

The chain rule for \mathcal{F} -differentiation

T. CHAOBANKOH, J. F. FEINSTEIN AND S. MORLEY

ABSTRACT. Let X be a perfect, compact subset of the complex plane, and let $D^{(1)}(X)$ denote the (complex) algebra of continuously complex-differentiable functions on X . Then $D^{(1)}(X)$ is a normed algebra of functions but, in some cases, fails to be a Banach function algebra. Bland and the second author ([3]) investigated the completion of the algebra $D^{(1)}(X)$, for certain sets X and collections \mathcal{F} of paths in X , by considering \mathcal{F} -differentiable functions on X .

In this paper, we investigate composition, the chain rule, and the quotient rule for this notion of differentiability. We give an example where the chain rule fails, and give a number of sufficient conditions for the chain rule to hold. Where the chain rule holds, we observe that the Faà di Bruno formula for higher derivatives is valid, and this allows us to give some results on homomorphisms between certain algebras of \mathcal{F} -differentiable functions.

Throughout this paper, we use the term *compact plane set* to mean a non-empty, compact subset of the complex plane, \mathbb{C} . We denote the set of all positive integers by \mathbb{N} and the set of all non-negative integers by \mathbb{N}_0 . Let X be a compact Hausdorff space. We denote the algebra of all continuous, complex-valued functions on X by $C(X)$ and we give $C(X)$ the uniform norm $|\cdot|_X$, defined by

$$|f|_X = \sup_{x \in X} |f(x)| \quad (f \in C(X)).$$

This makes $C(X)$ into a commutative, unital Banach algebra. A subset S of $C(X)$ *separates the points of X* if, for each $x, y \in X$ with $x \neq y$, there exists $f \in S$ such that $f(x) \neq f(y)$. A *normed function algebra* on X is a normed algebra $(A, \|\cdot\|)$ such that A is a subalgebra of $C(X)$, A contains all constant functions and separates the points of X , and, for each $f \in A$, $\|f\| \geq |f|_X$. A *Banach function algebra* on X is a normed function algebra on X which is complete. We say that such a Banach function algebra A is *natural* (on X) if every character on A is given by evaluation at some point of X . We refer the reader to [5] (especially Chapter 4) for further information on Banach algebras and Banach function algebras.

Let $D^{(1)}(X)$ denote the normed algebra of all continuously (complex) differentiable, complex-valued functions on X , as discussed in [6] and [7]. Furthermore, let $D^{(n)}(X)$ denote the normed algebra of all continuously n -times (complex) differentiable, complex-valued functions on X , and let $D^{(\infty)}(X)$ denote the algebra of continuous functions which have continuous (complex) derivatives of all orders.

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Dales and Davie ([6]) also introduced the algebras

$$D(X, M) := \left\{ f \in D^{(\infty)}(X) : \sum_{j=0}^{\infty} \frac{|f^{(j)}|_X}{M_j} < \infty \right\},$$

where $M = (M_n)_{n=0}^{\infty}$ is a suitable sequence of positive real numbers. The algebras $D(X, M)$ are called *Dales-Davie* algebras.

The usual norms on the algebras $D^{(n)}(X)$ ($n \in \mathbb{N}$) and $D(X, M)$ above need not be complete, so we often investigate the completion of these algebras. One approach to this was introduced by Bland and Feinstein [3], where they discussed algebras of \mathcal{F} -differentiable functions (see Section 2), and these algebras were investigated further in [7] and [12].

Kamowitz and Feinstein investigated the conditions under which composition with an infinitely differentiable map induces an endomorphism ([13, 9, 10]) or a homomorphism ([11]) between Dales-Davie algebras.

In this paper, we investigate composition, the chain rule, and the quotient rule for \mathcal{F} -differentiation. We give an example where the chain rule for \mathcal{F} -differentiation fails, and give a number of sufficient conditions for the chain rule to hold. We also prove a version of the quotient rule for \mathcal{F} -differentiable functions.

Where the chain rule holds, we observe that the Faà di Bruno formula for higher derivatives is valid, and this allows us to give some sufficient conditions, similar to those in [11], for composition with an infinitely \mathcal{F} -differentiable function to induce a homomorphism between the \mathcal{F} -differentiability versions of Dales-Davie algebras.

1. PATHS IN THE COMPLEX PLANE

We begin with a discussion of collections of paths in the complex plane.

Definition 1.1. A *path* in \mathbb{C} is a continuous function $\gamma : [a, b] \rightarrow \mathbb{C}$, where $a < b$ are real numbers. Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a path. The *parameter interval* of γ is the interval $[a, b]$. The *endpoints* of γ are the points $\gamma(a)$ and $\gamma(b)$, which we denote by γ^- and γ^+ , respectively. We denote by γ^* the *image* $\gamma([a, b])$ of γ . A *subpath* of γ is a path obtained by restricting γ to a non-degenerate, closed subinterval of $[a, b]$. If X is a subset of \mathbb{C} then we say that γ is a *path in X* if $\gamma^* \subseteq X$.

Let γ be a path in \mathbb{C} . We say that γ is a *Jordan path* if γ is an injective function.

Let $[a, b]$ be a non-degenerate closed interval. A *partition* \mathcal{P} of $[a, b]$ is a finite set $\{x_0, \dots, x_n\} \subseteq [a, b]$ such that $x_0 = a$, $x_n = b$ and $x_j < x_{j+1}$ for each $j \in \{0, 1, \dots, n-1\}$. If \mathcal{P} and \mathcal{P}' are partitions of $[a, b]$ then we say that \mathcal{P}' is *finer* than \mathcal{P} if $\mathcal{P} \subseteq \mathcal{P}'$.

