G1BCOF Examination 2004–2005

MODEL ANSWERS

1 (a) (i) The complex-valued function f is **complex-differentiable** at a if $\lim_{z \to a} \left(\dfrac{f(z) - f(a)}{z - a} \right)$ exists (and is in $\mathbb{C}$ – optional).

(ii) The function f is **analytic** at a if it is complex-differentiable at all points of some open disc centred at a. [Or any equivalent version using open sets or domains instead of discs.]

(b) Writing $f(x + \mathrm{i}y) = u(x, y) + \mathrm{i}v(x, y)$, the Cauchy–Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

(c) This can be done directly using the Cauchy–Riemann equations, or more quickly by quoting the result that Re(g) must be harmonic: $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} \right) \big(\mathrm{Re}(g)\big) = 0$ constantly if g is analytic.

Here we look at $(x^2 - y^2)^2 = x^4 - 2x^2y^2 + y^4$. We have

$$\frac{\partial^2}{\partial x^2}\big((x^2 - y^2)^2\big) = 12x^2 - 4y^2$$

$$\frac{\partial^2}{\partial y^2}\big((x^2 - y^2)\big) = 12y^2 - 4x^2.$$

Thus

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \big((x^2 - y^2)^2\big) = 8(x^2 + y^2),$$

which is not constantly 0. Thus no such analytic function g exists.

(d) (i) We use the standard series

$$\sin(u) = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \dots$$

so $\quad \sin(z^2) = z^2 - \dfrac{z^6}{3!} + \dfrac{z^{10}}{5!} - \dots.$

Also $\quad \dfrac{1}{1 + z} = 1 - z + z^2 - z^3 + \dots.$

We multiply these series, neglecting terms in z^8 or higher, to obtain

$$\frac{\sin(z^2)}{1+z} = z^2 - z^3 + z^4 - z^5 + z^6\left(1 - \frac{1}{3!}\right) - z^7\left(1 - \frac{1}{3!}\right)$$

$$+ \text{ terms in } z^8 \text{ and higher.}$$

(ii)

$$\begin{aligned} f^{(5)}(0) &= (5!) \times \text{coefficient of } z^5 \text{ in the Taylor series above} \\ &= -5!\ (= -120). \end{aligned}$$

(iii) The function f is analytic on and inside γ, so this is just $2\pi\mathrm{i}$ times the coefficient of z^7 in the Taylor series for f, i.e. $-2\pi\mathrm{i}\left(1 - \frac{1}{3!}\right)$ (or $-\frac{5}{3}\pi\mathrm{i}$).

2 (a) By Cauchy's theorem, the integral of the entire function $2z^2$ round the closed contour γ is 0. Thus the integral is just

$$\int_\gamma -3\bar{z}\ \mathrm{d}z = -3\int_\gamma \bar{z}\ \mathrm{d}z.$$

It was frequently mentioned verbally in lectures that the integral of $\bar{z}$ round a circle is $2\mathrm{i} \times$ (area of circle). Quoting this, the answer is $(-3) \times 2\mathrm{i} \times \pi \times 2^2 = -24\pi\mathrm{i}$.

Alternatively, use direct calculation; set

$$z = \gamma(t) = 5 + 2\mathrm{e}^{\mathrm{i}t} \quad (0 \le t \le 2\pi)$$

$$\bar{z} = 5 + 2\mathrm{e}^{-\mathrm{i}t}, \quad \gamma'(t) = 2\mathrm{i}\,\mathrm{e}^{\mathrm{i}t},$$

and the integral $\int_\gamma \bar{z}\ \mathrm{d}z$ becomes

$$\begin{aligned} \int_0^{2\pi} (5 + 2\mathrm{e}^{-\mathrm{i}t})2\mathrm{i}\mathrm{e}^{\mathrm{i}t}\ \mathrm{d}t = \int_0^{2\pi} (10\mathrm{i}\mathrm{e}^{\mathrm{i}t} + 4\mathrm{i})\ \mathrm{d}t &= [10\mathrm{e}^{\mathrm{i}t} + 4\mathrm{i}t]_0^{2\pi} \\ &= 8\pi\mathrm{i} \end{aligned}$$

(since $\mathrm{e}^{2\pi\mathrm{i}} = \mathrm{e}^0 = 1$).
Thus the original integral is $-3 \times 8\pi\mathrm{i} = -24\pi\mathrm{i}$.

