Integer points of entire functions

J.K. Langley

April 27, 2005

Abstract

A result is proved which implies the following conjecture of Osgood and Yang from 1976: if f and g are non-constant entire functions, such that $T(r, f) = O(T(r, g))$ as $r \rightarrow \infty$ and such that $g(z) \in \mathbb{Z}$ implies $f(z) \in \mathbb{Z}$, then there exists a polynomial G with coefficients in $\mathbb{Q}$, such that $G(\mathbb{Z}) \subseteq \mathbb{Z}$ and $f = G \circ g$.

Keywords: entire functions, forward differences, Wiman-Valiron theory.

MSC 2000: 30D20, 30D35.

1 Introduction

This paper will use standard notation from Nevanlinna theory [5], including $T(r, g)$ for the Nevanlinna characteristic of a function g meromorphic in the plane and, for an entire function f , $M(r, f) = \max\{|f(z)| : |z| = r\}$. The starting point of the paper is the following conjecture advanced by Osgood and Yang in [11].

Conjecture 1.1 *Let f and g be non-constant entire functions, such that $T(r, f) = O(T(r, g))$ as $r \rightarrow \infty$, and such that $g(z) \in \mathbb{Z}$ implies $f(z) \in \mathbb{Z}$. Then there exists a polynomial G with coefficients in $\mathbb{Q}$, such that $G(\mathbb{Z}) \subseteq \mathbb{Z}$ and $f = G \circ g$.*

This question arose in [11] from the problem of determining when

$$L = \frac{e^{2\pi i f} - 1}{e^{2\pi i g} - 1} \tag{1}$$

is entire. Conjecture 1.1 was proved in [11] in the special case where f and g are both polynomials. For the corresponding result in which f and g are rational functions, see [1] and [4] (see also [3, Lemma 3] and [7, Lemma 5]).

Integer points of entire functions have been considered by many authors going back at least to Pólya, who proved [12, p.55] that an entire function f with $M(r, f) = o(2^r)$ and taking integer values on $\mathbb{N}$ reduces to a polynomial. An analogous result for analytic functions of polynomial growth in a half-plane was established in [7, Lemma 5], and some applications to value distribution and differential equations appear in [7, 8, 9]. Results concerning entire functions taking integer values on subsets of $\mathbb{N}$, in terms of growth and densities, may be found in [2].

The results of the present paper will establish Conjecture 1.1 and rather more. In order to deduce that f is a function of g it turns out that a relatively mild growth condition suffices, and it is only necessary that $f(z) \in \mathbb{Z}$ whenever $g(z) \in \mathbb{N}$ and z lies in certain regions in which $g(z)$ is approximated by a suitable monomial. We will prove the following theorem.

Theorem 1.1 *Let $K > 1$ and $\eta > 0$. Let f, g be meromorphic with finitely many poles in the plane, such that g has a pole or essential singularity at ∞ . Suppose that there exists an unbounded set $J \subseteq [1, \infty)$ with the following properties. For each $r \in J$ there exist z_1 and a positive integer $N = N_r$ such that:*

$$|z_1| = r \quad \text{and} \quad g(z_1) \in (0, \infty); \quad (2)$$

$$g(z) = g(z_1)(z/z_1)^N(1 + o(1)) \quad \text{for all } z \text{ satisfying} \quad |\log(z/z_1)| \leq K\pi N^{-1}; \quad (3)$$

$$N \log^+ M(r, f) \log M(r, g) = o(g(z_1)); \quad (4)$$

$$f(z) \in \mathbb{Z} \quad \text{for all } z \text{ in } \{u \in \mathbb{C} : g(u) \in \mathbb{N}, |u| < r, |\log(u/z_1)| \leq 10\eta N^{-1}\}. \quad (5)$$

Then there exists an entire function G satisfying $f = G \circ g$ and such that

$$G(\mathbb{Z}) \subseteq \mathbb{Z} \quad \text{and} \quad \liminf_{R \rightarrow \infty} \frac{\log M(R, G)}{R} = 0. \quad (6)$$

The proof of Theorem 1.1 depends on applying the forward difference method to obtain local approximations for f in terms of g (see §2). The key idea is then to use harmonic measure to extend the domain of approximation. Clearly, if f, g and G are as in Theorem 1.1 and g has a pole in $\mathbb{C}$ then G must be a polynomial.