Definition 1.2. Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a path and let $c, d \in [a, b]$ with $c < d$. The *total variation* of γ over $[c, d]$ is

$$V_c^d(\gamma) := \sup \left\{ \sum_{j=0}^{n-1} |\gamma(x_{j+1}) - \gamma(x_j)| : \mathcal{P} = \{x_0, \dots, x_n\} \right\}$$

where the supremum is taken over all partitions \mathcal{P} of $[c, d]$. We say that γ is *rectifiable* if $V_a^b(\gamma) < \infty$, in which case we set $\Lambda(\gamma) := V_a^b(\gamma)$; otherwise it is *non-rectifiable*. The *length* of a rectifiable path γ is $\Lambda(\gamma)$.

For a detailed discussion of paths, total variation and path length, see [2, Chapter 6].

We say that a path γ is *admissible* if γ is rectifiable and contains no constant subpaths. Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a non-constant (but not necessarily admissible) rectifiable path. We define the *path length parametrisation* $\gamma^{\text{pl}} : [0, \Lambda(\gamma)] \rightarrow \mathbb{C}$ of γ to be the unique path satisfying $\gamma^{\text{pl}}(V_a^t(\gamma)) = \gamma(t)$ ($t \in [a, b]$); see, for example, [8, pp. 109-110] for details. We define the *normalised path length parametrisation* $\gamma^{\text{no}} : [0, 1] \rightarrow \mathbb{C}$ of γ to be the path such that $\gamma^{\text{no}}(t) = \gamma^{\text{pl}}(t\Lambda(\gamma))$ for each $t \in [0, 1]$. It is clear that γ^{pl} and γ^{no} are necessarily admissible paths and $(\gamma^{\text{pl}})^* = (\gamma^{\text{no}})^* = \gamma^*$. It is not hard to show, using [8, Theorem 2.4.18], that

$$\int_{\gamma} f(z) dz = \int_{\gamma^{\text{pl}}} f(z) dz = \int_{\gamma^{\text{no}}} f(z) dz,$$

for all $f \in C(\gamma^*)$. We shall use this fact implicitly throughout.

Definition 1.3. Let X be a compact plane set and let \mathcal{F} be a collection of paths in X . We define $\mathcal{F}^* := \{\gamma^* : \gamma \in \mathcal{F}\}$. We say that \mathcal{F} is *effective* if $\bigcup \mathcal{F}^* = X$, each path in \mathcal{F} is admissible, and every subpath of a path in \mathcal{F} belongs to \mathcal{F} . We denote by \mathcal{F}^{no} the collection $\{\gamma^{\text{no}} : \gamma \in \mathcal{F}\}$.

Let X be a compact plane set and let \mathcal{F} be a collection of paths in X . It is clear that $\mathcal{F}^* = (\mathcal{F}^{\text{no}})^*$.

We introduce the following definitions from [3, 6] and [7].

Definition 1.4. Let X be a compact plane set. We say that X is *uniformly regular* if there exists a constant $C > 0$ such that, for all $x, y \in X$, there exists a rectifiable path γ in X with $\gamma^- = x$ and $\gamma^+ = y$ such that $\Lambda(\gamma) \leq C|x - y|$. We say that X is *pointwise regular*, if for each $x \in X$, there exists a constant $C_x > 0$ such that, for each $y \in X$, there exists a path γ in X with $\gamma^- = x$ and $\gamma^+ = y$ such that $\Lambda(\gamma) \leq C_x|x - y|$. We say that X is *semi-rectifiable* if the union of the images of all rectifiable, Jordan paths in X is dense in X .

We also require the following definition from [7].

Definition 1.5. Let X be a compact plane set and let \mathcal{F} be an effective collection of paths in X . We say that X is *\mathcal{F} -regular at $x \in X$* if there exists a constant $C_x > 0$ such that for each $y \in X$ there exists $\gamma \in \mathcal{F}$ with $\gamma^- = x$, $\gamma^+ = y$ and $\Lambda(\gamma) \leq C_x|x - y|$. We say that X is *\mathcal{F} -regular* if X is \mathcal{F} -regular at each point $x \in X$.

We note that if X is a compact plane set which is \mathcal{F} -regular, for some effective collection \mathcal{F} of paths in X , then X is pointwise regular.

2. ALGEBRAS OF \mathcal{F} -DIFFERENTIABLE FUNCTIONS

In this section we discuss algebras of \mathcal{F} -differentiable functions as investigated in [3] and [7], along with algebras of \mathcal{F} -differentiable functions analogous to the Dales-Davie algebras introduced in [6].

Definition 2.1. Let X be a perfect compact plane set, let \mathcal{F} be a collection of rectifiable paths in X , and let $f \in C(X)$. A function $g \in C(X)$ is an *\mathcal{F} -derivative of f* if, for each $\gamma \in \mathcal{F}$, we have

$$\int_{\gamma} g(z) dz = f(\gamma^+) - f(\gamma^-).$$

If f has an \mathcal{F} -derivative on X then we say that f is \mathcal{F} -differentiable on X .

The following proposition summarises several properties of \mathcal{F} -derivatives and \mathcal{F} -differentiable functions on certain compact plane sets. Details can be found in [3] and [7].

