results for real series are also valid for complex series.

As usual, our convention is that $0^0 = 1$.

More details on the following standard examples and facts may be found in the full lecture notes.

Example

Let $|z| < 1$. Then $s_n = \sum_{k=0}^{n} z^k = \dfrac{1 - z^{n+1}}{1 - z} \to \dfrac{1}{1-z}$ as $n \to \infty$.

So $\sum_{k=0}^{\infty} z^k = \dfrac{1}{1-z}$ if $|z| < 1$.

Fact 1

If $\sum_{k=p}^{\infty} a_k$ converges, then $a_n \to 0$ as $n \to \infty$.

The converse is false, using the standard divergent real series (which is also a complex series)

$$\sum_{k=1}^{\infty} 1/k.$$

Fact 2

Suppose that $\sum_{k=p}^{\infty} a_k$ and $\sum_{k=p}^{\infty} b_k$ both converge, and that α,β are complex numbers.

Then $\sum_{k=p}^{\infty} (\alpha a_k + \beta b_k)$ converges, and equals $\alpha(\sum_{k=p}^{\infty} a_k) + \beta(\sum_{k=p}^{\infty} b_k)$.

So for example $\sum_{k=0}^{\infty} 2^{-k} + i3^{-k}$ converges, but $\sum_{k=1}^{\infty} 2^{-k} + i/k$ diverges.

Fact 3

Recall the following from last year: Suppose that a_k is real and non-negative.

Then $s_n = \sum_{k=p}^{n} a_k$ is a non-decreasing real sequence, and converges if and only if it is bounded above.

It is standard that if $p > 1$ the series $\sum_{k=1}^{\infty} 1/k^p$ converges.

Comparison test: if $0 \leqslant a_k \leqslant b_k$ and $\sum_{k=p}^{\infty} b_k$ converges then $\sum_{k=p}^{\infty} a_k$ converges

Fact 4

Suppose that $\sum_{k=p}^{\infty} |a_k|$ converges (in which case we say that $\sum_{k=p}^{\infty} a_k$ is absolutely convergent).

Then $\sum_{k=p}^{\infty} a_k$ converges.

Fact 5 (The Ratio Test)

Suppose that a_k is real and positive for $k \in \mathbb{N}$ and that $L = \lim_{n \to \infty} a_{n+1}/a_n$ exists.

(i) If $L > 1$ then a_n does not tend to 0 and so $\sum_{k=1}^{\infty} a_n$ diverges.

(ii) If $0 \leqslant L < 1$ then $\sum_{k=1}^{\infty} a_n$ converges.

There is no conclusion if $L = 1$.

Example: if $0 < t < 1$ then $\sum_{k=1}^{\infty} kt^{k-1}$ converges.

4.2 Power series

Consider the *power series*

$$F(z) = \sum_{k=0}^{\infty} c_k (z-\alpha)^k$$

$$= c_0 + c_1(z-\alpha) + c_2(z-\alpha)^2 + \ldots\ldots,$$

in which the centre α and the coefficients c_k are complex numbers. Obviously $F(\alpha) = c_0$.

To investigate convergence for $z \neq \alpha$ we let T_F be the set of non-negative real t having the property that $|c_k|t^k \to 0$ as $k \to +\infty$. Then $0 \in T_F$.

The radius of convergence R_F is defined as follows. If T_F is bounded above (this means that T_F has an upper bound P, a real number P with $x \leqslant P$ for all x in T_F), we let R_F be its l.u.b., i.e. the least real number which is an upper bound for T_F.

If T_F is not bounded above, then we set $R_F = \infty$.

In either case, the following is true. If $0 < r < R_F$ then r is not an upper bound for T_F and so there exists $s \in T_F$ with $s > r$.

Note that if $|z-\alpha| > R_F$ then $c_k(z-\alpha)^k$ cannot tend to 0 as $k \to \infty$ (since its modulus does not), and so $F(z)$ diverges.

