G1BCOF: Complex Functions 2004-5
Blow-by-blow account of the module so far

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Lecture 1: Motivation. The integral of a complex-valued function over an interval. Elementary examples and estimates.

Lecture 2: Paths, curves and contours. Smooth contours and their properties. Integrals of functions along smooth contours.

Lecture 3: Examples. Reversing contours. Simple smooth contours: Integral independent of parametrization (brief discussion). Fundamental estimate and applications.

Ex. Class: Revision exercises. Question and answer session. First introduction to complex exponentials and logarithms.

Lecture 4: Piecewise smooth contours (PSCs) and integrals. Stepwise curves. More on contour integrals. Simple PSCs and simple closed PSCs.

Lecture 5: Open sets. Domains. Partial derivatives.

Lecture 6: Sequences. Limits of sequences in $\mathbb{C}$. Connections between complex-valued functions of a complex variable and real-valued functions of two real variables. Function limits. Examples of limits.

Lecture 7: Continuity. Definition of the complex derivative. Examples: $\overline{z}$, z^2, polynomials (with complex coefficients) and rational functions (quotients of polynomials, where defined). Meaning of the complex derivative. **One-way implication:** Complex-differentiability implies continuity (converse is false, as shown by $\overline{z}$). Product rule, chain rule etc. Complex-differentiability implies that the Cauchy-Riemann equations hold. Converse holds **provided** the relevant partial derivatives exist at and near the point in question, and the partial derivatives are continuous at that point. (For most functions considered in this module, the continuity of the partial derivatives is obvious.)

Lecture 8: Revision of Cauchy-Riemann equations. Examples. Analytic functions: the distinction between being complex-differentiable at a point and being analytic at a point. Examples.

Lecture 9: More examples, including polynomials, rational functions (avoiding division by zero), the function considered in the Special Examples Class, $\exp(z) = \mathrm{e}^z$, $\sin z$ and $\cos z$. Entire functions (complex-differentiable on $\mathbb{C}$ and hence analytic on $\mathbb{C}$). Harmonic functions (functions satisfying Laplace's equation). The real and imaginary parts of analytic functions are harmonic.

Lecture 10: The complex logarithm: results from Special Examples class recalled, concerning solutions in w to the equation $\exp w = z$. Problems with $\arg$: no continuous argument defined all the way round a circle centred on the origin. Principal argument $\mathrm{Arg}(z)$ is discontinuous at points of the negative real axis. **Principal logarithm** Log(z) defined (on standard cut plane D_0), based on $\mathrm{Arg}(z)$. Derivative of Log. Other properties of Log.

Lecture 11: Powers of z, z^b (in standard cut plane, principal branch only). Discontinuity when b is not an integer. Integral of the derivative of an analytic function. In particular, the integral of an antidifferentiable function round a closed contour is 0.

Lecture 12: The Cauchy-Goursat theorem. Key ideas of proof discussed. For a detailed proof see full lecture notes. Examples. Star domains. Examples. Domains that are not quite star-shaped: all examples of domains in this module will be finite unions of star-shaped domains.

Lecture 13: An analytic function has an antiderivative in each star domain. Cauchy's theorem for star domains. The general Cauchy theorem. Cross-cuts (cancelling line segments) to split complicated contours into simpler ones. Initial discussion of the Cauchy integral formula.

Lecture 14: Annuli and integrals round circles. Discussion of statements and proofs of Cauchy integral formula, Liouville's Theorem, The Fundamental Theorem of Algebra. (Full details available in printed lecture notes.)

Lecture 15: Revision of convergence of complex sequences. Series with complex terms. Complex geometric series: important standard series for $1/(1-z)$ and $1/(1+z)$ when $|z| < 1$. Motivation for series expansions (computing integrals: examples). Revision of standard results for real series. Absolute convergence of complex series. The ratio test.

Lecture 16: Initial discussion and examples of power series (non-negative integer powers of $(z-\alpha)$) and Laurent series (possibly negative integer powers of $z-\alpha$). Basic properties of power series: Radius of convergence, disc of convergence. Term by term integration and differentiation within the disc of convergence. Examples: the geometric series and its derivative, and the integral of $1/(z^5(1-z))$ around $|z| = 1/2$.

Lecture 17: Using coefficients to find e.g. $f^{(9)}(0)$ when $f(z) = \sin(z^3)$. Series in negative powers of $(z-a)$: convergence OUTSIDE some disc. Examples. Statement and discussion of Laurent's theorem: expansion of $f(z)$ as $\sum_{k\in\mathbb{Z}} a_k(z-a)^k$ when f is analytic on an annulus A centred on a. Formula for coefficients.

Lecture 18: Examples of finding Laurent series. Guidance on methods and acceptable forms of answers: the answer **must** be in terms of powers $(z-a)^k$ where $k \in \mathbb{Z}$ and a is at the centre of the relevant annulus, and the answer must **not** be left in the form (series) times (series).

Lecture 19: Review of Laurent's theorem. Taylor's theorem. The derivative of an analytic function is analytic. Examples. The binomial theorem. Taylor's theorem for products. Examples.

Lecture 20: Function of a function (composition): care needed when substituting one series into another. Examples of calculations using these results. The functions $\sin(z)$ and $\cos(z)$ never take the values 0 or ± 1 when z is off the real axis. Examples on Taylor and Laurent series and calculating integrals.

Lecture 21: More examples on Taylor and Laurent series and associated integrals.

Lecture 22: Isolated singularities. Examples, classification. Residues of functions at isolated singularities. The Cauchy residue theorem: to find the integral (once anticlockwise round a simple closed piecewise smooth contour) you add up the residues over the (finitely many, isolated) singularities inside the contour and then multiply by $2\pi i$. Discussion of why this works with one or two singularities inside the contour. Examples.