Theorem 1.1 may be applied whenever g is locally approximated by a multiple of a power of z as in (3) and (4): this is the case, for example, for $g(z) = e^{-z} + z^4$ near the positive real axis. The fact that every entire function g has regions as in (3) and (4) follows from Wiman-Valiron theory [6], from which some standard facts are summarized in the following lemma. To establish the notation for this lemma let g be a non-constant entire function with Taylor series $g(z) = \sum_{n=0}^{\infty} a_n z^n$. Then for $r > 0$ the central index $\nu(r, g)$ is the largest n for which $|a_n| r^n = \max\{|a_m| r^m : m \geq 0\}$. The results of Lemma 1.1 are normally stated for transcendental entire functions, but obviously hold if g is a non-constant polynomial, in which case $\nu(r, g)$ for large r is the degree of g .

Lemma 1.1 ([6]) *Let g be a non-constant entire function, and let $0 < \delta < 1$. There exists a set $E \subseteq [1, \infty)$ of finite logarithmic measure with the following properties. For $r \in [1, \infty) \setminus E$ the central index $\nu(r, g)$ satisfies*

$$\nu(r, g) \leq (\log M(r, g))^{1+\delta}. \quad (7)$$

Further, there exists z_1 with

$$|z_1| = r \quad \text{and} \quad g(z_1) = |g(z_1)| \geq (1 - o(1))M(r, g), \quad (8)$$

such that, with $N = \nu(r, g)$,

$$M(s, g) = (s/r)^N M(r, g)(1 + o(1)) \quad \text{for all } s \text{ such that } |\log(s/r)| \leq N^{-2/3}, \quad (9)$$

and

$$g(z) = g(z_1)(z/z_1)^N(1 + o(1)) \quad \text{and} \quad |g(z)| \geq (1 - o(1))M(|z|, g) \quad (10)$$

for all z such that $|\log(z/z_1)| \leq N^{-2/3}$.

Lemma 1.1 is completely standard, apart from the condition that $g(z_1)$ is real and positive. However, for r in $[1, \infty) \setminus E$ and z^* with $|z^*| = r, |g(z^*)| = M(r, g)$, the results of [6] give (9), as well as

$$g(z) = g(z^*)(z/z^*)^N(1 + o(1)) \quad \text{for } |\log(z/z^*)| \leq N^{-5/8}.$$

Choosing z_1 with $|z_1| = r$ and $\arg(z_1/z^*) = O(1/N)$ and $g(z_1) \in (0, \infty)$ gives (8) and (10).

Theorem 1.2 *Let $0 < \delta < 1$ and $\eta > 0$. Let f, g be non-constant entire functions and suppose that there exists a set $H \subseteq [1, \infty)$, of infinite logarithmic measure, such that for each $r \in H$ there exists z_1 satisfying (8) and (10), with $N = \nu(r, g)$, and such that (5) holds. Assume further that*

$$\log M(r, f) = o\left(\frac{M(r, g)}{(\log M(r, g))^{2+\delta}}\right) \quad \text{for all } r \text{ in } H. \quad (11)$$

Then there exists an entire function G satisfying $f = G \circ g$ and (6).

If (11) is replaced by

$$\limsup_{r \rightarrow \infty, r \in H} \frac{T(r, f)}{T(r, g)} < \infty \quad (12)$$

then there exists a polynomial G with coefficients in $\mathbb{Q}$, such that $G(\mathbb{Z}) \subseteq \mathbb{Z}$ and $f = G \circ g$.

Conjecture 1.1 now follows from Theorem 1.2, since Lemma 1.1 shows that to every non-constant entire function g correspond H , z_1 and N as in (8) and (10).

An example illustrating the sharpness of Theorem 1.2 is given in §5. The obvious choice $f(z) = z \sin(\pi g(z))$ shows that the hypothesis (11) cannot be deleted in Theorem 1.2.

2 Application of the forward difference method

Define the forward differences in the standard way [12, p.52] for $n \in \mathbb{N}$ by

$$\Delta^0 h(a) = h(a), \quad \Delta h(a) = \Delta^1 h(a) = h(a+1) - h(a), \quad \Delta^{n+1} h(a) = \Delta^n h(a+1) - \Delta^n h(a).$$

It is easy to prove by induction that if $h(a), \dots, h(a+n)$ are integers then so is $\Delta^n h(a)$. For $x \in \mathbb{R}$ denote by $[x]$ the greatest integer not exceeding x .