Theorem 4.3 The main theorem on power series

Suppose that $F(z) = \sum_{k=0}^{\infty} c_k(z-\alpha)^k$ has positive radius of convergence R_F. Set $D = B(\alpha, R_F)$ (if $R_F = \infty$ then $D = \mathbb{C}$). Then

(i) F converges absolutely for z in D, and $F:D \to \mathbb{C}$ is a continuous function;

(ii) if γ is a PSC in D and $\phi(z)$ is continuous on γ we have

$$\int_\gamma F(z)\phi(z)\,dz = \sum_{k=0}^{\infty} \int_\gamma c_k(z-\alpha)^k \phi(z)\,dz,$$

i.e. we can integrate term by term. In particular

$$\int_\gamma F(z)\,dz = \sum_{k=0}^{\infty} c_k \int_\gamma (z-\alpha)^k dz$$

(iii) F is analytic on D and, in D,

$$F'(z) = \sum_{k=1}^{\infty} kc_k(z-\alpha)^{k-1}$$

$$= \sum_{k=0}^{\infty} (k+1)c_{k+1}(z-\alpha)^k.$$

Further, all derivatives $F^{(k)}$ exist on D, and $F^{(k)}(\alpha) = k!c_k$, i.e.

$$c_k = F^{(k)}(\alpha)/k!.$$

For full details of the proof, see the printed lecture notes. The **proof of Theorem 4.3 is non-examinable, but you are expected to know the statement and to be able to use it to solve problems.**

4.4 Series in negative powers

We need a similar result for series of form $G(z) = \sum_{k=1}^{\infty} c_k(z-a)^{-k}$.

We can regard this as a power series in $1/(z-a)$, and the following facts can be proved by setting $u = 1/(z-a)$ and $F(u) = \sum_{k=1}^{\infty} c_k u^k$. Obviously, $G(z) = F(1/(z-a))$.

Case 1: suppose that $R_F = 0$.

Then $F(u)$ converges only for $u = 0$, and so $G(z)$ diverges for every z in $\mathbb{C}$.

Case 2: suppose that $R_F > 0$.

Then $F(u)$ converges absolutely for $|u| < R_F$ and diverges for $|u| > R_F$.

Thus $G(z)$ converges absolutely for $|z-a| > S_G = 1/R_F$, and diverges for $|z-a| < S_G$.

Also, G is analytic, and can be differentiated term by term with

$$G'(z) = -(z-a)^{-2}F'(u)$$

$$= -(z-a)^{-2}\sum_{k=1}^{\infty} kc_k u^{k-1}$$

$$= \sum_{k=1}^{\infty} -kc_k(z-a)^{-k-1}$$

on the domain $D = \{z \in \mathbb{C} : S_G < |z-a| < \infty\}$.

As in 4.3(iii), if γ is a PSC in D and $\phi(z)$ is continuous on γ, then

$$\int_\gamma \phi(z)G(z)\,dz = \sum_{k=1}^{\infty}\int_\gamma \phi(z)c_k(z-a)^{-k}dz.$$

Finally, if $G(v)$ converges then $F(1/(v-a))$ converges, and so $F(u)$ converges for $|u| < |1/(v-a)|$ so $G(z)$ converges for $|z-a| > |v-a|$.

Example

Let w be a complex number. Then $\sum_{k=0}^{\infty}(w/z)^k = 1+(w/z)+(w/z)^2+.... =$ $1/(1-w/z)$ for $|z| > |w|$.

4.5 Laurent's theorem

Let $0 \leqslant R < S \leqslant \infty$, and let f be analytic in the annulus A given by $R < |z-a| < S$. Then there are constants $a_k, k \in \mathbb{Z}$, such that for all z in A we have

$$f(z) = \sum_{k \in \mathbb{Z}} a_k (z-a)^k$$

$$= \sum_{k=0}^{\infty} a_k (z-a)^k + \sum_{j=1}^{\infty} a_{-j} (z-a)^{-j}. \quad (1)$$

The series (i.e. both series) converge absolutely for z in A, and integrals of the form $\int_\gamma \phi(z) f(z)\, dz$ can be computed by integrating term by term:

For any PSC contour γ in A, i.e.

$$\int_\gamma \phi(z) f(z)\, dz = \sum_{k \in \mathbb{Z}} a_k \int_\gamma \phi(z)(z-a)^k dz$$

provided $\phi(z)$ is continuous on γ.

In particular, if $R < T < S$ then integrating once counter-clockwise gives

$$a_k = \frac{1}{2\pi i} \int_{|z-a| = T} f(z)(z-a)^{-k-1} dz, \quad (2)$$

so that there is just one Laurent series (1) representing $f(z)$ in A.

