G1BCOF Complex Functions 2004-2005: Problem Classes and Coursework

This handout contains all the coursework and problem class questions for the module. The questions were originally compiled by Professor Langley for his module G12CAN Complex Analysis (2002-3). Dr Feinstein has made very minor alterations.

Note that there is a separate sheet of questions for the special examples class in teaching week 2.

Coursework will be due for handing in at the end of the Friday lecture in Weeks 2, 4, 6, 8, 10. It does not form part of the assessment, but should give you useful feedback, and its completion is strongly advised in order to master the techniques of the module. If you have any queries about the marking of your work you should see Dr Feinstein.

Problem Classes will be fortnightly (in teaching weeks 3,5,7,9 and 11) on Mondays, 17:00 Pope A13.

You are welcome to discuss the homework problems with Dr Feinstein, provided that you have thought about them in advance. Solutions to all questions will be available from the module web page as the module progresses.

Coursework 1

Please do Questions 4 and 5 for homework and hand in at the **end of the Friday lecture** in week 2. The remaining problems are for your own practice and revision.

1. Let γ be the contour which describes twice counter-clockwise the circle with centre $2i$ and radius 1. Write down a parametrization of γ and determine $\int_\gamma z^2 dz, \quad \int_\gamma \bar{z} dz$.

2. Solve $z^6 + 2iz^3 + 1 = 0$. Express the roots in exponential ($re^{i\theta}$) form.

3. Determine the region given by $\mathrm{Re}(z) > 0$, $|\mathrm{Im}(1+z+z^2)| < 3$.

4. Let γ be the piecewise smooth contour which describes the stepwise curve consisting of, in order, the line segments from $-3-3i$ to $3-3i$, and from $3-3i$ to $3+3i$. Determine $\int_\gamma 2z + \bar{z}\, dz$.

5. Let γ be the contour consisting of the straight line from 0 to 2 followed by the semi-circular arc of radius 1, centre 1, counter-clockwise from 2 back to 0. Write down a parametrization for each part of γ, and determine $\int_\gamma \bar{z}\, dz$.

Problem Class 1

These are for the problem class in week 3.

1. Define $u(x,y)$ by $u(0,0) = 0$ and, if x,y aren't both zero, $u(x,y) = x^3y/(x^6+y^2)$. Show that $u(x,y)$ tends to 0 as (x,y) tends to (0,0) along any straight line through the origin (hint: consider the lines $x = 0$ and $y = cx$). Next, find a curve tending to $(0,0)$ along which $u(x,y)$ does not tend to 0 (this shows that $\lim_{(x,y)\to(0,0)} u(x,y)$ does not exist).

2. Sketch the following sets, where possible, and identify which are open. Of those which are open, which are domains? Justify answers briefly but clearly.
(i) $\{ z \in \mathbb{C}: |z| < 1 \text{ and } |z-1| < 1 \}$ (ii) $\{ z \in \mathbb{C}: |z| < 1 \text{ or } |z-4i| < 1 \}$
(iii) $\{ z \in \mathbb{C}: \mathrm{Re}(z) \geqslant 0 \text{ and } |z+1| < 2 \}$ (iv) $\{ z : z = x+iy,\ x,y \in \mathbb{R},\ |x| > |y| \}$
(v) $\{ z \in \mathbb{C}: \mathrm{Re}(z^2) > 0 \}$ (vi) $\{ z \in \mathbb{C}: |z+i| > |z-i| \}$ (vii) $\{ z \in \mathbb{C}: |z| \text{ is rational} \}$.

An acceptable convention is a "dotted" curve for a part of the boundary not in the set, "full" curve for parts which are in the set.

3. Let $\gamma(t) = \exp(it), 0 \leqslant t \leqslant 2\pi$. Determine the contour integral $\int_\gamma f(z)\,dz$ when $f(z) = f(x+iy)$ (x, y real) is given by (a) $\bar{z}$ (b) x^2+y^2+ixy.

Coursework 2

Please hand in solutions at the end of the Friday lecture in week 4.

