Integer points of analytic functions in a half-plane

Alastair N. Fletcher and J.K. Langley *

January 6, 2009

Abstract

It is shown that if f is an analytic function of sufficiently small exponential type in the right half-plane, which takes integer values on a subset of the positive integers having positive lower density, then f is a polynomial. MSC2000: 30D20, 30D35.

1 Introduction

A classical theorem of Pólya (see [14] and [21, p.55]) shows that 2^z is the slowest growing transcendental entire function which takes integer values at the non-negative integers. That is, let f be entire and take integer values on $\mathbb{N} \cup \{0\}$. Pólya shows that if

$$\limsup_{r \rightarrow \infty} \frac{M(r, f)}{2^r} < 1, \quad \text{where} \quad M(r, f) = \max\{|f(z)| : |z| = r\},$$

then f is a polynomial and, further, that if

$$M(r, f) = O(r^N 2^r) \tag{1}$$

as $r \rightarrow \infty$ for some $N > 0$, then there exist polynomials P_1 and P_2 such that $f(z) \equiv P_1(z)2^z + P_2(z)$. Further results on integer-valued entire functions may be found in [1, 2, 4, 6, 11, 12, 16, 17, 18, 19, 20].

*Both authors were supported by Engineering and Physical Sciences Research Council grant EP/D065321/1.

This article is concerned with similar results for analytic functions in a half-plane. It was proved in [8, Lemma 5] that if f is analytic of polynomial growth in the right half-plane and takes integer values at the positive integers, then f is a polynomial. This result has several applications to value distribution theory and differential equations [8, 9, 10]. In [13], an analogue of Pólya's result for a half-plane is given. That is, let f be analytic in the closed right half-plane $\Omega = \{z \in \mathbb{C} : \operatorname{Re}(z) \geq 0\}$ with maximum modulus

$$M_{\Omega}(r, f) = \max\{|f(z)| : z \in \Omega, |z| \leq r\}, \quad (2)$$

and assume that $f(n)$ is an integer for all sufficiently large positive integers n . If f satisfies (1) as $r \rightarrow \infty$ for some $N > 0$, with $M(r, f)$ replaced by $M_{\Omega}(r, f)$, then again there exist polynomials P_1 and P_2 with $f(z) \equiv P_1(z)2^z + P_2(z)$. Further, if f takes integer values at all the non-negative integers and

$$\limsup_{|z| \rightarrow \infty, z \in \Omega} \frac{|f(z)|}{2^{|z|}} < 1,$$

then f is a polynomial.

We remark that a result was proved in [22] for functions holomorphic on the product Ω^n of n half-planes and taking integer values on $\mathbb{N}^n$. This result contains [8, Lemma 5], but not the theorem from [13]. We are very grateful to the referee for drawing our attention to this reference and to others such as [1, 2, 23, 24].

In order to state our result some terminology will be required. Let f be analytic in Ω , and let $0 \leq \lambda < \infty$. Then f is of exponential type λ in Ω if

$$\limsup_{r \rightarrow \infty} \frac{\log^+ M_{\Omega}(r, f)}{r} = \lambda, \quad (3)$$

where $\log^+ x = \max\{0, \log x\}$ and $M_{\Omega}(r, f)$ is as in (2). This is of course in direct analogy with the definition of exponential type for entire functions. The main result to be proved is the following half-plane analogue of a theorem of Waldschmidt for entire functions [17].

Theorem 1.1. *Let d, J, λ satisfy*

$$0 < d < 1, \quad J \in \mathbb{N}, \quad \lambda > 0, \quad 16 \left(\frac{1 + \log(1 + J/2)}{J} \right) + 8(J-1)\lambda < d^2. \quad (4)$$

Let $E \subset \mathbb{N}$ have lower linear density

$$\underline{D}(E) = \liminf_{n \rightarrow \infty} \frac{|E \cap \{1, \dots, n\}|}{n} > d,$$

where $|X|$ denotes the number of elements of the set X . Let the function f be analytic of exponential type less than λ in the closed right half-plane Ω , and assume that $f(n) \in \mathbb{Z}$ for every $n \in E$. Then f is a polynomial.