Proposition 2.2. *Let X be a semi-rectifiable compact plane set and let \mathcal{F} be an effective collection of paths in X .*

- (a) *Let $f, g, h \in C(X)$ be such that g and h are \mathcal{F} -derivatives for f . Then $g = h$.*
- (b) *Let $f \in D^{(1)}(X)$. Then the usual complex derivative of f on X , f' , is an \mathcal{F} -derivative for f .*
- (c) *Let $f_1, f_2, g_1, g_2 \in C(X)$ be such that g_1 is an \mathcal{F} -derivative for f_1 and g_2 is an \mathcal{F} -derivative for f_2 . Then $f_1g_2 + g_1f_2$ is an \mathcal{F} -derivative for f_1f_2 .*
- (d) *Let $f_1, f_2, g_1, g_2 \in C(X)$ and $\alpha, \beta \in \mathbb{C}$ be such that g_1 is an \mathcal{F} -derivative for f_1 and g_2 is an \mathcal{F} -derivative for f_2 . Then $\alpha g_1 + \beta g_2$ is an \mathcal{F} -derivative for $\alpha f_1 + \beta f_2$.*

Let X be a semi-rectifiable compact plane set, and let \mathcal{F} be an effective collection of paths in X . In this setting we write $f^{[1]}$ for the unique \mathcal{F} -derivative of an \mathcal{F} -differentiable function and we will often write $f^{[0]}$ for f . We write $D_{\mathcal{F}}^{(1)}(X)$ for the algebra of all \mathcal{F} -differentiable functions on X . We note that, with the norm $\|f\|_{\mathcal{F},1} := |f|_X + |f^{[1]}|_X$ ($f \in D_{\mathcal{F}}^{(1)}(X)$), the algebra $D_{\mathcal{F}}^{(1)}(X)$ is a Banach function algebra on X ([7, Theorem 5.6]).

For each $n \in \mathbb{N}$, we define (inductively) the algebra

$$D_{\mathcal{F}}^{(n)}(X) := \{f \in D_{\mathcal{F}}^{(1)}(X) : f^{[1]} \in D_{\mathcal{F}}^{(n-1)}(X)\},$$

and, for each $f \in D_{\mathcal{F}}^{(n)}(X)$, we write $f^{[n]}$ for the n th \mathcal{F} -derivative of f . We note that, for each $n \in \mathbb{N}$, $D_{\mathcal{F}}^{(n)}(X)$ is a Banach function algebra on X (see [3]) when given the norm

$$\|f\|_{\mathcal{F},n} := \sum_{k=0}^n \frac{|f^{[k]}|_X}{k!} \quad (f \in D_{\mathcal{F}}^{(n)}(X)).$$

In addition, we define the algebra $D_{\mathcal{F}}^{(\infty)}(X)$ of all functions which have \mathcal{F} -derivatives of all orders; that is, $D_{\mathcal{F}}^{(\infty)}(X) = \bigcap_{n=1}^{\infty} D_{\mathcal{F}}^{(n)}(X)$. It is easy to see that, for each $n \in \mathbb{N}$, we have $D^{(n)}(X) \subseteq D_{\mathcal{F}}^{(n)}(X)$ and $D^{(\infty)}(X) \subseteq D_{\mathcal{F}}^{(\infty)}(X)$.

3. MAXIMAL COLLECTIONS AND COMPATIBILITY

We aim to prove a chain rule for \mathcal{F} -differentiable functions, but first we must investigate collections of paths further. Throughout this section, let X be a semi-rectifiable, compact plane set, and let \mathcal{A} be the collection of all admissible paths in X . In this section, we identify $D_{\mathcal{F}}^{(1)}(X)$ with the subset $S_{\mathcal{F}}$ of $C(X) \times C(X)$ consisting of all pairs (f, g) where $f \in D_{\mathcal{F}}^{(1)}(X)$ and g is the \mathcal{F} -derivative of f . We begin with a definition.

Definition 3.1. Let γ be an admissible path in X . Let $f, g \in C(X)$. We say that g is the γ -derivative of f if, for each subpath σ of γ , we have

$$\int_{\sigma} g(z) dz = f(\sigma^+) - f(\sigma^-).$$

Note that, in the above, if \mathcal{G} denotes the collection of all subpaths of γ , then \mathcal{G} is effective in γ^* , so \mathcal{G} -derivatives on γ^* are unique. Thus, if f has a γ derivative g on X , then $g|_{\gamma^*}$ is uniquely determined.

Definition 3.2. Let $S \subseteq C(X) \times C(X)$. We define

$$\mathbf{p}(S) := \{\gamma \in \mathcal{A} : \text{for all } (f, g) \in S, g \text{ is the } \gamma\text{-derivative of } f\}.$$

Let \mathcal{F} be an effective collection of paths in X . Then we write $\mathbf{m}(\mathcal{F})$ for $\mathbf{p}(S_{\mathcal{F}})$, where $S_{\mathcal{F}} = \{(f, f^{[1]}) : f \in D_{\mathcal{F}}^{(1)}(X)\}$ as above.

The following lemma follows quickly from the definition of $\mathbf{m}(\mathcal{F})$. We omit the details.

