G1BCOF: Complex Functions 2003-4

Solutions to questions 3-8 of Section 5.3

Let f be a function which has an isolated singularity at a. Recall that a is a **simple pole** of f if the Laurent series for f near a has the form $c_{-1}(z-a)^{-1} + c_0 + c_1(z-a)^1 + \cdots$ with $c_{-1} \neq 0$ (i.e. the Laurent series coefficients $c_{-2}, c_{-3}, \ldots$ are all 0, but $c_{-1} \neq 0$). If instead **all** of the negative Laurent coefficients $c_{-1}, c_{-2}, c_{-3}, \ldots$ are 0 then f has a **removable singularity** at a.

If f has either a simple pole or a removable singularity at a then the residue of f at a, $\mathrm{Res}(f,a)$, is equal to $\lim_{z\to a}(z-a)f(z)$.

Conversely, if this limit exists and is finite then the function f has either a simple pole or a removable singularity at a, and we have $\mathrm{Res}(f,a) = \lim_{z\to a}(z-a)f(z)$.

In particular, if $f(z) = 1/g(z)$ where g is analytic at a, $g(a) = 0$ and $g'(a) \neq 0$ then f has a simple pole at a, and, since $g(a) = 0$,

$$\mathrm{Res}(f,a) = \lim_{z\to a}(z-a)f(z) = \lim_{z\to a}(z-a)/(g(z)-g(a)) = 1/g'(a).$$

Note that these tricks only really help for simple poles

3 **Question** Evaluate

$$\int_{|z|=300} \frac{z-17}{(z-2)(z-4)}\,\mathrm{d}z\,.$$

Answer Here it is obvious that the function

$$f(z) = \frac{z-17}{(z-2)(z-4)}$$

has **simple poles** at 2 and 4, and these are both inside the contour. We have

$$\mathrm{Res}(f,2) = \lim_{z\to 2}(z-2)f(z) = 15/2$$

and

$$\mathrm{Res}(f,4) = \lim_{z\to 4}(z-4)f(z) = -13/2.$$

The sum of the residues at the singularities inside the contour is thus $15/2 - 13/2 = 1$ and so, by the Cauchy residue theorem, the integral is $2\pi i$ times the sum of residues, which is just $2\pi i$.

4 **Question** Evaluate

$$\int_{|z|=1} \frac{1}{\mathrm{e}^z - 1}\,\mathrm{d}z\,.$$

Answer The function $f(z) = 1/(\mathrm{e}^z - 1)$ has singularities at $z = 2n\pi i$ where $n \in \mathbb{Z}$ (these are the solutions of $\mathrm{e}^z = 1$).

The only singularity inside the contour is the one at 0. It is clear that this is a pole, but not obvious whether it is simple or not. However we have $f(z) = 1/g(z)$ with $g(z) = \mathrm{e}^z - 1$ and so $g(0) = 0$ but $g'(0) = 1 \neq 0$. It follows that f has a simple pole at 0 and the residue there is $\mathrm{Res}(f, 0) = 1/g'(0) = 1$. By the residue theorem the required integral is $2\pi i$ times the sum of the residues at the singularities inside the contour, and so the answer is again just $2\pi i$.

An alternative method is to use series. We can write $\mathrm{e}^z - 1 = z + z^2/2 + z^3/6 + \cdots = z(1 + z/2 + z^2/6 + \cdots) = z(1 + g(z))$ where g is analytic and $g(0) = 0$. Thus

$$\frac{1}{\mathrm{e}^z - 1} = \frac{1}{z(1+g(z))} = \frac{1}{z}\,\frac{1}{1+g(z)}.$$

Near 0 we know $|g(z)|$ is small, so we can expand as

$$\frac{1}{\mathrm{e}^z - 1} = \frac{1}{z}\left(1 - g(z) + g(z)^2 - g(z)^3 + \cdots\right)$$

and since $g(z) = z/2 + z^2/6 + \cdots$ we see easily that the coefficient of $1/z$ here is just 1. Thus the residue is 1, as above.

5 **Question** Evaluate

$$\int_{|z|=1} \frac{1}{(\mathrm{e}^z - 1)^2}\,\mathrm{d}z\,.$$

Answer Again the only singularity inside the contour is at $z = 0$. But this time we do not have a simple pole: in fact as $z \to 0$, $|z/(\mathrm{e}^z - 1)^2|$ becomes very large, and we can not use this to help us. Instead we return to series methods.