2 Lemmas used in the proof of Theorem 1.1

2.1 Linear forms

The following lemma is a slight modification of [6, Lemma I, p.11]: a proof is given for completeness.

Lemma 2.1. *Let $A \geq 1$ and $N \geq 2$ be integers. Suppose that $L_1, \dots, L_m$ are linear forms in the n variables $x_1, \dots, x_n$, with real coefficients $a_{j,k}$ for $j = 1, \dots, m$ and $k = 1, \dots, n$, that is,*

$$L_j = a_{j,1}x_1 + \dots + a_{j,n}x_n.$$

Suppose further that $n > m$ and

$$\max_{j,k} |a_{j,k}| \leq A.$$

Then there exist integers $x_1, \dots, x_n$, not all zero, such that

$$|L_j| \leq \frac{1}{N}$$

for $j = 1, \dots, m$, and

$$|x_k| \leq 2 \cdot (2nAN)^{\frac{m}{n-m}}$$

for $k = 1, \dots, n$.

Proof. Define X by

$$X = \left\lceil (2nAN)^{\frac{m}{n-m}} \right\rceil,$$

where $[x]$ denotes the greatest integer not exceeding x . An n -tuple of integers $(x_1, \dots, x_n)$, in which each x_k has absolute value no greater than X ,

gives rise to a point $(L_1, \dots, L_m)$ lying in the closed m -dimensional cube of centre $(0, \dots, 0)$ and side length $2nAX$. Divide this cube into $(2nAXN)^m$ closed subcubes each of side length $1/N$. The number of distinct n -tuples $(x_1, \dots, x_n)$ is evidently

$$(2X + 1)^n \geq (2 \cdot (2nAN)^{\frac{m}{n-m}} - 1)^n > (2nAN)^{\frac{nm}{n-m}} \geq (2nAXN)^m,$$

since if this is not the case then we get

$$(2nAN)^n < (2nAXN)^{n-m} \leq (2nAN)^{n-m} \cdot (2nAN)^m,$$

which is impossible. Hence there are distinct n -tuples giving rise to points $(L'_1, \dots, L'_m)$ and $(L''_1, \dots, L''_m)$ lying in the same subcube. But then we may write

$$\left| \sum_{k=1}^n a_{j,k} (x'_k - x''_k) \right| = |L'_j - L''_j| \leq \frac{1}{N}$$

for $j = 1, \dots, m$, where

$$|x'_k - x''_k| \leq 2X \leq 2 \cdot (2nAN)^{\frac{m}{n-m}}$$

and $x_k = x'_k - x''_k \neq 0$ for at least one k . This completes the proof. $\square$

2.2 An application of the maximum principle

Lemma 2.2. *Let d, M, L, K satisfy*

$$0 < d < 1, \quad M > 0, \quad 1 < K < L < \infty, \quad ML^2K < d^2(L - K). \quad (5)$$

Let $G \subseteq \mathbb{N}$ and let F be analytic in the closed right half-plane Ω such that $F(z) \in \mathbb{Z}$ for all $z \in G$. Let $s > 0$ be such that $M_\Omega(Ls, F) \leq e^{MLs}$ and F has $m \geq ds$ distinct zeros in $G \cap [1, s]$. Then $F(z) = 0$ for all $z \in G \cap [s, Ks]$.

Proof. Let $x_1, \dots, x_m$ be distinct zeros of F in $G \cap [1, s]$. For $0 < x \leq s$ let

$$p(z) = p(z, x) = \frac{z - x}{z + x}.$$

Then p satisfies

$$|p(z)| = 1 \quad (z \in i\mathbb{R}), \quad |p(z)| \geq \frac{Ls - x}{Ls + x} \quad (|z| = Ls),$$

and

$$|p(z)| \leq \frac{Ks - x}{Ks + x} \quad (z \in [s, Ks] \subseteq \mathbb{R}),$$

the last estimate following from monotonicity. Next, let

$$g(x) = \log \left[\left(\frac{Ls + x}{Ls - x} \right) \left(\frac{Ks - x}{Ks + x} \right) \right].$$

