G1BCOF Examination 2003–2004

MODEL ANSWERS

1 (a) By Cauchy's theorem the integral of the entire function $2z$ round the closed contour γ is 0. So the integral is just

$$\int_\gamma \bar{z}\, dz.$$

Writing

$$z = \gamma(t) = 4 + 3e^{it}, \quad dz = 3ie^{it},$$
$$\bar{z} = 4 + 3e^{-it}$$

and so

$$\begin{aligned}\int_\gamma \bar{z}\, dz &= \int_0^{2\pi} (4 + 3e^{-it})(3ie^{it})\, dt \\ &= \int_0^{2\pi} (12ie^{it} + 9i)\, dt \\ &= 18\pi i\end{aligned}$$

since $\displaystyle\int_0^{2\pi} e^{it}\, dt = 0.$

(b) The **Laurent series** for a function f which is analytic on an annulus $\{z \in \mathbb{C} \colon R_1 < |z - \alpha| < R_2\}$ is a series in positive and negative powers of $z - \alpha$

$$\sum_{n\in\mathbb{Z}} a_n (z-\alpha)^n$$

representing f in this annulus.
The formula for the coefficient a_n is

$$a_n = \frac{1}{2\pi i}\int_\gamma (z-\alpha)^{-(n+1)} f(z)\, dz$$

where γ is the circle $|z - \alpha| = \frac{1}{2}(R_1 + R_2)$ (for example) described once counterclockwise.

(c) (i) Here $|z^3| < 1$ so

$$\frac{1}{(1-z^3)} = 1 + z^3 + z^6 + z^9 + \dots$$

and

$$\begin{aligned}f(z) &= (1+z^2)(1 + z^3 + z^6 + \dots) \\ &= 1 + z^2 + z^3 + z^5 + z^6 + z^8 + \dots\end{aligned}$$

(ii) Here

$$\frac{1}{1-z^3} = \frac{-1}{z^3}\left(\frac{1}{1-\frac{1}{z^3}}\right) = -\frac{1}{z^3}\left(1+\frac{1}{z^3}+\frac{1}{z^6}+\dots\right) = -\frac{1}{z^3}-\frac{1}{z^6}-\frac{1}{z^9}-\dots$$

so

$$f(z) = -(1+z^2)\left(\frac{1}{z^3}+\frac{1}{z^6}+\frac{1}{z^9}+\dots\right) = -\left(\frac{1}{z}+\frac{1}{z^3}+\frac{1}{z^4}+\frac{1}{z^6}+\frac{1}{z^7}+\frac{1}{z^9}+\dots\right).$$

By Laurent's theorem (for example)

$$\begin{aligned}\int_\gamma f(z)\,dz &= 2\pi i \times \text{coefficient of } \frac{1}{z}\\ &= 2\pi i \times (-1) = -2\pi i.\end{aligned}$$

2 (a) The function $f(z)$ is

(i) complex differentiable at a if

$$\lim_{z \to a} \left(\frac{f(z) - f(a)}{z - a} \right)$$

exists.

(ii) f is **analytic at $\boldsymbol{a}$** if f is complex differentiable on some domain including a (**or** on an open set / open disk containing a).

(b) Writing $f(x + iy) = u(x, y) + iv(x, y)$, the Cauchy–Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

(c) This can be done directly using the Cauchy–Riemann equations, or quoting the result that $Re(g)$ must be harmonic: $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(Re(g)\right) = 0$, if g is analytic. But here $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left((x - y)^3\right) = 12(x - y)$, which is not the zero function.

(d) We write down the Taylor series for $f(z)$.

$$\begin{aligned}
f(z) &= z^2 \cos(z^3) \\
&= z^2 \left(1 - \frac{(z^3)^2}{2!} + \frac{(z^3)^4}{4!} - \frac{(z^3)^6}{6!} \dots \right) \\
&= z^2 \left(1 - \frac{z^6}{2} + \frac{z^{12}}{24} - \frac{z^{18}}{720} + \dots \right) \\
&= z^2 - \frac{z^8}{2} + \frac{z^{14}}{24} - \frac{z^{20}}{720} + \dots
\end{aligned}$$

so

(i) $f^{(8)}(0) = -\frac{8!}{2}$ (this form is okay).