1. Complex analysis is a very geometric subject, and one of the many uses of complex functions is to provide mappings between regions.
(i) The complex numbers w and z are related by $w = (1+z)/(1-z)$, with $z \neq 1$. Show that $\mathrm{Re}(w) > 0$ if and only if $|z| < 1$.
(ii) Determine the image of the right half plane $U = \{z \in \mathbb{C}: \mathrm{Re}(z) > 0\}$ under the mapping $w = z^2$. Hint: what happens to the argument?
(iii) Find a function mapping the disc $\{z \in \mathbb{C}: |z| < 1\}$ one-one onto (i.e. a bijection) the sector $\{u \in \mathbb{C}: |\arg u| < \pi/4\}$.

2. Use the Cauchy-Riemann equations to determine all points where $f = u+iv$ is complex differentiable, when u,v are as follows (x, y real):
(i) $u(x,y) = x^4+y^2$, $v(x,y) = xy^2$; (ii) $u(x,y) = \exp(x^2)\cos y$, $v(x,y) = \exp(x^2)\sin y$.
N.B. some care is required in solving the resulting equations!

Problem Class 2

These are for the problem class in week 5. Try to solve questions 1-3. If you finish early you can try questions 4 and 5.

1. Suppose that f is complex differentiable at every point in a domain D, and that $f(z)$ is real for every z in D. Explain why f must be constant on D.

2. Is there a function f complex differentiable in all of $\mathbb{C}$ such that the real part of $f(x+iy)$, when x,y are real, is (i) x^3-3xy^2+4y ; (ii) e^{x+y} ?
Hint: assume such an f, write $f = u+iv$, and see what Cauchy-Riemann tells you about v.

3. You may have heard of Julia sets (or the related Mandelbrot set) from books or TV programmes. In this question, we determine the Julia set for the function $f(z) = z^2$. For a given complex number z_0, we define the *iterative sequence* (z_n) by $z_{n+1} = f(z_n)$. (a) Determine the behaviour of the sequence when $|z_0| < 1$. (b) Do the same for starting values with $|z_0| > 1$.
(c) Let T be the unit circle $\{ z : |z| = 1 \}$. If we start on T, i.e. with $z_0 = e^{i\theta}$ (θ real), deduce a formula for z_n. Show that if θ is of the form $2\pi p/2^q$, with p, q positive integers, then $z_n = 1$ for all sufficiently large n.
(d) The precise definition of the Julia set of f is complicated, but we can define it in a way that we can work with as follows: a point b is in the Julia set if b is *unstable* i.e. there are points a arbitrarily close to b such that the behaviour of the sequence for $z_0 = a$ is very different to that for $z_0 = b$. Can you see what the Julia set is in this case?

4. Let γ be the simple piecewise smooth contour consisting of the straight line segment from 0 to 1, followed by that from 1 to 2 + i . Determine the integral $\int_\gamma e^{2z} dz$.

5. Another problem on finding Julia sets. Using the formula $\cos z = (e^{iz}+e^{-iz})/2$, show that if z is a non-real complex number then $\lim_{n \to \infty, n \in \mathbb{Z}} |\cos nz| = \infty$. Next, if w is a complex number, show that the equation $\cos z = w$ has at least one solution in $\mathbb{C}$ (this involves a quadratic equation).

Now set $T(\zeta) = 2\zeta^2 - 1$. Verify that $T(\cos z) = \cos 2z$ and find a formula for $T_n(\cos z)$ (here T_n means the function T applied n times).

Now let w be any complex number, not lying in the real interval $I = [-1, 1] \subseteq \mathbb{R}$. By choosing z with $\cos z = w$, show that $|T_n(w)| \to \infty$ as $n \to \infty$. Finally, for w in the interval I, show that $T(w) \in I$, and decide what the Julia set of T is.

Coursework 3

Please hand in solutions at the end of the Friday lecture in week 6.

1. Let $\alpha \in \mathbb{C}$. The function z^α is defined on the domain D_0 obtained by deleting from $\mathbb{C}$ the non-positive real axis, by $z^\alpha = \exp(\alpha \operatorname{Log} z)$. (i) Find all z in D_0 such that $z^i = i$. (ii) Determine the limits of z^i as z tends to -1 (a) from above (b) from below the negative real axis. (iii) Comment on the veracity of the following statement: "as $z \to 0$ in D_0, we have $z^i \to 0$, because 0 to any non-zero power is 0". (iv) Taking $z = e^{2\pi}, a = b = i$, show that $(z^a)^b$ need not equal z^{ab}.

