G1BCOF: Complex Functions, Solutions to Special Examples Class

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1 Revision Questions

Problem 1.1 Using the hint, we need to find the complex fourth roots of -16. First we write -16 in exponential form as $16\mathrm{e}^{i\pi}$.
(i) We use the standard formula given in the notes. In exponential form, the four complex fourth roots of the number $-16 = 16\mathrm{e}^{i\pi}$ are $16^{1/4}\mathrm{e}^{i(2k\pi+\pi)/4}$, $k = 0, 1, 2, 3$, i.e. $2\mathrm{e}^{i\pi/4}$, $2\mathrm{e}^{i(3\pi/4)}$, $2\mathrm{e}^{i(5\pi/4)}$ and $2\mathrm{e}^{i(7\pi/4)}$. Note that the last two arguments above are not the **principal** values of the argument. If you wish you can (for example) rewrite $2\mathrm{e}^{i(7\pi/4)}$ as $2\mathrm{e}^{i(-\pi/4)}$. (This is not required.)
(ii) We can plot these points on the Argand diagram and use standard right-angled triangles to find them in the form $x + iy$, as usual. The resulting four solutions have the form $\pm\sqrt{2} \pm \sqrt{2}\,i$.

Problem 1.2 Following the hint, we first note that $2i = 2\mathrm{e}^{i\pi/2}$, and the two complex square roots of this number are (using our formula for nth roots with $n = 2$) $\sqrt{2}\mathrm{e}^{i\pi/4}$ and $\sqrt{2}\mathrm{e}^{i(5\pi/4)}$. However, as always happens with square roots, the second square root is minus the first (remember also that $\mathrm{e}^{i\pi} = -1$). So the two square roots are $\pm\sqrt{2}\mathrm{e}^{i\pi/4}$. However $\mathrm{e}^{i\pi/4} = \cos(\pi/4) + i\sin(\pi/4) = (1/\sqrt{2}) + (1/\sqrt{2})i$, so the two square roots are simply $1 + i$ and $-1 - i$. (These may also be found 'by inspection' if you happen to remember or spot that $(1 + i)^2 = 2i$. The second square root must then be $-(1 + i)$.)
Now we use the quadratic formula to solve the equation

$$z^2 + (3 + 3i)z + 4i = 0 :$$

we have $a = 1$, $b = 3 + 3i$ and $c = 4i$. Informally the two answers are given by the formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 - 3i \pm \sqrt{(3 + 3i)^2 - 4i}}{2} = \frac{-3 - 3i \pm \sqrt{2i}}{2}.$$

What this means is that we use the two complex square roots of $2i$ (found above) to give us the two solutions $(-3 - 3i + (1 + i))/2 = -1 - i$ and $(-3 - 3i - (1 + i))/2 = -2 - 2i$.

2 The complex exponential function

Problem 2.1 (a) We know, for complex numbers a and b, that $|ab| = |a||b|$. Thus

$$|\exp(z)| = |\mathrm{e}^x\mathrm{e}^{iy}| = |\mathrm{e}^x||\mathrm{e}^{iy}| = \mathrm{e}^x,$$

because e^x is a positive real number and $|\mathrm{e}^{iy}| = 1$.

(b) Write $z = x + iy$ as usual. We have $w = \exp(z)$ so (by (a)) $|w| = \mathrm{e}^x > 0$. Thus (since e^x is never 0) $w \neq 0$, and $x = \ln(|w|)$, i.e. $\mathrm{Re}(z) = \ln(|w|)$, as required. Finally, set $r = |w| = \mathrm{e}^x$. Then $w = \exp(z) = r\,e^{iy}$, and we have found w in exponential form. So $\mathrm{Im}(z) = y$ must be an argument of w.

(Alternatively, you may observe that $w = (\mathrm{e}^x)\,\mathrm{e}^{iy}$ is already in exponential form, since $e^x > 0$. The rest follows from this.)

Problem 2.2 First note that the steps in 2.1 (b) are reversible: for $w \neq 0$, if $x = \ln(|w|)$ and y is an argument of w then $\exp(z) = w$.
(i) Write $w = 1 + i$ in exponential form as $\sqrt{2}\mathrm{e}^{i\pi/4}$. By above, the solutions of $\exp(z) = w$ are the numbers $z = x + yi$ where $x = \ln(|w|) = \ln(\sqrt{2}) = \frac{1}{2}\ln 2$, and y is an argument for w, so $y = \pi/4 + 2k\pi$ where $k \in \mathbb{Z}$. Thus the solutions $z = x + yi$ are $\frac{1}{2}\ln 2 + i(2k + \frac{1}{4})\pi$, where $k \in \mathbb{Z}$.
(ii) We may rewrite this equation in the form $\exp(z) = -1$, and then use the same methods as above. Now write $w = -1$ in exponential form as $\mathrm{e}^{i\pi}$ (or $1 \times \mathrm{e}^{\pi i}$). Then the solutions of $\exp(z) = w$ are the numbers $z = x + yi$ where $x = \ln(|w|) = \ln(1) = 0$, and y is an argument for w, so $y = \pi + 2k\pi = (2k + 1)\pi$ where $k \in \mathbb{Z}$. Thus the solutions z are all the odd integer multiples of πi.

Problem 2.3 As above, the solutions of the equation $\exp(z) = w$ have the form $x + iy$ where $x = \ln(|w|)$ and y is an argument of w. Since w has infinitely many different arguments, differing from each other by integer multiples of 2π, there are infinitely many different solutions, differing from each other by integer multiples of $2\pi i$.