Lemma 3.3. *Let $S, T \subseteq C(X) \times C(X)$. If $S \subseteq T$ then we have $\mathbf{p}(T) \subseteq \mathbf{p}(S)$. Let \mathcal{F}, \mathcal{G} be effective collections of paths in X . Then we have $D_{\mathbf{m}(\mathcal{F})}^{(1)}(X) = D_{\mathcal{F}}^{(1)}(X)$. Moreover, $S_{\mathcal{F}} \subseteq S_{\mathcal{G}}$ if and only if $\mathbf{m}(\mathcal{G}) \subseteq \mathbf{m}(\mathcal{F})$.*

Note that $S_{\mathcal{F}} \subseteq S_{\mathcal{G}}$ implies that $D_{\mathcal{F}}^{(1)}(X) \subseteq D_{\mathcal{G}}^{(1)}(X)$.

We now investigate some elementary operations on $\mathbf{m}(\mathcal{F})$. Let $\gamma_1 : [a, b] \rightarrow \mathbb{C}$, $\gamma_2 : [c, d] \rightarrow \mathbb{C}$ be paths such that $\gamma_1^+ = \gamma_2^-$. We write $\gamma_1 \dot{+} \gamma_2$ for the path given by

$$(\gamma_1 \dot{+} \gamma_2)(t) = \begin{cases} \gamma_1(a + 2t(b - a)), & t \in [0, 1/2), \\ \gamma_2(c + (2t - 1)(d - c)), & t \in [1/2, 1]. \end{cases}$$

We call the path $\gamma_1 \dot{+} \gamma_2$ the *join* of γ_1 and γ_2 . Note that, if γ_1 and γ_2 are admissible, then $\gamma_1 \dot{+} \gamma_2$ is admissible. The *reverse* of γ_1 , denoted by $-\gamma_1$, is given by $-\gamma_1(t) = \gamma_1(b - t(b - a))$ ($t \in [0, 1]$). Our notation for joining and reversing is not entirely standard and there are many ways to parametrise these paths.

Lemma 3.4. *Let \mathcal{F} be an effective collection of paths in X . Then $\mathbf{m}(\mathcal{F})$ has the following properties:*

- (a) *if $\gamma \in \mathbf{m}(\mathcal{F})$ then $-\gamma \in \mathbf{m}(\mathcal{F})$;*
- (b) *if $\gamma \in \mathbf{m}(\mathcal{F})$ then $\gamma^{\text{pl}}, \gamma^{\text{no}} \in \mathbf{m}(\mathcal{F})$;*
- (c) *if $\gamma_1, \gamma_2 \in \mathbf{m}(\mathcal{F})$ such that $\gamma_1^+ = \gamma_2^-$ then $\gamma_1 \dot{+} \gamma_2 \in \mathbf{m}(\mathcal{F})$.*

Proof. (a) This is clear from the definitions.

(b) This is clear from the definitions, and the discussions in Section 1.

(c) This is effectively [3, Theorem 4.5], and follows from the definitions. \square

We also make the following observation about collections of paths generated by a set in $C(X) \times C(X)$ and its closure in the norm given by $\|(f, g)\|_1 = |f|_X + |g|_X$ for each $(f, g) \in C(X) \times C(X)$.

Lemma 3.5. *Let $S \subseteq C(X) \times C(X)$. Then $\mathbf{p}(\overline{S}) = \mathbf{p}(S)$ where the closure of S is taken in the norm $\|\cdot\|_1$ on $C(X) \times C(X)$ as above.*

Proof. By Lemma 3.3, we have $\mathbf{p}(\overline{S}) \subseteq \mathbf{p}(S)$. Let (f_n, g_n) be a sequence in S such that $(f_n, g_n) \rightarrow (f, g) \in \overline{S}$ as $n \rightarrow \infty$. Let $\gamma \in \mathbf{p}(S)$. Then we have

$$f_n(\gamma^+) - f_n(\gamma^-) = \int_{\gamma} g_n(z) dz,$$

for all $n \in \mathbb{N}$. But now $f_n \rightarrow f$ and $g_n \rightarrow g$ uniformly as $n \rightarrow \infty$, so

$$f(\gamma^+) - f(\gamma^-) = \int_{\gamma} g(z) dz.$$

Thus $\gamma \in \mathfrak{p}(\overline{S})$. It follows that $\mathfrak{p}(S) \subseteq \mathfrak{p}(\overline{S})$ and this completes the proof. \square

We also require the following elementary lemma, which is a minor variant of a standard result. We include a proof for the convenience of the reader.

Lemma 3.6. *Let \mathcal{F} be an effective collection of paths in X . Let $\varphi \in D_{\mathcal{F}}^{(1)}(X)$ and let $\gamma : [a, b] \rightarrow X \in \mathcal{F}$. Then $\varphi \circ \gamma$ is rectifiable, and hence, if $\varphi \circ \gamma$ is non-constant, $(\varphi \circ \gamma)^{\text{pl}}$ is admissible.*

Proof. Let $\mathcal{P} = \{x_0, \dots, x_n\}$ be a partition of $[a, b]$. Set $\sigma := \varphi \circ \gamma$ and let γ_j be the subpath of γ obtained by restricting γ to $[x_j, x_{j+1}]$ for each $j \in \{0, \dots, n-1\}$. We have

$$V(\sigma, \mathcal{P}) := \sum_{j=0}^{n-1} |\sigma(x_{j+1}) - \sigma(x_j)| = \sum_{j=0}^{n-1} \left| \int_{\gamma_j} \varphi^{[1]}(z) dz \right| \leq |\varphi^{[1]}|_X \Lambda(\gamma).$$

It follows that $\Lambda(\sigma) = \sup V(\sigma, \mathcal{P}) \leq |\varphi^{[1]}|_X \Lambda(\gamma)$, where the supremum is taken over all partitions \mathcal{P} of $[a, b]$. As noted earlier, if σ is non-constant then σ^{pl} is admissible. This completes the proof. \square

We now introduce our notion of compatibility.