(b) The **Laurent series** for a function f which is analytic on an annulus $\{z \in \mathbb{C} \colon R_1 < |z - \alpha| < R_2\}$ is a series in positive and negative powers of $z - \alpha$,

$$\sum_{n \in \mathbb{Z}} a_n(z - \alpha)^n,$$

representing f in this annulus.
The formula for the coefficient a_n is

$$a_n = \frac{1}{2\pi \mathrm{i}} \int_\gamma (z-\alpha)^{-(n+1)} f(z)\ \mathrm{d}z$$

where γ is the circle $|z-\alpha| = \frac{1}{2}(R_1+R_2)$ (for example) described once counterclockwise.

(c) (i) For $|z|<1$, $|z^4|<1$ and we may expand

$$\begin{aligned}\frac{1}{1-z^4} &= 1 + z^4 + (z^4)^2 + \dots \\ &= 1 + z^4 + z^8 + \dots .\end{aligned}$$

This gives

$$\begin{aligned}\frac{1+2z^3}{1-z^4} &= (1+2z^3)(1+z^4+z^8+z^{12}+\dots) \\ &= 1 + 2z^3 + z^4 + 2z^7 + z^8 + \dots .\end{aligned}$$

(ii) For $|z|>1$, the above expansion is not valid. Instead write

$$\frac{1}{1-z^4} = \frac{1}{z^4\left(\frac{1}{z^4}-1\right)} = -\frac{1}{z^4}\left(\frac{1}{1-\frac{1}{z^4}}\right).$$

Since $|z|>1$, $\left|\dfrac{1}{z^4}\right| < 1$, and we have

$$\frac{1}{1-\frac{1}{z^4}} = 1 + \frac{1}{z^4} + \frac{1}{z^8} + \frac{1}{z^{12}} + \dots,$$

so

$$\frac{1}{1-z^4} = -\frac{1}{z^4} - \frac{1}{z^8} - \frac{1}{z^{12}} - \dots .$$

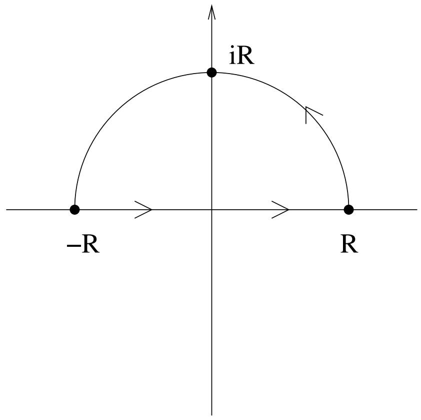
Finally, $\dfrac{1+2z^3}{1-z^4} =$

$$(1+2z^3)\left(-\frac{1}{z^4} - \frac{1}{z^8} - \frac{1}{z^{12}} - \dots\right)$$

$$= -\frac{2}{z} - \frac{1}{z^4} - \frac{2}{z^5} - \frac{1}{z^8} - \dots .$$

Finally, since γ lies in the annulus on which the second Laurent series is valid, the integral is just $2\pi\mathrm{i} \times$ the coefficient of $\dfrac{1}{z}$ in this latter series, i.e. $-4\pi\mathrm{i}$.

3 (a) The poles of the function $\dfrac{1}{(z^2+4)(z^2+9)}$ are at $\pm 2\mathrm{i}$ and $\pm 3\mathrm{i}$. Of these $2\mathrm{i}$ and $3\mathrm{i}$ lie inside the contour Γ_R and these are easily seen to be simple poles.



By the Cauchy residue theorem, the integral is $2\pi\mathrm{i}$ times $\big(\mathrm{Res}(f, 2\mathrm{i}) + \mathrm{Res}(f, 3\mathrm{i})\big)$ where $f(z) = \dfrac{1}{(z^2+4)(z^2+9)}$.

Writing $g(z) = \dfrac{1}{f(z)} = (z^2+4)(z^2+9)$, the residues at these (simple) poles are $\dfrac{1}{g'(2\mathrm{i})}$ and $\dfrac{1}{g'(3\mathrm{i})}$. (The fact that these are $\neq 0$ proves simplicity.)

$$\begin{aligned}
g'(z) &= 2z\big((z^2+4) + (z^2+9)\big) \\
&= 2z\ (2z^2+13) \\
g'(2\mathrm{i}) &= 4\mathrm{i}(5) = 20\mathrm{i} \\
g'(3\mathrm{i}) &= 6\mathrm{i}(-5) = -30\mathrm{i}.
\end{aligned}$$

Thus the integral is $(2\pi\mathrm{i})\left(\dfrac{1}{20\mathrm{i}} - \dfrac{1}{30\mathrm{i}}\right)$

$$\begin{aligned}
&= \pi\left(\frac{1}{10} - \frac{1}{15}\right) \\
&= \frac{\pi}{30}.
\end{aligned}$$