With $g(z)$ as in Question 4 above, we have

$$\frac{1}{(\mathrm{e}^z - 1)^2} = \frac{1}{z^2}\,\frac{1}{(1+g(z))^2} = \frac{1}{z^2}(1 - g(z) + g(z)^2 - g(z)^3 + \cdots)^2.$$

This means that the residue at 0 is the coefficient of z in $(1 - g(z) + g(z)^2 - g(z)^3 + \cdots)^2$. Since $g(z) = z/2 + z^2/6 + \cdots$, this is easily seen to be equal to the coefficient of z in $(1 - z/2)^2$ (no higher powers of g or z are relevant) and this is just -1. Thus the residue at the singularity is -1 and the integral is $-2\pi i$.

6 **Question** Let $R \geq 2$ and let Γ_R be the closed semicircular contour consisting of the straight line from $-R$ to R followed by the semicircular arc from R to $-R$ via iR. Calculate

$$\int_{\Gamma_R} \frac{\mathrm{e}^{iz}}{z^2+1}\,\mathrm{d}z\,.$$

Answer The function has singularities at $z = \pm i$, but only i lies inside the contour. The singularity there is a simple pole, and (factorizing $z^2 + 1 = (z+i)(z-i)$) the residue at i

is

$$\lim_{z\to i}(z-i)\frac{\mathrm{e}^{iz}}{z^2+1}=\lim_{z\to i}\frac{\mathrm{e}^{iz}}{z+i}=\frac{\mathrm{e}^{-1}}{2i}.$$

Thus the integral is just $(2\pi i)/(2i\mathrm{e})=\pi/\mathrm{e}$.

7 **Question** Determine

$$\lim_{R\to\infty}\int_0^R\frac{\cos x}{x^2+1}\,\mathrm{d}x\,.$$

Answer We use the information obtained in Question 6 about Γ_R. Let γ_R be the semicircular arc from R to $-R$ via iR. Note that on γ_R we have $\mathrm{Im}(z)\geq 0$ and so $\mathrm{Re}(iz)\leq 0$. From this we have $|\mathrm{e}^{iz}|\leq 1$ for z on γ_R, while $|1+z^2|\geq|z^2|-1=R^2-1$, so that

$$\left|\frac{\mathrm{e}^{iz}}{z^2+1}\right|\leq 1/(R^2-1)$$

for z on γ_R. Since the length of γ_R is πR, we see that the modulus of the integral along γ_R of $\dfrac{\mathrm{e}^{iz}}{z^2+1}$ is at most $\pi R/(R^2-1)$, and this tends to 0 as $R\to\infty$.

Since the total integral round Γ_R remains constant at π/e, the conclusion is that the integral along the straight line from $-R$ to R must converge to π/e as $R\to\infty$. Thus we have proved that

$$\int_{-\infty}^{\infty}\frac{\mathrm{e}^{ix}}{x^2+1}\,\mathrm{d}x=\lim_{R\to\infty}\int_{-R}^{R}\frac{\mathrm{e}^{ix}}{x^2+1}\,\mathrm{d}x=\pi/\mathrm{e}.$$

Taking real parts, we obtain

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{\cos x}{x^2+1}\,\mathrm{d}x=\pi/\mathrm{e},$$

and since $\cos x/(x^2+1)$ is an even function we have

$$\lim_{R\to\infty}\int_0^R\frac{\cos x}{x^2+1}\,\mathrm{d}x=\pi/(2\mathrm{e}).$$

Note that the answer is a real number! Any other kind of answer here would have to be wrong.

You may ask **why do we use** e^{iz} **instead of** $\cos z$ **here?** Perhaps surprisingly, the answer is that $|\cos z|$ is too large on the γ_R. Remember that $\cos z=(\mathrm{e}^{iz}+\mathrm{e}^{-iz})/2$, and although the first term is small on γ_R, the second term gets very big (look at e.g. $z=iR$).

8 **Question** Evaluate

$$\int_{-\infty}^{\infty}\frac{1}{x^2+2x+6}\,\mathrm{d}x\,.$$

Answer Using the semicircular contour method, with Γ_R and γ_R as above, and integrating the function $1/(z^2+2z+6)$, we see again that the integral over γ_R tends to

0 as $R \to \infty$ (the details are an exercise) and so the answer is simply equal to the integral of $1/(z^2 + 2z + 6)$ around any of the large closed semicircular contours Γ_R above.

We now look at the singularities of $1/(z^2 + 2z + 6)$. The solutions of $z^2 + 2z + 6 = 0$ are found using the quadratic formula to be at $z = -1 \pm i\sqrt{5}$, and only $-1 + i\sqrt{5}$ lies inside the contour. There is a simple pole there, and (using your favourite method) the residue is found to be $1/(2i\sqrt{5})$. Thus the integral we want is $2\pi i$ times this, i.e. $\pi/\sqrt{5}$. This is, of course, a positive real number. If you get any other kind of number here you must have made a mistake, since $1/(x^2 + 2x + 6) > 0$ for all real x.