Lemma 2.1 *Let the real numbers η, ε, S and R satisfy the five conditions:*

$$0 < \eta < \frac{1}{4}; \quad 0 < \varepsilon < \frac{\eta}{48}; \quad 12e^S 3^{-S} < 1; \quad 1 \leq S < \varepsilon R; \quad R \in \mathbb{N}. \quad (13)$$

Let F be an analytic function in the region

$$\Sigma = \{w \in \mathbb{C} : |\log(w/R)| < 2\eta\}, \quad (14)$$

that satisfies

$$F(\Sigma \cap \mathbb{N}) \subseteq \mathbb{Z} \quad \text{and} \quad \log |F(w)| \leq S \quad \text{for all } w \text{ in } \Sigma. \quad (15)$$

Then there exists a polynomial

$$P_q(w) = P(w) = F(R) + (w - R)\Delta F(R) + \dots + \frac{(w - R) \dots (w - R - q + 1)}{q!} \Delta^q F(R) \quad (16)$$

of degree at most $q = [S]$, with coefficients satisfying

$$\left| \frac{\Delta^m F(R)}{m!} \right| \leq 6e^S \quad (17)$$

for $0 \leq m \leq q$, such that $P(\mathbb{Z}) \subseteq \mathbb{Z}$ and

$$|F(w) - P(w)| < 12e^S 3^{-\varepsilon R}, \quad (18)$$

for all w in the region

$$(1 - \varepsilon)R < |w| < (1 + \varepsilon)R, \quad |\arg w| < \varepsilon. \quad (19)$$

The proof of Lemma 2.1 requires a number of intermediate lemmas. Throughout this section, F is the function of Lemma 2.1.

Lemma 2.2 *Let C be the contour describing once counter-clockwise the boundary of the region*

$$|\log(z/R)| \leq \eta,$$

where R and η are as in Lemma 2.1, and suppose that w lies inside C , and that $n \in \mathbb{N}$ with $R + n < Re^\eta$. Then

$$F(w) - P_n(w) = \frac{1}{2\pi i} \int_C \frac{(w - R) \dots (w - R - n) F(t)}{(t - R) \dots (t - R - n)(t - w)} dt, \quad (20)$$

where $P_n(w)$ is a polynomial of degree at most n , which equals $F(w)$ at the $n + 1$ points $R, R + 1, \dots, R + n$, and is given by

$$P_n(w) = F(R) + (w - R)\Delta F(R) + \dots + \frac{(w - R) \dots (w - R - n + 1)}{n!} \Delta^n F(R). \quad (21)$$

Moreover, the polynomial P_n maps $\mathbb{Z}$ into $\mathbb{Z}$, and the coefficients in (21) satisfy (17) for $0 \leq m \leq \varepsilon R$.

Proof. (20) and (21) are completely standard: see, for example, [12, pp.52-53]. Since R and the forward differences $\Delta^j F(R), j = 0, \dots, n$, are integers, by (13) and (15), and since

$$\frac{p(p-1) \dots (p-m+1)}{m!} \in \mathbb{Z} \quad \text{for } p \in \mathbb{Z} \quad \text{and } m \in \mathbb{N}, \quad (22)$$

the inclusion $P_n(\mathbb{Z}) \subseteq \mathbb{Z}$ follows.

It remains to prove that the coefficients satisfy (17) for $0 \leq m \leq \varepsilon R$. The m 'th forward difference is given by [12, pp.52-53]

$$\frac{\Delta^m F(R)}{m!} = \frac{1}{2\pi i} \int_C \frac{F(t)}{(t-R) \dots (t-R-m)} dt,$$

in which $\log |F(t)| \leq S$ by (15). Now (13) gives

$$|e^u - 1| \geq \eta - \eta^2 - \eta^3 - \dots \geq \eta - 2\eta^2 > \eta/2 \quad \text{for } |u| = \eta,$$

and the the image of the circle $|u| = \eta$ under e^u has length at most $2\pi\eta e < 6\pi\eta$. Thus

$$\text{dist}(C, R) > \eta R/2 > 24\varepsilon R > 24 \quad \text{and} \quad \text{length}(C) \leq 6\pi R\eta. \quad (23)$$

Hence $|t - R| > \frac{1}{2}\eta R$ and $|t - R - j| > \frac{1}{4}\eta R > 1$ for $1 \leq j \leq \varepsilon R$ and $t \in C$, which gives (17). $\square$

Lemma 2.3 *Let w lie in the region (19), and let the contour C and the polynomial P_n be as in Lemma 2.2. Then w lies inside C and, with ε as in (13),*

$$|F(w) - P_n(w)| < 12e^S 3^{-n-1} \quad \text{for } n \leq \varepsilon R.$$

Proof. Let t lie on C . Then (13), (19) and (23) give

$$|w - R| < 3\varepsilon R < \text{dist}(C, R) \quad \text{and} \quad |t - w| \geq \frac{1}{2}\eta R - 3\varepsilon R \geq \frac{1}{4}\eta R.$$