(b) We use the fundamental estimate. On the contour we have $|z| = R$, $|z^2| = R^2$, and so $|z^2 + 4| \geq R^2 - 4$, $|z^2 + 9| \geq R^2 - 9$.
Since $R \geq 10$, this gives

$$\left|\frac{1}{(z^2+4)(z^2+9)}\right| \leq \frac{1}{(R^2-4)(R^2-9)}.$$

The length of the contour is πR, so

$$\left|\int_{\gamma_R} \frac{1}{(z^2+4)(z^2+9)}\,\mathrm{d}z\right| \leq \frac{\pi R}{(R^2-4)(R^2-9)} \to 0 \quad \text{as } R \to \infty.$$

Thus, by the sandwich theorem,

$$\left|\int_{\gamma_R} \frac{1}{(z^2+4)(z^2+9)}\,\mathrm{d}z\right| \to 0 \quad \text{as } R \to \infty, \quad \text{i.e.}$$

$$\int_{\gamma_R} \frac{1}{(z^2+4)(z^2+9)}\,\mathrm{d}z \to 0 \quad \text{as } R \to \infty.$$

(c) Note that

$$\int_{\Gamma_R} \frac{1}{(z^2+4)(z^2+9)}\,\mathrm{d}z = \int_{-R}^{R} \frac{1}{(x^2+4)(x^2+9)}\,\mathrm{d}x + \int_{\gamma_R} \frac{1}{(z^2+4)(z^2+9)}\,\mathrm{d}z.$$

Thus

$$\int_{-R}^{R} \frac{1}{(x^2+4)(x^2+9)}\,\mathrm{d}x = \frac{\pi}{30} - \int_{\gamma_R} \frac{1}{(z^2+4)(z^2+9)}\,\mathrm{d}z \to \frac{\pi}{30} \quad \text{as } R \to \infty$$

(by (a) and (b)). This gives

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+4)(x^2+9)}\,\mathrm{d}x = \frac{\pi}{30}.$$

Aliter: $\displaystyle \frac{1}{(x^2+4)(x^2+9)} = \frac{1}{5}\left(\frac{1}{(x^2+4)} - \frac{1}{(x^2+9)}\right).$

But $\displaystyle \int_{-\infty}^{\infty} \frac{1}{x^2+4}\,\mathrm{d}x = \left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_{-\infty}^{\infty} = \frac{\pi}{2}$

and $\displaystyle \int_{-\infty}^{\infty} \frac{1}{x^2+9}\,\mathrm{d}x = \left[\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]_{-\infty}^{\infty} = \frac{\pi}{3}.$

Thus the integral is $\displaystyle \frac{1}{5}\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{\pi}{30}.$

4 (a) The equation $\mathrm{e}^z - 1 = 0$ has solutions $z = 2n\pi\mathrm{i}$ for $n \in \mathbb{Z}$. Thus the only singularity in the contour is at 0. This is a simple pole because $\mathrm{e}^0 = \frac{\mathrm{d}}{\mathrm{d}z}(\mathrm{e}^z - 1)\Big|_{z=0} \neq 0$, and the residue at 0 is $\frac{1}{\frac{\mathrm{d}}{\mathrm{d}z}(\mathrm{e}^z-1)|_{z=0}} = \frac{1}{e^0} = 1$. The integral is thus $2\pi\mathrm{i}\,\mathrm{Res}\left(\frac{1}{\mathrm{e}^z - 1}, 0\right) = 2\pi\mathrm{i}$.

[Or use series methods, writing $\mathrm{e}^z - 1 = z + \frac{z^2}{2} + \cdots = z\left(1 + \frac{z}{2} + \ldots\right)$ etc.]

(b) $\mathrm{e}^{2z} + 1 = 0$ has solutions where $2z = (2k+1)\pi\mathrm{i}$ for some $k \in \mathbb{Z}$. The only singularities inside the contour are at $z = \pm\frac{\pi}{2}\mathrm{i}$. Again these are simple poles:
$\frac{\mathrm{d}}{\mathrm{d}z}(\mathrm{e}^{2z} + 1) = 2\mathrm{e}^{2z}$,
so $\mathrm{Res}\left(\frac{1}{\mathrm{e}^{2z}+1}, \frac{\pi}{2}\mathrm{i}\right) = \frac{1}{2\mathrm{e}^{\pi\mathrm{i}}} = -\frac{1}{2}$
and $\mathrm{Res}\left(\frac{1}{\mathrm{e}^{2z}+1}, -\frac{\pi}{2}\mathrm{i}\right) = \frac{1}{2\mathrm{e}^{-\pi\mathrm{i}}} = -\frac{1}{2}$.
By the Cauchy residue theorem the integral is $2\pi\mathrm{i}(-\frac{1}{2} + -\frac{1}{2}) = -2\pi\mathrm{i}$.