Then, for $0 \leq x \leq s$,

$$\begin{aligned} g'(x) &= \frac{1}{Ls + x} + \frac{1}{Ls - x} - \frac{1}{Ks + x} - \frac{1}{Ks - x} \\ &= \frac{2Ls}{L^2s^2 - x^2} - \frac{2Ks}{K^2s^2 - x^2} \\ &= \frac{2s^3KL(K - L) + 2x^2s(K - L)}{(L^2s^2 - x^2)(K^2s^2 - x^2)} \\ &\leq \frac{2(K - L)}{LKs} \end{aligned}$$

and hence

$$g(x) \leq \frac{2x(K - L)}{LKs}.$$

The function

$$F_1(z) = F(z) \prod_{j=1}^m \frac{1}{p(z, x_j)}$$

is analytic in Ω and satisfies

$$|F_1(z)| \leq M_\Omega(Ls, F) \prod_{j=1}^m \frac{Ls + x_j}{Ls - x_j}$$

on the boundary of the region given by $z \in \Omega, |z| \leq Ls$, and this estimate also holds for $z \in [s, Ks]$, by the maximum principle. For $z \in [s, Ks]$ it

therefore follows that

$$\begin{aligned}
|F(z)| &\leq M_\Omega(Ls, F) \prod_{j=1}^m \left[\left(\frac{Ls + x_j}{Ls - x_j} \right) \left(\frac{Ks - x_j}{Ks + x_j} \right) \right] \\
&= M_\Omega(Ls, F) \exp \left(\sum_{j=1}^m g(x_j) \right) \\
&\leq M_\Omega(Ls, F) \exp \left(\frac{2(K-L)}{LKs} \sum_{j=1}^m x_j \right) \\
&\leq M_\Omega(Ls, F) \exp \left(\frac{2(K-L)}{LKs} \cdot \frac{m(m+1)}{2} \right) \\
&\leq \exp \left(MLs + \frac{d^2(K-L)s}{LK} \right) < 1,
\end{aligned}$$

using (5) and the fact that the x_j are distinct positive integers, which proves the lemma. $\square$

In order to apply Lemma 2.2, it is necessary for a given d to choose M , L and K with (5) in mind. Evidently if

$$ML^2 < d^2(L-1)$$

then K may be chosen with $K-1$ small and positive so that (5) is satisfied. Since elementary calculus gives

$$q(L) = \frac{L-1}{L^2} \leq q(2) = \frac{1}{4}$$

for $1 < L < \infty$, the appropriate condition is $4M < d^2$.

Lemma 2.3. *Let $0 < d < 1$ and $0 < 4M < d^2$. Let $G \subseteq \mathbb{N}$ and let F be analytic in Ω such that $F(z) \in \mathbb{Z}$ for all $z \in G$. Let $S > 0$ be such that*

$$Q(r) = |G \cap [0, r]| \geq dr \quad \text{and} \quad M_\Omega(r, F) \leq e^{Mr}$$

for all $r \geq S$, and assume that $F(z) = 0$ for all z in $G \cap [1, S]$. Then $F(z) = 0$ for all $z \in G$.

Proof. Choose $L = 2$ and $K \in (1, 2)$ such that (5) is satisfied. Then F has at least dS distinct zeros in $G \cap [1, S]$. Applying Lemma 2.2 with $s = S$ then shows that $F(z) = 0$ for all $z \in G \cap [S, KS]$, from which it follows at once that F has at least $Q(KS) \geq dKS$ distinct zeros in $[1, KS]$. Hence Lemma 2.2 may again be applied, this time with $s = KS$. Repetition of this argument proves Lemma 2.3. $\square$

2.3 The Nevanlinna characteristic in a half-plane

This section provides a brief overview of a half-plane characteristic analogous to the Nevanlinna characteristic in the plane, the details of which may be found in [7, p.38]. Let f be meromorphic in the closed upper half-plane

$$\overline{\mathbb{H}} = \{z \in \mathbb{C} : \operatorname{Im}(z) \geq 0\}$$

with poles at $\rho_n e^{i\psi_n}$, where $\rho_n \geq 0$ and $0 \leq \psi_n \leq \pi$. The counting function of the poles is

$$c(r, f) = \sum_{1 < \rho_n \leq r} \sin \psi_n,$$

and the integrated counting function takes the form

$$C(r, f) = 2 \int_1^r c(t, f) \left(\frac{1}{t^2} + \frac{1}{r^2} \right) dt = 2 \sum_{1 < \rho_n \leq r} \left(\frac{1}{\rho_n} - \frac{\rho_n}{r^2} \right) \sin \psi_n.$$