Definition 3.7. Let Y be a semi-rectifiable compact plane set, let \mathcal{F} be an effective collection of paths in X and let \mathcal{G} be an effective collection of paths in Y . Let $\varphi \in D_{\mathcal{F}}^{(1)}(X)$ such that $\varphi(X) \subseteq Y$. We say that φ is \mathcal{F} - \mathcal{G} -compatible if, for each $\gamma \in \mathcal{F}$, either $\varphi \circ \gamma$ is constant or we have $(\varphi \circ \gamma)^{\text{pl}} \in \mathfrak{m}(\mathcal{G})$.

Let X, Y be semi-rectifiable compact plane sets, let \mathcal{F} be an effective collection of paths in X , and \mathcal{G} be an effective collection of paths in Y . If $\mathfrak{m}(\mathcal{G})$ is the collection of all admissible paths in Y then, for any $\varphi \in D_{\mathcal{F}}^{(1)}(X)$ with $\varphi(X) \subseteq Y$, φ is \mathcal{F} - \mathcal{G} -compatible.

Example 3.8. Let $X := \{x + iy \in \mathbb{C} : x, y \in [0, 1]\}$. Let \mathcal{F} be the collection of all line segment paths in X parallel to the real axis and let \mathcal{G} be the collection of all line segment paths in X parallel to the imaginary axis. Set $\varphi(z) := z$ ($z \in X$). Then $\varphi : X \rightarrow X$ is continuously differentiable on X and $\varphi(X) = X$. Clearly $\varphi \circ \gamma \in \mathcal{F}$ for all $\gamma \in \mathcal{F}$. It is not hard to show that, if $\gamma \in \mathfrak{m}(\mathcal{F})$, then $\gamma^* \in \mathcal{F}^*$ and, if $\gamma \in \mathfrak{m}(\mathcal{G})$, then $\gamma^* \in \mathcal{G}^*$. Thus $\mathfrak{m}(\mathcal{F}) \cap \mathfrak{m}(\mathcal{G}) = \emptyset$ and it follows that φ is not \mathcal{F} - \mathcal{G} -compatible.

Let X, Y be semi-rectifiable compact plane sets, let \mathcal{F} be an effective collection of paths in X , and let \mathcal{G} be an effective collection of paths in Y . If $D^{(1)}(Y)$ is dense in $D_{\mathcal{G}}^{(1)}(Y)$ then, by Lemma 3.5, the collection $\mathfrak{m}(\mathcal{G})$ is the collection of all admissible paths in Y . So, by the comments following Definition 3.7, any function $\varphi \in D_{\mathcal{F}}^{(1)}(X)$ such that $\varphi(X) \subseteq Y$ is automatically \mathcal{F} - \mathcal{G} -compatible.

4. COMPOSITION OF \mathcal{F} -DIFFERENTIABLE FUNCTIONS

We now discuss an analogue of the chain rule for \mathcal{F} -differentiable functions. The following lemma is an \mathcal{F} -differentiability version of the usual change of variable formula.

Lemma 4.1. *Let X be a semi-rectifiable, compact plane set and let \mathcal{F} be an effective collection of paths in X . Let $\varphi \in D_{\mathcal{F}}^{(1)}(X)$ and let $\gamma : [a, b] \rightarrow X \in \mathcal{F}$. Then, for each $f \in C(\varphi(\gamma^*))$, we have*

$$(1) \quad \int_{\gamma} (f \circ \varphi) \varphi^{[1]}(z) \, dz = \int_{\varphi \circ \gamma} f(z) \, dz.$$

Proof. By Lemma 3.6, $\sigma := \varphi \circ \gamma$ is a rectifiable path so that the integral on the right-hand side of (1) exists. Fix $f \in C(\sigma^*)$ and let $\varepsilon > 0$. Set $M := |\varphi^{[1]}|_X \Lambda(\gamma)$ and let $h := f \circ \sigma : [a, b] \rightarrow \mathbb{C}$. Since h is uniformly continuous, there exists $\delta > 0$ such that, for each $s, t \in [a, b]$ with $|s - t| < \delta$, we have $|h(t) - h(s)| < \varepsilon/(2M)$. Choose a partition $\mathcal{P}_0 = \{t_0^{(0)}, \dots, t_m^{(0)}\}$ of $[a, b]$ such that

$$\max_{0 \leq j \leq m-1} |t_{j+1} - t_j| < \delta.$$

For any partition $\mathcal{P} = \{t_0, \dots, t_n\}$ finer than \mathcal{P}_0 and, for each $j \in \{0, \dots, n-1\}$, let $\gamma_j^{(\mathcal{P})}$ be the restriction of γ to $[t_j, t_{j+1}]$. We have

$$(2) \quad T(\mathcal{P}) := \left| \sum_{j=0}^{n-1} \int_{\gamma_j^{(\mathcal{P})}} (f(\varphi(z)) - h(s_j)) \varphi^{[1]}(z) \, dz \right| < \frac{\varepsilon}{2},$$

for any $s_j \in [t_j, t_{j+1}]$ ($j = 0, 1, \dots, n-1$).

