G1BCOF: Complex Functions 2003-4
Blow-by-blow account of the module

Lecturer: Dr J. Feinstein

Lecture 1: Motivation. The integral of a complex-valued function over an interval. Elementary examples and estimates.

Lecture 2: Paths, curves and contours. Smooth contours and their properties. Integrals of functions along smooth contours.

Lecture 3: Examples. Fundamental estimate and applications. Simple smooth contours: integral independent of parametrization (brief discussion). Piecewise smooth contours (PSCs) and integrals. Stepwise curves.

Lecture 4: More on contour integrals. Simple PSCs and simple closed PSCs. Open sets.

Lecture 5: Domains. Partial derivatives. Sequences. Limits of sequences in $\mathbb{C}$. Connections between complex-valued functions of a complex variable and real-valued functions of two real variables. Function limits.

Lecture 6: Examples of limits. Continuity. Definition of the complex derivative.

Lecture 7: Examples: $\overline{z}$, z^2, polynomials and rational functions. Meaning of the complex derivative. Product rule, chain rule etc. Complex differentiable implies Cauchy-Riemann equations hold. Converse holds provided the partial derivatives exist and are continuous near the point in question. Examples.

Lecture 8: Analytic functions. Examples, including $\exp(z) = \mathrm{e}^z$, $\sin z$, $\cos z$. Entire functions.

Lecture 9: The complex logarithm: principal logarithm Log(z) (on standard cut plane). Derivative of Log. Powers of z, z^b (in standard cut plane, principal branch only). Discontinuity when b is not an integer. Integral of the derivative of an analytic function.

Lecture 10: The integrals of an antiderivative round a closed contour is 0. Statement of Cauchy-Goursat. Key ideas of proof discussed. For a detailed proof see full lecture notes. Examples.

Lecture 11: Revision of statement of Cauchy-Goursat. Statements and discussion of the general Cauchy theorem and Cauchy's integral formula. Star domains. Examples. Discussion of ways to deal with domains that are not quite star-shaped. An analytic function has an antiderivative in each star domain. Cauchy's theorem for star domains.

Lecture 12: The general Cauchy theorem. Cross-cuts (cancelling line segments) to split complicated contours into simpler ones. The Cauchy integral formula.

Lecture 13: Liouville's Theorem. The Fundamental Theorem of Algebra.

(Non-examinable application of analytic functions to fluids: see printed notes).

Motivation for series expansions (computing integrals). Series with complex terms. Revision of tests for real series. Complex geometric series. Absolute convergence of complex series. Series representations for the function $\dfrac{1}{z(1-z)}$ in the regions $|z| < 1$, $|z| > 1$. The typical annulus centred on 0: $\{z \in \mathbb{C} : R_1 < |z| < R_2\}$.

Lecture 14: Use of ratio test to prove absolute convergence for complex series. Basic properties of power series: Radius of convergence. Term by term integration and differentiation. Examples: the geometric series and its derivative, and the integral of $1/(z^2(1-z))$ around $|z| = 1/2$.

Lecture 15: Using coefficients to find e.g. $f^{(9)}(0)$ when $f(z) = \sin(z^3)$. Series in negative powers of $(z-a)$: convergence OUTSIDE some disk. Statement and discussion of Laurent's theorem: expansion of $f(z)$ as $\sum_{k\in\mathbb{Z}} a_k(z-a)^k$ when f is analytic on an annulus A centred on a. Formula for coefficients. Special case where $f(z) = (z-a)^n$ (for some $n \in \mathbb{Z}$) checked. Examples.

Lecture 16: More examples and applications of Laurent's theorem. Taylor's theorem as a special case of Laurent. The derivative of an analytic function is analytic. Examples.

Lecture 17: The binomial theorem. Taylor's theorem for products, compositions. Examples of calculations using these results.

Lecture 18: The functions $\sin(z)$ and $\cos(z)$ never take the values 0 or ± 1 when z is off the real axis. More examples on Taylor and Laurent series and calculating integrals.

Lecture 19: More examples on Laurent series. Isolated singularities. Examples, classification.

Lecture 20: Revision of types of isolated singularities. Residues of functions at isolated singularities. Discussion of the residue theorem: to find the integral (once anticlockwise round a simple piecewise smooth contour) you add up the residues over the (finitely many, isolated) singularities inside the countour and then multiply by $2\pi i$. Examples indicating why this works with one or two singularities inside the contour.

Lecture 21: Statement of the Cauchy residue theorem. Deduction of the Cauchy integral formula from the residue theorem. Simple poles: short cuts using cover-up rule or $1/g'(a)$ when $f(z) = 1/g(z)$ has a simple pole at a. **These tricks are only valid for simple poles**.

Lecture 22: More examples of calculations using the Cauchy residue theorem. Real integrals via residues, using the semi-circular contour.

Lecture 23: Review session with questions and answers. See also the Frequently Asked Questions document available online at

`http://www.maths.nott.ac.uk/personal/jff/G1BCOF/#handouts`