The analogue of the Nevanlinna proximity function consists of the following two functions:

$$\begin{aligned} A(r, f) &= \frac{1}{\pi} \int_1^r \left(\frac{1}{t^2} - \frac{1}{r^2} \right) [\log^+ |f(t)| + \log^+ |f(-t)|] dt; \\ B(r, f) &= \frac{2}{\pi r} \int_0^\pi \log^+ |f(re^{i\phi})| \sin \phi d\phi. \end{aligned}$$

The half-plane characteristic is then given by

$$S(r, f) = A(r, f) + B(r, f) + C(r, f)$$

and satisfies, for non-constant f and $a \in \mathbb{C}$,

$$S\left(r, \frac{1}{f-a}\right) = S(r, f) + O(1) \tag{6}$$

as $r \rightarrow \infty$.

The following lemma uses the half-plane characteristic and is in the spirit of Carlson's theorem [5]. For generalisations in other directions see [23, 24].

Lemma 2.4. *Let $E \subseteq i\mathbb{N} = \{i, 2i, \dots\}$ have lower density D . Let f be analytic in $\overline{\mathbb{H}}$, of exponential type $\lambda < \pi D$, with $f(z) = 0$ for all $z \in E$. Then $f \equiv 0$.*

Here the lower density of E and exponential type relative to the upper half-plane are defined in straightforward analogy with §1.

Proof. Assume that f is not identically zero. As $r \rightarrow \infty$,

$$B(r, f) \leq \frac{2}{\pi r} \int_0^\pi (\lambda + o(1)) r \sin \phi \, d\phi = O(1)$$

and

$$A(r, f) \leq \frac{1}{\pi} \int_1^r \left(\frac{1}{t^2} - \frac{1}{r^2} \right) 2(\lambda + o(1)) t \, dt + O(1) \leq \frac{2(\lambda + o(1))}{\pi} \log r.$$

Since f has no poles in $\overline{\mathbb{H}}$ applying (6) with $a = 0$ now gives

$$S(r, 1/f) \leq A(r, f) + B(r, f) + O(1) \leq \frac{2(\lambda + o(1))}{\pi} \log r. \quad (7)$$

But since the lower density of E is D we have

$$c(r, 1/f) \geq \sum_{n \in \mathbb{N} \cap (1, r], in \in E} 1 \geq (D - o(1))r$$

as $r \rightarrow \infty$. Integrating this yields

$$S(r, 1/f) \geq C(r, 1/f) \geq 2 \int_1^r (D - o(1)) t \left(\frac{1}{t^2} + \frac{1}{r^2} \right) dt \geq 2(D - o(1)) \log r$$

as $r \rightarrow \infty$, which on combination with (7) gives $\lambda \geq \pi D$, a contradiction. Therefore f must be identically zero. $\square$

2.4 A class of polynomials

The following lemma summarises some basic facts from [17] concerning a class of polynomials which are key to the proof of Theorem 1.1.

Lemma 2.5. *Define polynomials $p_0, p_1, \dots$ by*

$$p_0(z) = 1, \quad p_1(z) = z, \quad p_h(z) = \frac{z(z-1)\dots(z-h+1)}{h!} \quad (h = 2, 3, \dots). \quad (8)$$

Then $p_h(\mathbb{Z}) \subseteq \mathbb{Z}$ and for $R > 0$ and $H \in \mathbb{N}$ we have

$$|p_h(z)| \leq e^H \left(\frac{R}{H} + 1 \right)^H \quad \text{for } |z| \leq R, \quad h = 0, \dots, H. \quad (9)$$

Proof. It is easy to check that $p_h(\mathbb{Z}) \subseteq \mathbb{Z}$. To prove (9) write, following [17],

$$|p_h(z)| \leq \frac{(R+H)^h}{h!} \leq \frac{H^h}{h!} \left(\frac{R}{H} + 1 \right)^H \leq e^H \left(\frac{R}{H} + 1 \right)^H.$$