Further, if j is an integer with $0 \leq j \leq \varepsilon R$, then

$$\left| \frac{w - R - j}{t - R - j} \right| \leq \frac{4\varepsilon R}{\eta R/2 - \varepsilon R} < \frac{4\varepsilon R}{\eta R/4} = \frac{16\varepsilon}{\eta} < \frac{1}{3},$$

and the lemma follows using (15), (20) and (23) again. $\square$

Lemma 2.4 *Let R , S and ε be as in (13), and let the polynomial P_n be as in Lemma 2.2.*

(i) *If $[S] \leq n \leq \varepsilon R$ then $F(R-1) = P_n(R-1)$.*

(ii) *If $[S] + 1 \leq n \leq \varepsilon R$ then $\Delta^n F(R) = 0$.*

Proof. The point $w = R - 1$ lies in the region (19), by (13). Hence Lemma 2.3 gives, for $[S] \leq n \leq \varepsilon R$,

$$|F(R - 1) - P_n(R - 1)| < 12e^S 3^{-n-1} < 12e^S 3^{-S} < 1,$$

using (13) again. Part (i) is thus proved, since $R - 1$ is an integer by (13), and $F(R - 1)$ and $P_n(R - 1)$ are both integers by (15) and Lemma 2.2.

Part (ii) now follows using part (i), since for $[S] + 1 \leq n \leq \varepsilon R$,

$$F(R - 1) = P_{n-1}(R - 1) = P_n(R - 1) = P_{n-1}(R - 1) + (-1)^n \Delta^n F(R).$$

□

To complete the proof of Lemma 2.1, let w lie in the region (19), and let R , S and ε be as in (13). Let $P = P_q$ be given by (16), with $q = [S]$, and let $n = [\varepsilon R]$. Then $q \leq n$ by (13), and (17) follows from Lemma 2.2. Finally, Lemma 2.3 and part (ii) of Lemma 2.4 give

$$|F(w) - P(w)| = |F(w) - P_q(w)| = |F(w) - P_n(w)| < 12e^S 3^{-n-1} < 12e^S 3^{-\varepsilon R}.$$

□

3 Proof of Theorem 1.1

Let f, g be as in the hypotheses, and assume without loss of generality that η is small. If f is a rational function with $f(\infty)$ finite then (2), (3) and (5) show that f takes a fixed integer value n_0 infinitely often and so is constant. Assume henceforth that f either is transcendental or has a pole at ∞ . Choose $R_0 \in (2, \infty)$, so large that

$$f^{-1}(\{\infty\}) \cup g^{-1}(\{\infty\}) \subseteq B\left(0, \frac{1}{2}R_0\right), \quad (24)$$

where $B(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$. Then the function

$$v(z) = \begin{cases} M(R_0, g), & \text{if } |z| \leq R_0, \\ \max\{|g(z)|, M(R_0, g)\}, & \text{if } |z| > R_0, \end{cases}$$

is subharmonic in the plane, and

$$M(r, g) = \max\{|g(z)| : |z| = r\} = \max\{|g(z)| : R_0 \leq |z| \leq r\} \quad (25)$$

for large r . Similar considerations apply to $M(r, f)$.

Lemma 3.1 *With the notation of Theorem 1.1, let $r \in J$ be large. Then there exists z_0 satisfying*

$$|z_0| = r' < r \quad \text{and} \quad -5\eta N^{-1} < \log |z_0/z_1| < -4\eta N^{-1} \quad \text{and} \quad |\arg(z_0/z_1)| \leq \eta N^{-1}, \quad (26)$$

such that

$$R = g(z_0) \in \mathbb{N} \quad \text{and} \quad g(z_0) \geq (1 - o(1))e^{-5\eta}g(z_1) \geq \frac{1}{2}g(z_1), \quad (27)$$

and such that

$$g(z) = g(z_0)(z/z_0)^N(1 + o(1)) \quad \text{for all } z \text{ such that } |\log(z/z_0)| \leq 4\eta N^{-1}. \quad (28)$$

Moreover, there exists a simply connected domain $\Omega = \Omega_r$ mapped univalently by g onto the region Σ defined by (14), and

$$\Omega \subseteq \{z \in \mathbb{C} : |\log(z/z_0)| \leq 4\eta N^{-1}\} \subseteq B(0, r). \quad (29)$$

Finally

$$S = \log M(r, f) = o(R), \quad (30)$$

and on Ω the function f may be written in the form $f(z) = F(g(z))$ with F analytic on $g(\Omega) = \Sigma$ and satisfying (15).