(ii) (More than one method here.) For example

$$f'(z) = 2z - 4z^7 + \frac{14}{24} z^{13} \dots$$

and

$$\frac{f'(z)}{z^{14}} = \frac{2}{z^{13}} - \frac{4}{z^7} + \left(\frac{14}{24}\right) \cdot \frac{1}{z} - \dots$$

and so

$$\begin{aligned}
\int \frac{f'(z)}{z^{14}} \, dz &= (2\pi i) \times \left(\text{coeff. } \frac{1}{z}\right) \\
&= (2\pi i) \times \frac{14}{24} = \frac{7}{6}\pi i.
\end{aligned}$$

3 (a) Writing z in exponential form as $r\exp\big(i\,\mathrm{Arg}(z)\big)$ (Arg is the principal argument),

$$\begin{aligned}\mathrm{Log}(z) &= \ln r + i\,\mathrm{Arg}(z)\\ &= \ln|z| + i\,\mathrm{Arg}(z).\end{aligned}$$

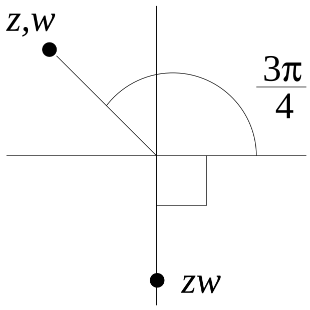
For $z \in D_0$ we have $\exp\big(\mathrm{Log}(z)\big) = z$ (but not $\mathrm{Log}\big(\exp(z)\big) = z$).

(b) Take $z = w = -1 + i \in D_0$. Then $zw = -2i \in D_0$. We have

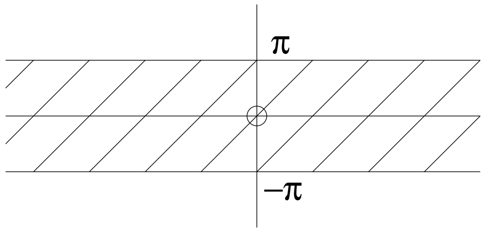
$$\mathrm{Log}(z) = \mathrm{Log}(w) = \ln(\sqrt{2}) + \left(\frac{3\pi}{4}\right)i$$

while

$$\mathrm{Log}(zw) = \ln(2) - \frac{\pi}{2}i \neq \mathrm{Log}(z) + \mathrm{Log}(w).$$



(c) $\mathrm{Log}(D_0)$ is the infinite horizontal strip
$\{x + iy\colon x \in \mathbb{R},\ -\pi < y < \pi\} = \{z \in \mathbb{C}\colon\ |\mathrm{Im}(z)| < \pi\}$.



(Boundary not included.)

(d) Note that $i = \exp(i\pi/2)$.

$$z^{2i} = i \iff \exp\big(2i\,\mathrm{Log}(z)\big) = \exp(i\pi/2)$$

$$\iff 2i\,\mathrm{Log}(z) = i\pi/2 + 2k\pi i$$

for some $k \in \mathbb{Z}$,

$$\iff \mathrm{Log}(z) = \frac{\pi}{4} + k\pi$$

$$\iff z = \exp\left(\frac{\pi}{4} + k\pi\right) \quad \text{with } k \in \mathbb{Z}.$$

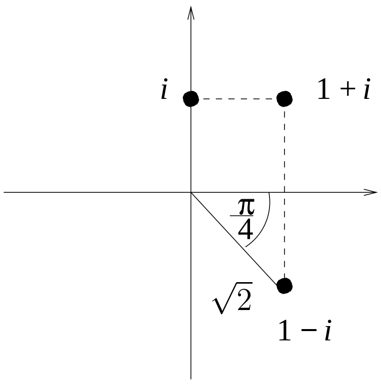
All these solutions $\exp\left(\frac{\pi}{4} + k\pi\right)$ lie in D_0.