For the next question, you should use **Cauchy's integral formula (which will be discussed in lectures):** let f be analytic on and inside the simple closed piecewise smooth contour γ. Let $w \in \mathbb{C}$ with w not on γ. Then, integrating once counter-clockwise,

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-w}\, dz$$

is $f(w)$ if w lies inside γ, and is 0 if w lies outside γ.

2. Let γ be the circle $|z| = 2$, described once counter-clockwise. Use Cauchy's integral formula to determine $\int_\gamma f(z)\, dz$ when $f(z)$ is given by the following.
(a) $e^{3z}/(z-1)$ (b) $e^{3z}/(z^2-1)$ (c) $(\sin z)/(z^2-6z+5)$.

Hint: in at least one of these you need partial fractions.

Problem Class 3

These are for the problem class in week 7. Try to answer questions 1, 2 and 3. If you finish early you can try question 4.

1. Determine ALL complex numbers z for which the series $F(z)$ converges, when $F(z)$ is given by (i) $\sum_{k=1}^{\infty} k^i z^k$ (ii) $\sum_{k=0}^{\infty} (z/2)^k (k^2+1)^{-1}$.

2. Let γ be the circle $|z| = 1$, described once counter-clockwise. For each of the following functions, (i) determine the largest annulus $\{ z \in \mathbb{C} : R < |z| < S \}$ containing γ on which f is analytic (ii) determine a series representation for $f(z)$ in powers of z, valid on the annulus, and (iii) use the series to calculate the integral around γ of the function.
(a) $f(z) = 1/z(2-z)$ (b) $f(z) = 1/(2-z)(z-1/2)$.

3. (Revision) Where are these functions analytic: (i) $\cos(1/(z-2))$ (ii) $1/\sin z$ (iii) $z^{17} \exp(\exp(1/z))$? Remember that the quotient, product and chain rules are all true for analytic functions (in notes).

4. Statement: "The function $f(z) = \bar{z}$ is analytic on the circle $|z| = 1$ because for $|z| = 1$ we have $\bar{z} = 1/z$ and $g(z) = 1/z$ is analytic except at 0". Is this correct?

Coursework 4

Please hand in solutions at the end of the Friday lecture in week 8.

1. Let $f(z) = (1-z)^{-2} + (z-3)^{-1}$. Determine the Laurent series of f in each of the following annuli. (i) $0 < |z| < 1$ (ii) $1 < |z| < 3$ (iii) $2 < |z-1| < +\infty$ (iv) $0 < |z-3| < 2$.
Use your answers to compute the integral once counter-clockwise of the function f around the following circles. (a) $|z| = 2$ (b) $|z-1| = 4$ (c) $|z-3| = 1$.

2. Let f and g be analytic on a domain D, such that $g(z) = \overline{f(z)}$ for all z in D. Prove that f is constant.

For the remaining question sheets, you will sometimes need to use Cauchy's residue theorem

Cauchy's residue theorem

Let γ be a simple closed piecewise smooth contour (finishes where it starts but otherwise passes through no point twice). Let γ and its interior lie in the domain D.

Let f be analytic on D, apart possibly from finitely many isolated singularities α_j, none of which lie on γ. Then, integrating once counter-clockwise around γ, $\int_\gamma f(z)\,dz = 2\pi i \sum \operatorname{Res}(f, \alpha_j)$, in which the sum is taken over those α_j which lie **inside** γ. To find $\operatorname{Res}(f, \alpha_j)$, choose an annulus $0 < |z - \alpha_j| < \rho$ on which f is analytic, compute the Laurent series $f(z) = \sum_{k \in \mathbb{Z}} a_k (z - \alpha_j)^k$ which represents f there, and the residue is a_{-1}.

Problem Class 4

These are for the problem class in week 9. Try to answer questions 1-3. If you finish early you can try question 4.

1. Let $f(z) = \sin(\sin(z^2))$. Determine the Taylor series about the origin of $f(z)$, up to and including the term in z^{10}. Hence determine the integral once counter-clockwise around $|z| = 1$ of each of the following functions.
(i) $f(z)z^{-11}$ (ii) $f'(z)z^{-6}$ (iii) $f(z)^2 z^{-9}$ (iv) $f(z^4)z^{-25}$.