Proof. The existence of z_0 satisfying (26), (27) and (28) follows at once from (2) and (3) since η is small, and (28) in turn implies the existence of Ω satisfying (29). The function F is defined on $g(\Omega)$ by $F = f \circ g^{-1}$, and (15), with $S = \log M(r, f)$, follows from (5), the remark following (25), (26) and (29). Finally, (30) follows from (4) and (27). $\square$

Let ε be positive, but small compared to η , and use c to denote a positive constant, not necessarily the same at each occurrence, but not depending on $\varepsilon, \eta, K, r, f$ or g . For large $r \in J$ Lemma 2.1 may then be applied, since the conditions (13) are satisfied, using (27) and (30).

Lemma 3.2 *Let J be as in the hypotheses of Theorem 1.1 and let $P_q = P$ be as in Lemma 2.1. If $r \in J$ is sufficiently large then there exists an arc A_r of the circle $|z| = r'$, where r' is given by (26), such that A_r has angular measure at least $c\varepsilon N^{-1}$ and*

$$|f(z) - P_q(g(z))| < \exp(-c\varepsilon R) \quad \text{for all } z \text{ in } A_r. \quad (31)$$

Further, with R_0 as in (24),

$$|P_q(g(z))| \leq cSe^S(3M(r, g))^S \quad \text{for } R_0 \leq |z| \leq r'. \quad (32)$$

Proof. The estimate (31) follows at once from (18) and (30), since if the positive constant c is chosen small enough the image under g of the arc $z = z_0 e^{i\theta}$, $-c\varepsilon N^{-1} \leq \theta \leq c\varepsilon N^{-1}$, lies in the region (19), by (28). Next, if $w = g(z)$, with $R_0 \leq |z| \leq r'$ and $j = 0, 1, \dots, q$, then

$$|w - R - j| \leq M(r, g) + R + S \leq M(r, g) + R(1 + o(1)) \leq 3M(r, g),$$

using (25), (26), (27) and (30). Thus (32) follows from (16) and (17). $\square$

Lemma 3.3 *Let $\sigma > 0$. Then, with the hypotheses of Lemma 3.2, if $r \in J$ is sufficiently large,*

$$\max\{|f(z) - P_q(g(z))| : R_0 \leq |z| \leq r/2\} < \sigma. \quad (33)$$

Proof. By (16) and (24) there exist $c > 0$ and a monic polynomial h_q , with degree at most $cq \leq cS$ and with all its zeros in $|z| < \frac{1}{2}R_0$, such that the function

$$u(z) = \log |h_q(z)(f(z) - P_q(g(z)))|$$

is subharmonic on the disc $B(0, r')$. Then, for $z \in B(0, r')$,

$$u(z) \leq cS \log M(r, g) + 2cS \log r \leq c \log M(r, f) \log M(r, g) = o(N^{-1}R),$$

by (4), (27), (30), (32) and the maximum principle. On the other hand, (4), (27) and (31) give

$$u(z) < -c\varepsilon R + 2cS \log r < -c\varepsilon R \quad \text{for } z \text{ in } A_r,$$

in which A_r is the arc of Lemma 3.2. Next, if $|z| \leq r/2$ then the harmonic measure of the arc A_r with respect to $B(0, r')$, evaluated at z , satisfies $\omega(z, A_r, B(0, r')) \geq c\varepsilon N^{-1}$ by Poisson's formula. Applying the two-constants theorem [10, p.42] to u now gives

$$u(z) \leq o(N^{-1}R) - c\varepsilon^2 N^{-1}R < -c\varepsilon^2 N^{-1}R \quad \text{for } |z| \leq r/2.$$

But $|h_q(z)| \geq 1$ for $|z| \geq R_0$, since $R_0 > 2$, and (33) follows using (4) and (27), provided r is sufficiently large. $\square$

Choose a sequence $r_n \rightarrow \infty, r_n \in J$. Then (33) gives polynomials G_n , each satisfying $G_n(\mathbb{Z}) \subseteq \mathbb{Z}$, such that

$$\max\{|f(z) - G_n(g(z))| : R_0 \leq |z| \leq s_n = r_n/2\} = o(1) \quad \text{as } n \rightarrow \infty. \quad (34)$$

The proof of Theorem 1.1 is completed by the next lemma.