Now fix a partition $\mathcal{P} = \{t_0, \dots, t_n\}$ of $[a, b]$ finer than \mathcal{P}_0 such that, viewing the integral in the right-hand side of (1) as a Riemann-Stieltjes integral on $[a, b]$, we have

$$(3) \quad \left| \sum_{j=0}^{n-1} h(s_j) (\sigma(t_{j+1}) - \sigma(t_j)) - \int_{\sigma} f(z) \, dz \right| < \frac{\varepsilon}{2},$$

for any choice of $s_j \in [t_j, t_{j+1}]$ ($j = 0, 1, \dots, n-1$).

We now *claim* that, for this partition \mathcal{P} , we have

$$(4) \quad \left| \int_{\gamma} f(\varphi(z)) \varphi^{[1]}(z) \, dz - \sum_{j=0}^{n-1} h(s_j) (\sigma(t_{j+1}) - \sigma(t_j)) \right| < \frac{\varepsilon}{2}$$

for any choice of $s_j \in [t_j, t_{j+1}]$ ($j = 0, 1, \dots, n-1$).

For the remainder of the proof, for each $j \in \{0, \dots, n-1\}$, fix $s_j \in [t_j, t_{j+1}]$ and let $S := \sum_{j=0}^{n-1} h(s_j) (\sigma(t_{j+1}) - \sigma(t_j))$.

By the definition of $\varphi^{[1]}$, we have

$$\sum_{j=0}^{m-1} h(s_j) \int_{\gamma_j^{(\mathcal{P})}} \varphi^{[1]}(z) \, dz = S.$$

We also have

$$\left| \int_{\gamma} f(\varphi(z)) \varphi^{[1]}(z) \, dz - \sum_{j=0}^{n-1} h(s_j) \int_{\gamma_j^{(\mathcal{P})}} \varphi^{[1]}(z) \, dz \right| = T(\mathcal{P}),$$

where $T(\mathcal{P}) < \varepsilon/2$ as in (2).

But now, by (2) and (3), we have

$$\left| \int_{\gamma} f(\varphi(z)) \varphi^{[1]}(z) \, dz - \int_{\sigma} f(w) \, dw \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This holds for all $\varepsilon > 0$ and any choice of the s_j , so the result follows. \square

We can now state and prove a version of the chain rule for \mathcal{F} -differentiable functions.

Theorem 4.2. *Let X, Y be semi-rectifiable, compact plane sets, let \mathcal{F} be an effective collection of paths in X , and let \mathcal{G} be an effective collection of paths on Y . Let $\varphi \in D_{\mathcal{F}}^{(1)}(X)$ with $\varphi(X) \subseteq Y$. Suppose that φ is \mathcal{F} - \mathcal{G} -compatible. Then, for all $f \in D_{\mathcal{G}}^{(1)}(Y)$, $f \circ \varphi$ is \mathcal{F} -differentiable and $(f \circ \varphi)^{[1]} = (f^{[1]} \circ \varphi)\varphi^{[1]}$.*

Proof. Fix $f \in D_{\mathcal{G}}^{(1)}(Y)$ and $\gamma \in \mathcal{F}$. Then, by Lemma 4.1, we have

$$\int_{\gamma} (f^{[1]} \circ \varphi)(z)\varphi^{[1]}(z) dz = \int_{\varphi \circ \gamma} f^{[1]}(z) dz.$$

Since φ is \mathcal{F} - \mathcal{G} -compatible, we have $\varphi \circ \gamma \in \mathfrak{m}(\mathcal{G})$ and so

$$\int_{\varphi \circ \gamma} f^{[1]}(z) dz = f((\varphi \circ \gamma)^+) - f((\varphi \circ \gamma)^-).$$

But $(\varphi \circ \gamma)^+ = \varphi(\gamma^+)$ and $(\varphi \circ \gamma)^- = \varphi(\gamma^-)$. Thus $f \circ \varphi$ is \mathcal{F} -differentiable and has \mathcal{F} -derivative $(f^{[1]} \circ \varphi)\varphi^{[1]}$. This completes the proof. \square

As a corollary we obtain the quotient rule for \mathcal{F} -differentiable functions. This was originally proved by means of repeated bisection in [4].

Corollary 4.3. *Let X be a semi-rectifiable compact plane set, let \mathcal{F} be an effective collection of paths in X , and let $f, g \in D_{\mathcal{F}}^{(1)}(X)$ such that $0 \notin g(X)$. Then we have $f/g \in D_{\mathcal{F}}^{(1)}(X)$ and*

$$(f/g)^{[1]} = (gf^{[1]} - fg^{[1]})/g^2.$$

Proof. We first show that $h := 1/g \in D_{\mathcal{F}}^{(1)}(X)$ and that we have $h^{[1]} = -g^{[1]}/g^2$. Since we have $0 \notin g(X)$, the function $\varphi(z) := 1/z$ ($z \in g(X)$) is continuous and complex-differentiable on $g(X)$, i.e. $\varphi \in D^{(1)}(g(X))$. Let \mathcal{G} be the collection of all admissible paths in $g(X)$. Then we have $\varphi \in D^{(1)}(g(X)) \subseteq D_{\mathcal{G}}^{(1)}(g(X))$, and g is \mathcal{F} - \mathcal{G} -compatible by the comments following Definition 3.7. By Theorem 4.2, $\varphi \circ g \in D_{\mathcal{F}}^{(1)}(X)$ with $(\varphi \circ g)^{[1]} = (\varphi^{[1]} \circ g)g^{[1]}$. However, $\varphi^{[1]}$ is just the ordinary complex derivative of φ , and so $\varphi^{[1]} \circ g = -1/g^2$. Thus $h^{[1]} = -g^{[1]}/g^2$. The result now follows from the product rule for \mathcal{F} -derivatives, Proposition 2.2(c). \square

By combining Theorem 4.2 with our comments at the end of Section 3, we obtain the following corollary.

