WEAK SEQUENTIAL COMPLETENESS OF UNIFORM ALGEBRAS

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ABSTRACT. We prove that a uniform algebra is weakly sequentially complete if and only if it is finite-dimensional.

1. The Result

A uniform algebra is reflexive if and only if it is finite-dimensional. However, the only mention of this that we have found in the literature is at the very end of the paper [3] where the result is obtained as a consequence of the general theory developed in that paper concerning a representation due to Asimow [1] of a uniform algebra as a space of affine functions. The purpose of the present paper is to give a simple direct proof of the stronger fact that a uniform algebra is weakly sequentially complete if and only if it is finite-dimensional. Since it is trivial that every finite-dimensional Banach space is weakly sequentially complete, the substance of our result is the following.

Theorem 1.1. *No infinite-dimensional uniform algebra is weakly sequentially complete.*

As an immediate consequence we have the following.

Corollary 1.2. *For a uniform algebra A, the following are equivalent.*
- (a) *A is weakly sequentially complete.*
- (b) *A is reflexive.*
- (c) *A is finite-dimensional.*

One can also consider what are sometimes called *nonunital uniform algebras*. These algebras are roughly the analogues on noncompact locally compact Hausdorff spaces of the uniform algebras on compact Hausdorff spaces. (The precise definition is given in the next section.)

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Every nonunital uniform algebra is, in fact, a maximal ideal in uniform algebra, and hence is, in particular, a codimension 1 subspace of a uniform algebra. Since it is easily proven that the failure of weak sequential completeness is inherited by finite codimensional subspaces, it follows at once that the above results hold also for nonunital uniform algebras.

It should be noted that the above results do *not* extend to general semisimple commutative Banach algebras. For instance the Banach space ℓ^p of pth-power summable sequences of complex numbers is a Banach algebra under coordinatewise multiplication and is of course well-known to be reflexive for $1 < p < \infty$; for $p = 1$ the space is nonreflexive but is weakly sequentially complete [6, p. 140]. Also for G a locally compact abelian group, the Banach space $L^1(G)$ is a Banach algebra with convolution as multiplication and is nonreflexive but is weakly sequentially complete [6, p. 140]. All of these Banach algebras are nonunital, with the exception of the algebras $L^1(G)$ for G a discrete group. However, adjoining an identity in the usual way where necessary, one obtains from them unital Banach algebras with the same properties with regard to reflexivity and weak sequential completeness.

In the next section, which can be skipped by those well versed in basic uniform algebra and Banach space concepts, we recall some definitions. The proof of Theorem 1.1 is then presented in the concluding section.

2. Definitions

For X a compact Hausdorff space, we denote by $C(X)$ the algebra of all continuous complex-valued functions on X with the supremum norm $\|f\|_X = \sup\{|f(x)| : x \in X\}$. A *uniform algebra* on X is a closed subalgebra of $C(X)$ that contains the constant functions and separates the points of X. We say that the uniform algebra A on X *natural* if X is the maximal ideal space of A, that is, if the only (nonzero) multiplicative linear functionals on A are the point evaluations at points of X. For Y a noncompact locally compact Hausdorff space, we denote by $C_0(Y)$ the algebra of continuous complex-valued functions on Y that vanish at infinity, again with the supremum norm. By a *nonunital uniform algebra* B on Y we mean a closed subalgebra of $C_0(Y)$ that strongly separates points in the sense that for every pair of distinct points x and y in Y there is a function f in B such that $f(x) \neq f(y)$ and $f(x) \neq 0$. If B is a nonunital uniform algebra on Y, then the linear span of B and the constant functions on Y forms a unital Banach algebra that can be identified with a uniform algebra A on the one-point compactification of Y, and under this identification B

is the maximal ideal of A consisting of the functions in A that vanish at infinity.

Let A be a uniform algebra on a compact Hausdorff space X. A closed subset E of X is a *peak set* for A if there is a function $f \in A$ such that $f(x) = 1$ for all $x \in E$ and $|f(y)| < 1$ for all $y \in X \setminus E$. Such a function f is said to *peak on* E. A *generalized peak set* is an intersection of peak sets. A point p in X is a *peak point* if the singleton set $\{p\}$ is a peak set, and p is a *generalized peak point* if $\{p\}$ is a generalized peak set. For Λ a bounded linear functional on A, we say that a regular Borel measure μ on X represents Λ if $\Lambda(f) = \int f \, d\mu$ for every $f \in A$.