Lemma 3.4 *The polynomials G_n in (34) have a subsequence converging locally uniformly on $\mathbb{C}$ to an entire function G which satisfies $f = G \circ g$ and (6).*

Proof. Let T be large, and choose $r \in J$ and z_1 satisfying (2), (3) and (4) with $g(z_1) > 2T$. Then (3) and the fact that $K > 1$ show that $g(\zeta)$ assumes in $R_0 \leq |\zeta| < r$ all values w lying on a circle $|w| = T' \geq g(z_1)/2 > T$. If n is so large that $s_n \geq r$ then (34) gives

$$M(T, G_n) \leq M(T', G_n) \leq \max\{|G_n(g(z))| : R_0 \leq |z| \leq r\} \leq \max\{|f(z)| : R_0 \leq |z| \leq r\} + 1.$$

Hence the G_n form a normal family on $\mathbb{C}$. Assume without loss of generality that the sequence G_n converges locally uniformly on $\mathbb{C}$ to an entire function G . Then $G(\mathbb{Z}) \subseteq \mathbb{Z}$, and $f = G \circ g$ by (34). Moreover, using the remark following (25),

$$M(T', G) \leq \max\{|G(g(z))| : R_0 \leq |z| \leq r\} = \max\{|f(z)| : R_0 \leq |z| \leq r\} = M(r, f),$$

and so (4) gives

$$\log M(T', G) \leq \log M(r, f) = o(g(z_1)) = o(T').$$

□

4 Proof of Theorem 1.2

Let f, g, δ, η and the set H be as in the hypotheses. Assume without loss of generality that $H \cap E = \emptyset$, with E the exceptional set of Lemma 1.1, and for large $r \in H$ choose z_1 as in the statement of the theorem. Then (7), (8), (10) and (11) show that (2), (3) and (4) are satisfied, with $N = \nu(r, g)$. Since f satisfies (5) it follows at once from Theorem 1.1 that there exists an entire function G satisfying $f = G \circ g$ and (6). This proves the first part of Theorem 1.2.

Suppose next that (11) is replaced by (12). Applying the Borel lemma [5, p.38] in the standard way to $h(s) = T(e^s)$, where $T(r) = T(r, f)$, shows that

$$T(re^{1/T(r)}) = O(T(r)) \quad \text{for } r \in [1, \infty) \setminus E_0, \quad (35)$$

in which E_0 has finite logarithmic measure and may be assumed without loss of generality to have empty intersection with H . Now (12), (35) and standard estimates give, for $r \in H$,

$$\log M(r, f) = O(T(r)T(re^{1/T(r)})) = O(T(r))^2 = O(T(r, g))^2 = O(\log M(r, g))^2.$$

Thus f satisfies (11) and so there exists an entire function G with $G(\mathbb{Z}) \subseteq \mathbb{Z}$ and $f = G \circ g$. Then G is a polynomial, by (12) and [5, p.54], and it is easy to see that G has rational coefficients. $\square$

Corollary 4.1 *Let f and g be non-constant entire functions satisfying (12), where H has infinite logarithmic measure, and suppose that the function L defined by (1) is entire. Then there exists a polynomial G with coefficients in $\mathbb{Q}$, such that $G(\mathbb{Z}) \subseteq \mathbb{Z}$ and $f = G \circ g$.*

5 An example

It seems worth observing that the condition (6) does not force an entire function G to be a polynomial. To see this, for $n \in \mathbb{N}$ let

$$R_n(z) = \frac{z(z^2 - 1) \dots (z^2 - n^2)}{(2n + 1)!} = \frac{(z + n) \dots z \dots (z - n)}{(2n + 1)!}. \quad (36)$$

Thus $R_n(\mathbb{Z}) \subseteq \mathbb{Z}$ by (22). Next, let ψ be a non-decreasing positive function on $[1, \infty)$ with $\lim_{s \rightarrow \infty} \psi(s) = \infty$ and $\psi(s) \leq s$, and choose integers $n_k, \mu_k, \lambda_k, k = 1, 2, \dots$, such that

$$1 \leq n_k < \psi(\mu_k) \leq \mu_k < \lambda_k^{o(1)} < \lambda_k < n_{k+1} \quad \text{and} \quad \lambda_k^{2\lambda_k} < 2^{n_{k+1} - \lambda_k}. \quad (37)$$

Since, for $n \in \mathbb{N}$,

$$\left| \frac{z^2 - n^2}{2n(2n + 1)} \right| \leq \begin{cases} s^2, & \text{if } \max\{n, |z|\} \leq s, \\ \frac{1}{2}, & \text{if } |z| \leq n, \end{cases}$$

it follows on writing

$$\frac{R_{n_{k+1}}(z)}{R_{n_k}(z)} = \left(\frac{z^2 - (n_k + 1)^2}{(2n_k + 2)(2n_k + 3)} \right) \dots \left(\frac{z^2 - \lambda_k^2}{2\lambda_k(2\lambda_k + 1)} \right) \dots \left(\frac{z^2 - n_{k+1}^2}{2n_{k+1}(2n_{k+1} + 1)} \right)$$

that, using (37),

$$\left| \frac{R_{n_{k+1}}(z)}{R_{n_k}(z)} \right| \leq \lambda_k^{2(\lambda_k - n_k)} 2^{\lambda_k - n_{k+1}} \leq \lambda_k^{-2n_k} \leq \frac{1}{2} \quad \text{for } |z| \leq \lambda_k. \quad (38)$$