(e) This is best done by antidifferentiation. Since the contour lies in D_0, and $\frac{d}{dz}\big(\mathrm{Log}(z)\big) = \frac{1}{z}$ on D_0,

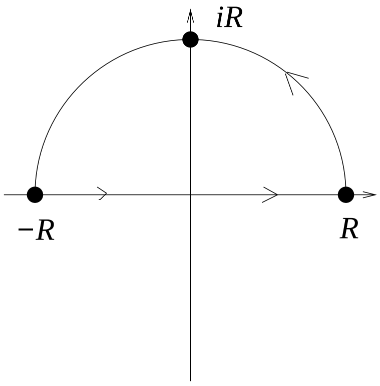
$$\int_\gamma \frac{1}{z}\, dz = \mathrm{Log}(i) - \mathrm{Log}(1-i)$$

(the difference in Log at the endpoints)

$$\begin{aligned} &= \frac{i\pi}{2} - \left(\ln(\sqrt{2}) - \frac{\pi}{4} i\right) \\ &= -\ln(\sqrt{2}) + \frac{3\pi}{4} i. \end{aligned}$$



4 (a)



The poles of $\frac{1}{z^4+4}$ are the zeros of z^4+4, i.e. the 4th roots of -4. These are $\pm 1 \pm i$. Of these $\pm 1 + i$ are inside the contour, and so by the Cauchy residue theorem the integral is

$$2\pi i \left(\mathsf{Res}\left(\frac{1}{z^4+4}, -1+i\right) + \mathsf{Res}\left(\frac{1}{z^4+4}, 1+i\right)\right).$$

These residues may be found using various short cuts, e.g. since $\frac{d}{dz}(z^4+4) = 4z^3$,

$$\text{Res}\left(\frac{1}{z^4+4}, -1+i\right) = \frac{1}{4(-1+i)^3} = \frac{-1+i}{-16}$$

while

$$\text{Res}\left(\frac{1}{z^4+4}, 1+i\right) = \frac{1}{4(1+i)^3} = \frac{1+i}{-16}.$$

Thus the integral is

$$2\pi i\left(\frac{-1+i}{-16} + \frac{1+i}{-16}\right)$$
$$= 2\pi i\left(\frac{-i}{8}\right) = \frac{\pi}{4}.$$

(b) We use the fundamental estimate. On the contour we have $|z| = R$, so $|z|^4 + 4 \geq R^4 - 4 > 0$ and

$$\left|\frac{1}{z^4+4}\right| \leq \frac{1}{R^4-4}.$$

The length of the contour is πR, so

$$\left|\int_{\gamma_R} \frac{1}{z^4+4}\, dz\right| \leq \frac{\pi R}{R^4-4} \to 0 \quad \text{as} \quad R \to \infty.$$

Thus, by the Sandwich Theorem

$$\left|\int_{\gamma_R} \frac{1}{z^4+4}\, dz\right| \to 0 \quad \text{as} \quad R \to \infty, \quad \text{i.e.}$$

$$\int_{\gamma_R} \frac{1}{z^4+4}\, dz \to 0 \quad \text{as} \quad R \to \infty.$$

(c) Note that

$$\int_{\Gamma_R} \frac{1}{z^4+4}\, dz = \int_{-R}^{R} \frac{1}{x^4+4}\, dx + \int_{\gamma_R} \frac{1}{z^4+4}\, dz.$$