Corollary 4.4. *Let X be a semi-rectifiable, compact plane set and let \mathcal{F} be an effective collection of paths in X . Let $f, g \in D_{\mathcal{F}}^{(1)}(X)$ such that $g(X) \subseteq X$. Suppose that $D^{(1)}(X)$ is dense in $D_{\mathcal{F}}^{(1)}(X)$. Then $f \circ g \in D_{\mathcal{F}}^{(1)}(X)$ and $(f^{[1]} \circ g)g^{[1]}$ is the \mathcal{F} -derivative of $f \circ g$.*

This last result was proved in the first author's PhD thesis ([4]) using the quotient rule, under the apparently stronger condition that the set of rational functions with no poles on X be dense in $D_{\mathcal{F}}^{(1)}(X)$. See the final section of this paper for an open problem related to this.

By applying Theorem 4.2 inductively, we obtain the Faà di Bruno formula for the composition of n -times \mathcal{F} -differentiable functions.

Corollary 4.5. *Let X, Y be semi-rectifiable, compact plane sets, let \mathcal{F} be an effective collection of paths in X , and let \mathcal{G} be an effective collection of paths in Y . Let $n \in \mathbb{N}$, and let $\varphi \in D_{\mathcal{F}}^{(n)}(X)$ with $\varphi(X) \subseteq Y$. Suppose that φ is \mathcal{F} - \mathcal{G} -compatible. Then, for all $f \in D_{\mathcal{G}}^{(n)}(Y)$, $f \circ \varphi \in D_{\mathcal{F}}^{(n)}(X)$ and, for each $k \in \{1, 2, \dots, n\}$, we have*

$$(f \circ \varphi)^{[k]} = \sum_{i=0}^k (f^{[i]} \circ \varphi) \sum \frac{k!}{a_1! \cdots a_k!} \prod_{j=1}^k \left(\frac{\varphi^{[j]}}{j!} \right)^{a_j},$$

where the inner sum is over all $a_1, \dots, a_k \in \mathbb{N}_0$ such that

$$a_1 + \cdots + a_k = i \quad \text{and} \quad a_1 + 2a_2 + \cdots + ka_k = k.$$

5. HOMOMORPHISMS

We now discuss some algebras of infinitely \mathcal{F} -differentiable functions, analogous to the algebras $D(X, M)$ introduced by Dales and Davie in [6] (see also the introduction of the present paper). In particular, we describe some sufficient conditions under which a function can induce a homomorphism between these algebras. These conditions are similar to those discussed by Feinstein and Kamowitz in [9]. We begin with some definitions from [1, 6] and [9].

Definition 5.1. Let $M = (M_n)_{n=0}^{\infty}$ be a sequence of positive real numbers. We say that M is an *algebra sequence* if $M_0 = 1$ and, for all $j, k \in \mathbb{N}_0$, we have

$$\binom{j+k}{j} \leq \frac{M_{j+k}}{M_j M_k}.$$

We define $d(M) := \lim_{n \rightarrow \infty} (n! / M_n)^{1/n}$ and we say that M is *non-analytic* if $d(M) = 0$.

Let X be a perfect compact plane set and let $M = (M_n)_{n=0}^{\infty}$ be an algebra sequence. Then the set of all rational functions with no poles on X is contained in $D(X, M)$ if and only if M is non-analytic.

We now discuss the algebras $D_{\mathcal{F}}(X, M)$ as introduced in [3].

Definition 5.2. Let X be a semi-rectifiable, compact plane set and let \mathcal{F} be an effective collection of paths in X . Let $M = (M_n)_{n=0}^{\infty}$ be an algebra sequence. We define the normed algebra

$$D_{\mathcal{F}}(X, M) := \left\{ f \in D_{\mathcal{F}}^{(\infty)}(X) : \sum_{j=0}^{\infty} \frac{|f^{[j]}|_X}{M_j} < \infty \right\}$$

with pointwise operations and the norm

$$\|f\| := \sum_{j=0}^{\infty} \frac{|f^{[j]}|_X}{M_j} \quad (f \in D_{\mathcal{F}}(X, M)).$$

The proof that the $D_{\mathcal{F}}(X, M)$ are indeed algebras is similar to the proof of Theorem 1.6 of [6]. In fact, since $D_{\mathcal{F}}^{(1)}(X)$ is complete with the conditions above, it follows that $D_{\mathcal{F}}(X, M)$ is a Banach function algebra; this is noted in [3].

Unfortunately, it is not known in general whether the Banach function algebras $D_{\mathcal{F}}^{(n)}(X)$ and $D_{\mathcal{F}}(X, M)$ are natural on X , although some sufficient conditions are given in [4]. We note that a necessary condition for $D_{\mathcal{F}}(X, M)$ to be natural is that $M = (M_n)_{n=0}^{\infty}$ be a non-analytic algebra sequence.