Further,

$$|R_n(z)| \leq \frac{s(2s^2)^n}{(2n+1)!} \leq \frac{(2s)^{2n+1}}{(2n+1)!} \quad \text{for } |z| = s \geq n \in \mathbb{N}. \quad (39)$$

Set

$$G(z) = \sum_{k=1}^{\infty} R_{n_k}(z). \quad (40)$$

Let

$$m \in \mathbb{N} \quad \text{and} \quad \mu_m \leq |z| = s \leq \lambda_m. \quad (41)$$

Then (37), (38) and (39) give

$$\sum_{k=1}^{\infty} |R_{n_k}(z)| \leq \sum_{k=1}^{m-1} |R_{n_k}(z)| + |R_{n_m}(z)| \left(1 + \frac{1}{2} + \frac{1}{4} + \dots \right) \leq 2 \sum_{k=1}^m |R_{n_k}(z)| \leq 2e^{2s}. \quad (42)$$

In particular the series (40) converges locally uniformly in $\mathbb{C}$, and G is entire with $G(\mathbb{Z}) \subseteq \mathbb{Z}$.

To prove that G is transcendental, define h_m and G_m by

$$h_m(z) = \sum_{k=m+1}^{\infty} R_{n_k}(z), \quad G(z) = G_m(z) + h_m(z).$$

Then $G_m(z)$ is a polynomial of degree $2n_m + 1$ with $G_m^{(2n_m+1)}(z) \equiv 1$. Since $h_m(j) = 0$ for $j = -n_m - 1, -n_m, \dots, n_m, n_m + 1$ by (36), repeated application of Rolle's theorem shows that there exists $x_m \in (-n_m - 1, n_m + 1)$ with $h_m^{(2n_m+1)}(x_m) = 0$ and hence $G^{(2n_m+1)}(x_m) = 1$.

For z satisfying (41), inequalities (37), (38), (39) and (42) imply that

$$|G(z)| \leq 2 \sum_{k=1}^m |R_{n_k}(z)| \leq 2m(2s)^{2n_m+1} \leq 2n_m(2s)^{2n_m+1},$$

which gives, for some $c > 0$,

$$\log M(s, G) \leq (2n_m + 1) \log 2s + \log 2n_m \leq c\psi(s) \log s \quad \text{for } \mu_m \leq s \leq \lambda_m. \quad (43)$$

Thus G may have arbitrarily slow lower growth, subject to being transcendental, even on a set of upper logarithmic density 1.

Finally, the choice $g(z) = e^z$, $f(z) = G(e^z)$, together with (43), gives

$$T(r, f) \leq \log M(r, f) \leq \log M(M(r, g), G) \leq c\psi(M(r, g)) \log M(r, g) \leq c\pi\psi(e^{\pi T(r, g)}) T(r, g)$$

for r in the union H of the intervals $[\log \mu_m, \log \lambda_m]$, and H has infinite logarithmic measure, by (37). Since $\psi \circ \exp$ can grow arbitrarily slowly, the second part of Theorem 1.2 is sharp.

6 Functions taking integer values at the same points

The following result addresses the natural question of what may be deduced if two non-constant entire functions take integer values at the same points.

Theorem 6.1 *Let f and g be non-constant entire functions with the following property. There exist $\varepsilon \in (0, 1)$ and a set $H \subseteq [1, \infty)$ of infinite logarithmic measure such that for $r \in H$:*

(i) $f(\zeta) \in \mathbb{Z}$ for all ζ satisfying

$$re^{-\varepsilon/\nu(r,g)} < |\zeta| < r \quad \text{and} \quad g(\zeta) \in \mathbb{N} \quad \text{and} \quad |g(\zeta)| > (1 - \varepsilon)M(|\zeta|, g); \quad (44)$$

(ii) $g(\zeta) \in \mathbb{Z}$ for all ζ satisfying

$$re^{-\varepsilon/\nu(r,f)} < |\zeta| < r \quad \text{and} \quad f(\zeta) \in \mathbb{N} \quad \text{and} \quad |f(\zeta)| > (1 - \varepsilon)M(|\zeta|, f).$$

Then there exists $q \in \mathbb{Z}$ such that $f = \pm g + q$.