Finally, the series (1) can be differentiated term by term in A.

Example: find the Laurent series of $f(z) = 1/z(z-i)^2$ in

(i) $0 < |z| < 1$

(ii) $1 < |z| < \infty$

(iii) $1 < |z-i| < \infty$.

Find the Laurent series of $1/(z+1)(z+2)$ in $1 < |z| < 2$.

Using the geometric series $1/(1-u) = 1+u+u^2+........$ and its differentiated versions, we can get Laurent series for rational functions fairly straightforwardly.

The next problem is to obtain series for more complicated functions.

We begin by considering what happens when f is in fact analytic in $|z-a| < S$.

Theorem 4.6 (Taylor's theorem)

Let a be a complex number, let $0 < S \leqslant \infty$, and suppose that f is analytic in $|z-a| < S$.

Then f can be differentiated as many times as we like on $|z-a| < S$ and for $|z-a| < S$ we have (Taylor series)

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^k.$$

In particular, all these derivatives $f^{(k)}(a)$ exist.

Proof of Taylor's theorem

We derive Taylor's theorem from Laurent's theorem, with $R = 0$. We get a series (1) valid for $0 < |z-a| < S$, and the coefficients a_k are given by (2).

But, if $k \in \mathbb{Z}, k < 0$, it follows that $-k-1 \geqslant 0$ and so $f(z)(z-a)^{-k-1}$ is analytic in $|z-a| < S$. Hence $a_k = 0$ for $k < 0$ by Cauchy's theorem.

This gives $f(z) = \sum_{k=0}^{\infty} a_k(z-a)^k$ for $|z-a| < S$, and this is a power series with radius of convergence at least S.

In particular, we see from 4.3 that $f^{(k)}(a)/k!$ exists and equals a_k.

4.7 Remarks and Examples

1. It follows from Taylor's theorem that if $f: D \to \mathbb{C}$ is analytic on the domain $D \subseteq \mathbb{C}$ then so is f'.

To see this, take any w in D and just note that Taylor's theorem shows that $f^{(k)}(w)$ exists for each non-negative integer k.

2. There are functions $g: \mathbb{R} \to \mathbb{R}$ for which the kth (real) derivative $d^k g/dx^k$ exists for every k, but which do not always equal their Taylor series.

For example, let $g(0) = 0$, with $g(x) = \exp(-1/x^2)$ for $x \neq 0$.

3. Taylor's theorem tells us that, for all z,

$$e^z = 1 + z + z^2/2! + ...,$$

$$\sin z = z - z^3/3! + z^5/5! - .....$$

Also, if $F(z) = \sum_{k=0}^{\infty} c_k(z-\alpha)^k$ has $R_F > 0$ then F is its own Taylor series about α.

In particular, the standard series

$$1/(1-z) = 1 + z + z^2 + ...,$$

$$1/(1-z)^2 = 1 + 2z + 3z^2 + ...$$

are valid for $|z| < 1$ and very useful.

4. The binomial theorem. Suppose that b is a complex number, and consider $(1+z)^b$ for $|z| < 1$.

It is not immediately clear how to *define* this. However, for $|z| < 1$, the number $1+z$ will lie in the domain of definition of the principal logarithm Log, and so we define (for $|z|<1$)

$$(1+z)^b = h(z) = \exp(\, b \operatorname{Log}(1+z))$$

This function h is then analytic in $|z| < 1$ by the chain rule.

We also have

$$h'(z) = h(z)b/(1+z)$$

$$= h(z)b\exp(-\operatorname{Log}(1+z)) = b(1+z)^{b-1}.$$

Thus $h'(0) = b$, $h''(0) = b(b-1)$, and $h^{(k)}(0) = b(b-1)\ldots(b-k+1)$ for every positive integer k.

Taylor's theorem then gives (for $|z|<1$)

$$(1+z)^b = 1+bz+z^2b(b-1)/2$$

$$+z^3b(b-1)(b-2)/3!+.....,$$

which is the binomial theorem.

If b is a positive integer, the series terminates and the expansion is valid for all z.