Definition 5.3. Let X be a semi-rectifiable, compact plane set, and let \mathcal{F} be an effective collection of paths in X . Let $f \in D_{\mathcal{F}}^{(\infty)}(X)$. We say that f is \mathcal{F} -analytic if

$$\limsup_{k \rightarrow \infty} \left(\frac{|\varphi^{[k]}|_X}{k!} \right)^{1/k} < \infty.$$

Note that a function $f \in D_{\mathcal{F}}^{(\infty)}(X)$ which is \mathcal{F} -analytic need not be analytic (in the sense of extending to be analytic on a neighbourhood of X). Let X and \mathcal{F} be as in Example 3.8, let $M = (M_n)_{n=0}^{\infty}$ be an algebra sequence, and let $f \in D_{\mathcal{F}}(X, M)$ with $f \notin D(X, M)$ such that f is \mathcal{F} -analytic. Then f is not analytic. For example, we may take $f(z) = \operatorname{Im}(z)$ here, so that $f^{[1]}$ is identically 0.

We now give the main result of this section. No detailed proof is required, since once the Faà di Bruno formula is established the calculations are identical to those from [13] and [10] (see also [9]). Note that we do not assume the naturality of the algebras here.

Theorem 5.4. *Let X, Y be semi-rectifiable, compact plane sets and let $n \in \mathbb{N}$. Let $\varphi \in D_{\mathcal{F}}^{(\infty)}(X)$ such that $\varphi(X) \subseteq Y$. Suppose that φ is \mathcal{F} -analytic. Let \mathcal{F} be an effective collection of paths on X and let \mathcal{G} be an effective collection of paths on Y such that φ is \mathcal{F} - \mathcal{G} -compatible. Let $M = (M_n)_{n=0}^{\infty}$ be a non-analytic algebra sequence.*

- (a) *If $|\varphi^{[1]}|_X < 1$ then φ induces a homomorphism from the algebra $D_{\mathcal{G}}(Y, M)$ into $D_{\mathcal{F}}(X, M)$.*
- (b) *If the sequence $(n^2 M_{n-1}/M_n)$ is bounded and $|\varphi^{[1]}|_X \leq 1$ then φ induces a homomorphism from $D_{\mathcal{G}}(Y, M)$ into $D_{\mathcal{F}}(X, M)$.*

Note that, in the above, $D_{\mathcal{F}}(X, M)$ and $D_{\mathcal{G}}(Y, M)$ are always Banach function algebras, so we do not need to make any additional completeness assumptions.

When $(n^2 M_{n-1}/M_n)$ is unbounded, the condition that $|\varphi^{[1]}|_X \leq 1$ may no longer be sufficient for φ to induce a homomorphism from $D_{\mathcal{G}}(Y, M)$ into $D_{\mathcal{F}}(X, M)$. The following example is from [11].

Example 5.5. Let $I = [0, 1]$ and let \mathcal{F} be the collection of all admissible paths in I . For each $n \in \mathbb{N}$, let $M_n = (n!)^{3/2}$ and let $M = (M_n)_{n=0}^{\infty}$. Then M is a non-analytic algebra sequence such that $n^2 M_{n-1}/M_n \rightarrow \infty$ as $n \rightarrow \infty$. Moreover, since I is uniformly regular, we have $D_{\mathcal{F}}(I, M) = D(I, M)$, and $D_{\mathcal{F}}(I, M)$ is natural on I by [5, Theorem 4.4.16]. Let $\varphi(t) := (1 + t^2)/2$ ($t \in I$). Then φ is an \mathcal{F} -analytic map from I into I , $|\varphi'|_X \leq 1$, and φ is \mathcal{F} - \mathcal{F} -compatible. However, by [9, Theorem 3.2], φ does not induce a homomorphism from $D_{\mathcal{F}}(I, M)$ into $D_{\mathcal{F}}(I, M)$.

6. OPEN PROBLEMS

We conclude with some open problems related to the content of this paper.

Question 6.1. Can the assumption that φ be \mathcal{F} -analytic in Theorem 5.4 be weakened or removed altogether?

We next ask two questions about maximal collections of paths.

Question 6.2. Let X be a semi-rectifiable compact plane set, and let \mathcal{F} be the collection of all Jordan paths in X . Is it necessarily true that $\mathfrak{m}(\mathcal{F})$ is the collection of all admissible paths in X ?

Question 6.3. Let Γ be an admissible path in \mathbb{C} , and set $X = \Gamma^*$. Let \mathcal{F} be the collection of all subpaths of Γ . Does $\mathfrak{m}(\mathcal{F})$ necessarily include all Jordan paths in X ?

Let X be a semi-rectifiable compact plane set. Our final questions concern the density of rational functions and differentiable functions in the algebras $D^{(1)}(X)$ and $D_{\mathcal{F}}^{(1)}(X)$, respectively. Some partial results were obtained in [3] and [7].

Question 6.4. Is the set of all rational functions with no poles on X always dense in $D^{(1)}(X)$?

Question 6.5. Is $D^{(1)}(X)$ always dense in $D_{\mathcal{F}}^{(1)}(X)$ when \mathcal{F} is the collection of all admissible paths in X ?

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(T. Chaobankoh) DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, CHIANG MAI UNIVERSITY, CHIANG MAI, 50200, THAILAND

(J. F. Feinstein and S. Morley) SCHOOL OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF NOTTINGHAM, UNIVERSITY PARK, NOTTINGHAM, NG7 2RD, UK

E-mail address, T. Chaobankoh: tanadon.c@cmu.ac.th

E-mail address, J. F. Feinstein: joel.feinstein@nottingham.ac.uk

E-mail address, S. Morley: pmxsm9@nottingham.ac.uk