Proof. Let $F(z) = \sin(\pi f(z))$. Then

$$T(r, F) \leq \log M(r, F) \leq cM(r, f), \quad (45)$$

denoting by c a positive constant, not necessarily the same at each occurrence. Applying the Borel lemma [5, p.38] gives (35), this time with $T(r) = T(r, F)$, in which the exceptional set E_0 has finite logarithmic measure. Without loss of generality both E_0 and the exceptional set E of Lemma 1.1 may be assumed to have empty intersection with H . This gives

$$n(r, 1/F) \leq cT(r)N(re^{1/T(r)}, 1/F) \leq cT(r, F)^2 \quad \text{for } r \in H. \quad (46)$$

Let η be small and positive, and for large $r \in H$ choose z_1 satisfying (8) and (10), with N the central index of g . Then (5) is satisfied, and (9) and (10) show that there exist at least $cM(r, g)$ points ζ satisfying (44). On combination with (45) and (46) this leads to

$$cM(r, g) \leq n(r, 1/F) \leq cT(r, F)^2 \leq cM(r, f)^2$$

from which it follows, applying the same argument with f and g interchanged, that

$$\log M(r, g) \leq c \log M(r, f) \leq c \log M(r, g) \quad \text{for } r \in H.$$

Applying Theorem 1.2 twice now gives entire functions G and G_1 , each mapping $\mathbb{Z}$ into $\mathbb{Z}$, such that $f = G \circ g$, $g = G_1 \circ f$. Here $G_1 = G^{-1}$ and so G is a linear function $G(\zeta) = p\zeta + q$. Since $G(0), G(1), G_1(0), G_1(1)$ are integers it follows that $q \in \mathbb{Z}$ and $p = \pm 1$. $\square$

The paper concludes with a simple result for meromorphic functions.

Theorem 6.2 *Let f and g be functions meromorphic on a plane domain U , such that g has a pole at $a \in U$ and $f(z) \in \mathbb{Z}$ whenever $z \in U$ and $g(z) \in \mathbb{N}$. Then f is a polynomial in g .*

Proof. Choose $V = B(a, r)$ such that f has no poles in $0 < |z - a| \leq r$. Then $w = g(z)$ maps a subdomain W of V conformally onto a half-plane $H = \{w \in \mathbb{C} : \operatorname{Re}(w) > M > 1\}$. Set $f(z) = F(g(z))$ for $z \in W$. Then there exists $N > 0$ such that $|F(w)| \leq |w|^N$ on H , and $F(w) \in \mathbb{Z}$ for $w \in H \cap \mathbb{N}$. By [7, Lemma 5], F is a polynomial. $\square$

The author thanks Gary Gundersen for useful correspondence and for drawing his attention to references [1] and [4], and the referee for some very helpful suggestions to improve the readability of the paper.

References

- [1] I.N. BAKER, Solution to Advanced Problem 6410, *Amer. Math. Monthly* 91, May 1984, 317.
- [2] A.O. GELFOND, *Transcendental and algebraic numbers* (Dover, New York, 1960).
- [3] G.G. GUNDERSEN, When two entire functions and also their first derivatives have the same zeros, *Indiana Univ. Math. J.* 30, No. 2 (1981), 293-303.
- [4] G.G. GUNDERSEN and S. OSBORN, Advanced Problem 6410, *Amer. Math. Monthly* 89, Dec. 1982, 787-788.
- [5] W.K. HAYMAN, *Meromorphic functions* (Clarendon Press, Oxford, 1964).
- [6] W.K. HAYMAN, The local growth of power series: a survey of the Wiman-Valiron method, *Canad. Math. Bull.* 17 (1974), 317-358.

- [7] J.K. LANGLEY, On second order linear differential polynomials, *Results Math.* 26 (1994), 51-82.
- [8] J.K. LANGLEY, Two results related to a question of Hinkkanen, *Kodai Math. J.* 19 (1996), 52-61.
- [9] J.K. LANGLEY, Quasiconformal modifications and Bank-Laine functions, *Arch. Math. (Basel)* 71 (1998), 233-239.
- [10] R. NEVANLINNA, *Eindeutige analytische Funktionen*, 2. Auflage, Grundlehren Math. Wiss. 46 (Springer, Berlin, 1953).
- [11] C.F. OSGOOD and C.C. YANG, On the quotient of two integral functions, *J. Math. Anal. Appl.* 54 (1976), 408-418.
- [12] J.M. WHITTAKER, *Interpolatory function theory*, Cambridge Tract No. 33 (Cambridge University Press 1935).

J.K. Langley
School of Mathematical Sciences
University of Nottingham
Nottingham, NG7 2RD

jkl@maths.nott.ac.uk