2. Determine $\int_{|z| = \pi} z^{-1}(1 - \cos z)^{-1}\,dz$, integrating once counter-clockwise.

3. Let Γ be the rectangle with vertices $100 \pm 25i, -100 \pm 25i$. Use Cauchy's residue theorem to determine the integral once counter-clockwise around Γ of the following functions, in each case identifying the singularities inside Γ and showing how you obtain the residues.
(a) $\dfrac{z+4}{z(z-4)}$ (b) $\dfrac{z+4}{z^2(z-4)}$ (c) $\dfrac{e^{\sin z}}{5-z}$ (d) $\dfrac{e^{\sin z}}{(5-z)^2}$ (e) $z^5\left(e^{1/z^2} - 1\right)$.

4. (optional) This question explains how to classify isolated singularities in terms of the Laurent series. We assume WLOG that the singularity is at 0.

Let f be analytic in the annulus $0 < |z| < \rho$, with Laurent series $f(z) = \sum_{k \in \mathbb{Z}} a_k z^k$ there.

(a) Show that we have a finite limit $\lim_{z \to 0} f(z)$ if and only if $a_k = 0$ for all $k < 0$. (Hint: "if" is easy; for "only if" use the formula for a_k).

(b) Show that if $a_k \neq 0$ for some, but finitely many, negative k then $|f(z)| \to +\infty$ as $z \to 0$. (Hint: the lowest power dominates).

(c) Conversely, show that if $|f(z)| \to +\infty$ as $z \to 0$ then $1/f$ has a removable singularity at 0, and by Taylor's theorem applied to $1/f$ we have $a_k \neq 0$ for some, but finitely many, negative k.

It now follows, of course, that 0 is an essential singularity if and only if $a_k \neq 0$ for infinitely many negative k.

Coursework 5

Please hand in solutions at the end of the Friday lecture in week 10.

1. (a) Suppose that $f(z)$ is analytic in $B(0, R)$ and that $f(z) = -f(-z)$ for all z there (i.e. f is an odd function). Show that $f^{(m)}(0) = 0$ for m even.

(b) Using the equation $dw/dz = 1 + w^2$, or otherwise, find the Taylor series of $\tan z$ about 0, up to and including the term in z^7. What is the largest disc of centre 0 in which you would expect this series to represent $\tan z$?

(c) Determine the integral once counter-clockwise around $|z| = \pi/4$ of (i) $z^{-6} \tan z$ and (ii) $z^{-7} \sec^2 z$.

2. Determine the location and nature of the singularities of $f(z) = (\operatorname{Log} z)/(z^2 - 3z + 2)$ in the domain D_0 ($\mathbb{C}$ with the non-positive real axis deleted). Here a pole at α means $|f(z)| \to +\infty$ as $z \to \alpha$, a removable singularity means that $f(z)$ tends to a finite limit as $z \to \alpha$, and an essential singularity is when there is no limit at all as $z \to \alpha$.

3. Determine the integral of $\sec z$ once counter-clockwise around the ellipse given by $z = x + iy$, $(x/\pi)^2 + y^2 = 1$.

Problem Class 5

These are for the problem class in week 11. You may wish to use Cauchy's residue theorem (stated on page 6).

1. (a) Identify the singularities of the function $f(z) = \dfrac{e^{iz}}{z^2+4z+5}$ and determine the residue of f at each singularity.

(b) Let Γ_R be the semi-circular contour consisting of the straight line from $-R$ to $+R$ and the arc γ_R given by $z = Re^{i\theta}$, $0 \leqslant \theta \leqslant \pi$, where R is a large positive number. Evaluate $\int_{\Gamma_R} f(z)\,dz$ using residues. Show that $|e^{iz}| \leqslant 1$ on γ_R and that $\int_{\gamma_R} f(z)\,dz$ tends to 0 as $R \to +\infty$. Hence determine the value of $\int_{-\infty}^{+\infty} \dfrac{\cos x}{x^2+4x+5}\,dx$. Justify all steps carefully.

2. Use residues to determine $\int_{-\infty}^{+\infty} \dfrac{x^2+1}{(x^2+4)^2}\,dx$.

3. By writing $z = e^{i\theta}$, determine $\int_0^{2\pi} \dfrac{1}{2+\cos\theta}\,d\theta$. Hint: $\cos\theta = (e^{i\theta}+e^{-i\theta})/2$.

A further practice problem, similar to Qns. 1 and 2, may be found at **http://www.maths.nott.ac.uk/personal/jkl/canex1.pdf**