5. The Cauchy product. Suppose that $F(z), G(z)$ are both analytic in $|z-a| < S$, with Taylor series

$$F(z) = a_0 + a_1(z-a) + \dots\dots ,$$

$$G(z) = b_0 + b_1(z-a) + \dots\dots ,$$

there. Then $H(z) = F(z)G(z)$ is analytic in the same disc. If we multiply the Taylor series of F by that of G we get

$$(a_0 + a_1(z-a) + \dots\dots)(b_0 + b_1(z-a) + \dots\dots) =$$

$$= a_0 b_0 + (a_1 b_0 + a_0 b_1)(z-a) + \dots$$

$$+ (a_k b_0 + \dots + a_0 b_k)(z-a)^k + \dots$$

and it is not hard to show that this really is the Taylor series for H.

6. Find the Taylor series of $(\cos z)/(1-z^2)$ in $|z| < 1$.

7. Evaluate $\int \exp(z^2) z^{-17} dz$ with the integral once counter-clockwise around $|z| = 1$.

8. Function of a function. Suppose that $f(u)$ is analytic in $|u-b| < r$ and that $g(z)$ is analytic in $|z-a| < s$, with $g(a) = b$.

Then if z is close enough to a we have $|g(z)-b| < r$, and so $f(g(z)) = h(z)$ is analytic in $|z-a| < t$, for some $t > 0$.

Suppose that

$$f(u) = a_0 + a_1(u-b) + ..., \quad |u-b| < r,$$

and

$$g(z) = b + b_1(z-a) + ..., \quad |z-a| < s.$$

If we set $u = g(z)$ and substitute the series for g into that for $f(u)$, we get

$$a_0 + a_1(b_1(z-a) + b_2(z-a)^2 + ...)$$

$$+ a_2(b_1(z-a) + b_2(z-a)^2 + \ldots)^2$$

$$+ a_3(b_1(z-a) + b_2(z-a)^2 + \ldots)^3 + ...$$

$$= a_0 + a_1 b_1(z-a) + (z-a)^2(a_1 b_2 + a_2 b_1^2) +$$

$$(z-a)^3(a_1 b_3 + 2a_2 b_1 b_2 + a_3 b_1^3) + ...$$

when we gather up powers of $z-a$.

The following theorem tells us that this is the Taylor series.

A theorem on the Taylor series of a composition: Suppose that g is analytic at a and f is analytic at $b = g(a)$. Then the composition h defined by $h(z) = f(g(z))$ is analytic at a, by the chain rule, and the Taylor series of h about a is obtained by substituting the Taylor series of g about a into the Taylor series of f about b and gathering up powers of $z-a$.

Warning: this only works if $g(a) = b$.

Using this difficult but important theorem on the Taylor series of a composition we return to our consideration of examples.

9. Find the integral once counter-clockwise around $|z| = 1$ of $1/z^4(1 - \sin z)$.

10. The same, for $z^{-4}(1 + \cos z)^{-1}$.

11. The same, for $\exp(1/z)$.

12. Same again, for $z^{-8}\sin(z^3)$.

13. Find the integral once counter-clockwise around $|z| = 4$ of $z^{-6}(1-z)^{-1}$.

14. Find the Laurent series of $1/(z^2-4)$ in (a) $|z| < 1$ (b) $0 < |z-2| < 4$ (c) $4 < |z+2| < \infty$ (d) $10 < |z| < \infty$.

15. Calculate $\int_{|z| = 3} e^{1/z} z^4 \, dz$, the integral being once counter-clockwise.

Section 5 : Singularities and the residue calculus

5.1 Singularities

We say that the complex-valued function f has an isolated singularity at a if f is not defined at a but there is some $s > 0$ such that f is analytic in the punctured disc $\{ z \in \mathbb{C} : 0 < |z - a| < s \}$.

The singularity can be classified according to how f behaves as z approaches a.

Examples

1. $f(z) = \dfrac{\cos z}{z}.$

Clearly, 0 is a problem point for this function. As $z \to 0$, we easily see that $|f(z)| \to \infty$.

We say that f has a pole at 0.

A pole is an isolated singularity a with the property that $|f(z)| \to \infty$ as $z \to a$.

2. $g(z) = \dfrac{\sin z}{z}$.

Here the behaviour is not so obvious. However, for $z \neq 0$ we can write

$$g(z) = z^{-1}(z - z^3/6 + ...) = 1 - z^2/6 + ...$$

The RHS is now a power series, converging for all $z \neq 0$, and so for all z.