(c) As in (a), the only pole inside the contour is at 0, but this time the pole is **not** simple. To find $\mathrm{Res}\left(\frac{1}{(\mathrm{e}^z-1)^2}, 0\right)$ we can use series methods:

$$\begin{aligned}
\mathrm{e}^z - 1 &= z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots \\
&= z\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \ldots\right) \\
&= z\,h(z), \quad \text{say.}
\end{aligned}$$

Thus, near 0, we may write

$$\begin{aligned}
\frac{1}{(\mathrm{e}^z-1)^2} &= \frac{1}{\big(z\,h(z)\big)^2} = \frac{1}{z^2}\frac{1}{h(z)^2} \\
&= \frac{1}{z^2}\frac{1}{\left(1 + \left(\frac{z}{2!} + \frac{z^2}{3!} + \ldots\right)\right)^2} \\
&= \frac{1}{z^2}\left(1 - \left(\frac{z}{2!} + \frac{z^2}{3!} + \ldots\right) + \ldots\right)^2.
\end{aligned}$$

We want the coefficient of $\frac{1}{z}$, which is simply -1, so the integral is $(2\pi\mathrm{i}) \times (-1) = -2\pi\mathrm{i}$.

5 (a) (i) Writing z in exponential form as $r\exp\big(\mathrm{i}\,\mathrm{Arg}(z)\big)$ (where Arg is the principal argument)

$$\begin{aligned}\mathsf{Log}\, z &= \ln r + \mathrm{i}\,\mathrm{Arg}(z)\\ &= \ln|z| + \mathrm{i}\,\mathrm{Arg}(z).\end{aligned}$$

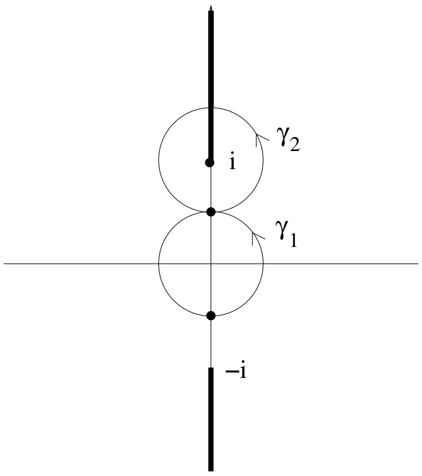
For $z \in D_0$ we have $\exp\big(\mathsf{Log}(z)\big) = z$ (but **not** $\mathsf{Log}\big(\exp(z)\big) = z$).

(ii) $z^a = \exp\big(a\,\mathrm{Log}(z)\big)$.

(iii)
$$\begin{aligned} z^{a+b} &= \exp\big((a+b)\mathrm{Log}(z)\big)\\ &= \exp\big(a\,\mathrm{Log}(z) + b\,\mathrm{Log}(z)\big)\\ &= \exp\big(a\,\mathrm{Log}(z)\big)\exp\big(b\,\mathrm{Log}(z)\big) \quad \text{(by properties of } \exp\text{)}\\ &= z^a z^b, \quad \text{as required.}\end{aligned}$$

(b) (i) $z^2+1 \in D_0$ **unless** $z^2 = -x$ for some $x \in [1,\infty)$. This happens if and only if $z = \mathrm{i}y$ for some $y \in \mathbb{R}$ with $|y| \geq 1$.
Thus $z^2+1 \in D_0$ for all $z \in \mathbb{C}$ except for $z = \mathrm{i}y$ for $y \in \mathbb{R}$, $|y| \geq 1$.

(ii)



indicates $f(z) \notin D_0$

Although the argument is not a sensible method, it **is** valid for γ_1, because $f(z) \in D_0$ for all z on γ_1, which makes the claim $\dfrac{\mathrm{d}}{\mathrm{d}z}\big(\mathrm{Log}\big(f(z)\big)\big) = \dfrac{f'(z)}{f(z)}$ valid there. The integral is thus 0.

The argument is false for γ_2: it breaks down because $f(\frac{3}{2}\mathrm{i}) \notin D_0$ and so $\mathrm{Log}\big(f(z)\big)$ is not complex differentiable on γ_2.

The correct answer is

$$2\pi\mathrm{i}\,\mathsf{Res}\left(\frac{2z}{z^2+1}, \mathrm{i}\right) = 2\pi\mathrm{i}\left(\frac{2\mathrm{i}}{\mathrm{i}+\mathrm{i}}\right) = 2\pi\mathrm{i}.$$