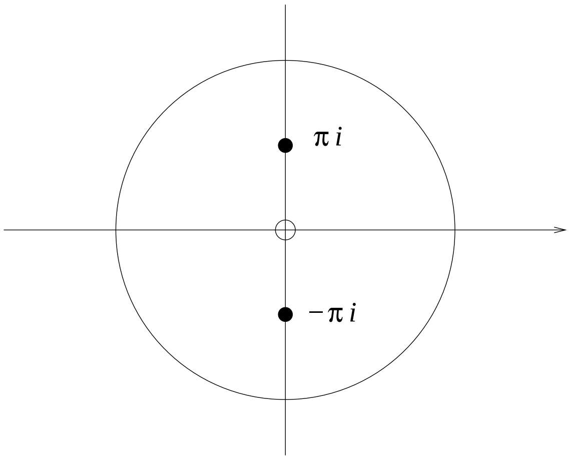
Thus

$$\int_{-R}^{R} \frac{1}{x^4+4}\, dx = \frac{\pi}{4} - \int_{\gamma_R} \frac{1}{z^4+4}\, dz$$
$$\to \frac{\pi}{4} \quad \text{as} \quad R \to \infty$$

(by (a) and (b)).

This gives us $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^4+4}\, dx = \frac{\pi}{4}.$

5 (a) The poles of this function are at 0 and at $(2k+1)\pi i$ where k is an integer (since $e^{(2k+1)\pi i} = -1$). Of these, 0 and $\pm\pi i$ lie inside the contour.



Near 0 we can write $\frac{1}{e^z+1}$

$$= \frac{1}{\left(2 + z + \frac{z^2}{2} + \dots\right)} = \frac{1}{2}\left(\frac{1}{1 + \left(\frac{z}{2} + \frac{z^2}{4} + \dots\right)}\right)$$

$$= \frac{1}{2}\left(1 - \left(\frac{z}{2} + \frac{z^2}{4} + \dots\right) + \left(\frac{z}{2} + \frac{z^2}{4} + \dots\right)^2 \dots\right)$$

and the coefficient of z here is $-\frac{1}{4}$, so the residue of $\frac{1}{z^2}\left(\frac{1}{e^z+1}\right)$ at 0 is $-\frac{1}{4}$.
At $\pm\pi i$ we have simple poles and, since $\frac{d}{dz}(e^z+1) = e^z$, the residues there are

$$\frac{1}{(\pi i)^2 \exp(\pi i)} \quad \text{and} \quad \frac{1}{(-\pi i)^2 \exp(-\pi i)}$$

and both of these are $\frac{1}{\pi^2}$. Thus, by the Cauchy residue theorem, the integral is

$$2\pi i\left(-\frac{1}{4} + \frac{2}{\pi^2}\right) = -\frac{\pi}{2} i + \frac{4}{\pi} i.$$

(b) Write $g(x+iy) = u(x,y) + iv(x,y)$ and $f(x+iy) = U(x,y) + iV(x,y)$. Then the question gives us that $g(x+iy) = f(2x+3iy)$, i.e.

$$u(x,y) + iv(x,y) = U(2x,3y) + iV(2x,3y).$$

Taking real and imaginary parts gives us $u(x,y) = U(2x,3y)$ and $v(x,y) = V(2x,3y)$.
We also have the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{1}$$

and

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \quad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}. \tag{2}$$

By the chain rule, we have

$$\frac{\partial u}{\partial x}(x, y) = 2\frac{\partial U}{\partial x}(2x, 3y) \tag{3}$$

and

$$\frac{\partial u}{\partial y}(x, y) = 3\frac{\partial U}{\partial y}(2x, 3y) \tag{4}$$

while

$$\frac{\partial v}{\partial x}(x, y) = 2\frac{\partial V}{\partial x}(2x, 3y) \tag{5}$$

and

$$\frac{\partial v}{\partial y}(x, y) = 3\frac{\partial V}{\partial y}(2x, 3y). \tag{6}$$

Combining these with the Cauchy Riemann equations we see that (using (1), (3), (2) and (6))

$$2\frac{\partial V}{\partial y}(2x, 3y) = 3\frac{\partial V}{\partial y}(2x, 3y)$$

and (using (1), (2), (4) and (5))

$$-3\frac{\partial V}{\partial x}(2x, 3y) = -2\frac{\partial V}{\partial x}(2x, 3y)$$

giving us $\frac{\partial V}{\partial x} = 0$, $\frac{\partial V}{\partial y} = 0$, and so $\frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} = 0$ also, and f is constant.
Hence g is also constant.