If we set $g(0) = 1$, then g becomes analytic at 0 as well, and we have removed the singularity.

A removable singularity a of a function h is an isolated singularity with the property that $\lim_{z \to a} h(z)$ exists and is finite.

3. Now try $H(z) = \dfrac{\sin z}{z^2}$.

For $z \neq 0$ we have

$$H(z) = z^{-1} - z/6 + z^3/5! - .....$$

As $z \to 0$ the term $|z^{-1}| \to \infty$, while the rest of the series tends to 0. This is again a pole.

4. $F(z) = e^{1/z}$.

This is an altogether worse kind of singularity, called essential. $F(z)$ has no limit of any kind as $z \to 0$. It is interesting to look at the behaviour as z tends to zero along the real and imaginary axes.

5. $T(z) = \operatorname{cosec}(1/z)$.

Here 0 is not an isolated singularity at all, but a much bigger problem.

The function has singularities at all the points where $1/z$ is an integer multiple of π.

When we DO have an isolated singularity at a, we can use our knowledge of Laurent series to help us calculate integrals.

In the next section we see how to do this systematically, in terms of residues.

Residues

If f has an isolated singularity at a, we compute the Laurent series

$$\sum_{k=-\infty}^{\infty} c_k (z-a)^k$$

which represents f on some annulus A_ρ given by $0 < |z-a| < \rho$ on which f is analytic.

Provided that $\rho > 0$ and f is analytic on A_ρ, the coefficients don't depend on ρ (since the Laurent series for a given function and annulus is unique).

The **residue** of f at a is defined to be c_{-1}, the coefficient of $(z-a)^{-1}$ in this Laurent series.

We also denote the residue of f at a by Res(f,a).

Note that if $0 < t < \rho$ then, as we saw before,

$$\int_{|z-a| = t} f(z)\,dz = 2\pi i c_{-1}$$

$$= 2\pi i Res(f,a),$$

when we integrate once counter-clockwise.

Examples

1. Let γ be the circle $|z| = 2$ described once counter-clockwise. Determine $\int_{\gamma} \frac{\sin z}{(z-1)^2}\, dz$.

2. Let Γ be the contour which describes once counter-clockwise the square with vertices at $\pm 10 \pm 10i$. Determine $\int_{\Gamma} \frac{1}{z^2(z+1)}\, dz$.

5.2 Cauchy's residue theorem

Suppose that γ is a simple closed piece-wise smooth contour described in the positive (i.e. counter-clockwise) sense, and let $D \subseteq \mathbb{C}$ be a domain which contains γ and its interior. Suppose that f is analytic in D apart from a finite set of isolated singularities, none of which lie on γ.

Let $\alpha_1, \ldots, \alpha_n$ be the singularities which lie INSIDE γ. Then

$$\int_\gamma f(z)\, dz = 2\pi i \left(\sum_{j=1}^{n} \operatorname{Res}(f, \alpha_j) \right).$$

We stress that the integral must be once around γ in the positive sense.

To compute the residue $\text{Res}(f, \alpha_j)$, we look at the Laurent series of f which represents f in an annulus $0 < |z-\alpha_j| < \rho$, for some $\rho > 0$. The residue is just the coefficient of $(z-\alpha_j)^{-1}$ in this series.

5.3 Examples

1. Keeping the notation of 5.2, suppose there are no singularities α_j. Then the integral is 0 (this is an even stronger form of Cauchy's theorem than 3.5).

2. The Cauchy integral formula.

3. Consider $\int_{|z| = 300} \dfrac{z-17}{(z-2)(z-4)}\, dz.$

4. Consider $\int_{|z| = 1} \frac{1}{e^z - 1}\, dz$.

5. Consider $\int_{|z| = 1} \frac{1}{(e^z - 1)^2}\, dz$.

6. Let γ be the semicircular contour through $-R, R$ and iR, and calculate $\int_\gamma e^{iz}/(z^2+1)\,dz$.

7. Determine

$$\lim_{R \to \infty} \int_0^R (\cos x)/(x^2+1)\,dx.$$

8. Evaluate $\int_{-\infty}^{\infty} 1/(x^2+2x+6)\,dx$.