A Banach space A is *reflexive* if the canonical embedding of A in its double dual A^{**} is a bijection. The Banach space A is *weakly sequentially complete* if every weakly Cauchy sequence in A is weakly convergent in A. More explicitly the condition is this: for each sequence (x_n) in A such that (Λx_n) converges for every Λ in the dual space A^*, there exists an element x in A such that $\Lambda x_n \to \Lambda x$ for every Λ in A. It is trivial that every reflexive space is weakly sequentially complete.

3. The Proof

Our proof of Theorem 1.1 hinges on the following lemma.

Lemma 3.1. *Every infinite-dimensional natural uniform algebra has a peak set that is not open.*

Proof. Let A be a natural uniform algebra on a compact Hausdorff space X.

Consider first the case when A is a proper subalgebra of $C(X)$. In that case, by the Bishop antisymmetric decomposition [2, Theorem 2.7.5] there is a maximal set of antisymmetry E for A such that the restriction $A|E$ is a proper subset of $C(E)$. Then of course E has more than one point, and because A is natural, E is connected [5, p. 119]. Since every maximal set of antisymmetry is a generalized peak set, and every generalized peak set contains a generalized peak point (see the proof of [2, Corollary 2.4.6]), E contains a generalized peak point p. Choose a peak set P for A such that $p \in P$ but $P \not\supseteq E$. Then $P \cap E$ is a proper nonempty closed subset of the connected set E and hence is not open in E. Thus P is not open in X.

In case $A = C(X)$, it follows from Urysohn's lemma that the peak sets of A are exactly the closed G_δ-sets in X (see for instance [4, Section 33, exercise 4]). Thus the proof is completed by invoking the following lemma. □

Lemma 3.2. *Every infinite compact Hausdorff space X contains a closed G_δ-set that is not open.*

Proof. Let $\{x_n\}$ be a countably infinite subset of X. For each $n = 1, 2, 3, \ldots$ choose by the Urysohn lemma a continuous function $f_n : X \to [0,1]$ such that $f_n(x_k) = 0$ for $k < n$ and $f(x_n) = 1$. Let $F : X \to [0,1]^\omega$ be given by $F(x) = \big(f_n(x)\big)_{n=1}^{\infty}$. Then $F(x_m) \neq F(x_n)$ for all $m \neq n$. Thus the collection $\{F^{-1}(t) : t \in [0,1]^\omega\}$ is infinite. Each of the sets $F^{-1}(t)$ is a closed G_δ-set because F is continuous and $[0,1]^\omega$ is metrizable. Since these sets form an infinite family of disjoint sets that cover X, they cannot all be open, by the compactness of X. □

Proof of Theorem 1.1. Let A be an infinite-dimensional uniform algebra on a compact Hausdorff space X. Since A is isometrically isomorphic to a natural uniform algebra via the Gelfand transform, we can assume without loss of generality that A is natural. By Lemma 3.1 there exists a peak set P for A that is not open. Choose a function $f \in A$ that peaks on P.

For a bounded linear functional Λ on A, and a regular Borel measure μ on X that represents Λ, we have by the Lebesgue dominated convergence theorem that

$$\Lambda(f^n) = \int f^n \, d\mu \to \mu(P) \quad \text{as} \quad n \to \infty. \tag{1}$$

Thus the sequence $(f^n)_{n=1}^\infty$ in A is weakly Cauchy. Furthermore (1) shows that, regarded as a sequence in the double dual A^{**}, the sequence $(f^n)_{n=1}^\infty$ is weak*-convergent to a functional $\Phi \in A^{**}$ that satisfies the equation $\Phi(\Lambda) = \mu(P)$ for every functional $\Lambda \in A^*$ and every regular Borel measure μ that represents Λ.

For $x \in X$, denote the point mass at x by δ_x. Denote the characteristic function of the set P by χ_P. Then

$$\Phi(\delta_x) = \chi_P(x) \tag{2}$$

while for any function $h \in A$ we have

$$\int h \, d\delta_x = h(x). \tag{3}$$

Since P is not open in X, the characteristic function χ_P is not continuous and hence is not in A. Consequently, equations (2) and (3) show that the functional $\Phi \in A^{**}$ is not induced by an element of A. We conclude that the weakly Cauchy sequence $(f^n)_{n=1}^\infty$ is not weakly-convergent in A. □

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