□

2.5 Algebraic functions mapping integers to integers

Proposition 2.6. *Let the algebraic function f be analytic in Ω and satisfy $f(E) \subseteq \mathbb{Z}$ for some set $E \subseteq \mathbb{N}$ of positive lower density. Then f is a polynomial.*

To prove Proposition 2.6, let E and f be as in the hypotheses, and assume that the lower density of E exceeds $D > 0$. We assert that f maps the positive real axis into $\mathbb{R}$. To see this observe that the functions $\overline{f(\bar{z})}$ and $f(z) - \overline{f(\bar{z})}$ are algebraic because f is. Since $f(z) \in \mathbb{R}$ for $z \in E$ and since an algebraic function having a sequence of zeros tending to infinity must vanish identically, the assertion follows.

Again since f is algebraic there exists a positive integer m such that, for all sufficiently large r ,

$$M_\Omega(r, f) \leq r^m. \quad (10)$$

Let n and N be integers with n/m and N/n large, and in particular with

$$DN \geq n + 1. \quad (11)$$

Lemma 2.7. *There exist arbitrarily large $r \in \mathbb{N}$ such that*

$$|E \cap \{r, r+1, \dots, r+N-1\}| \geq n+1. \quad (12)$$

Proof. Assume that there exists $p_0 \in \mathbb{N}$ such that, for every $p \geq p_0$,

$$|E \cap \{Np, \dots, N(p+1)-1\}| \leq n.$$

Since the lower density of E exceeds D this gives, for large p ,

$$DNp \leq |E \cap \{1, \dots, Np\}| \leq (p - p_0)n + O(1) \leq (n + o(1))p,$$

which contradicts (11). $\square$

Let ε be small and positive and choose a large positive integer r satisfying (12). Let $\Gamma = \Gamma_r$ be the circle of centre r , radius εr , described once counter-clockwise. Choose distinct

$$a_0, \dots, a_n \in E \cap \{r, r+1, \dots, r+N-1\}. \quad (13)$$

Then $a_0, \dots, a_n$ lie inside Γ , since r is large.

For $k = 0, \dots, n$ it follows from Cauchy's integral formula and the identity

$$\frac{1}{t-z} = \frac{1}{t-a_0} + \frac{z-a_0}{(t-a_0)(t-a_1)} + \dots + \frac{(z-a_0)\dots(z-a_k)}{(t-a_0)\dots(t-a_k)(t-z)},$$

which is easily proved by induction, that

$$f(z) = P_k(z) + \frac{1}{2\pi i} \int_{\Gamma} \frac{(z-a_0)\dots(z-a_k) f(t) dt}{(t-a_0)\dots(t-a_k)(t-z)} \quad (14)$$

for z inside Γ , where

$$P_k(z) = A_0 + A_1(z-a_0) + \dots + A_k(z-a_0)\dots(z-a_{k-1}) \quad (15)$$

is given by

$$A_j = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t) dt}{(t-a_0)\dots(t-a_j)}. \quad (16)$$

Thus $P_k(z)$ is the interpolating polynomial of degree at most k which equals $f(z)$ at the $k+1$ points $a_0, \dots, a_k$ [6, p.103].

Next, let

$$Q = \prod_{0 \leq j < k \leq n} |a_k - a_j| \leq C = N^{(n+1)^2}, \quad (17)$$

and observe that C is independent of r . Since $f(a_j) \in \mathbb{Z}$ for $j = 0, \dots, n$ it follows from (16) and the residue theorem that

$$QA_j \in \mathbb{Z} \quad \text{for } j = 0, \dots, n. \quad (18)$$

On the other hand since r is large (13) gives

$$|t - a_j| \geq \frac{\varepsilon r}{2}$$

for $t \in \Gamma$. Thus combining (10), (16) and (17) yields for $m < j \leq n$, again since r is large,

$$|QA_j| \leq \frac{C(\varepsilon r)(2r)^m}{(\varepsilon r/2)^{j+1}} < \frac{1}{2},$$

which in conjunction with (18) gives $A_j = 0$.

Recalling (14) and the definition (15) of P_k it now follows that $P_m = P_n$ and that $f - P_m$ has $n + 1$ zeros $a_0, \dots, a_n$ in the interval $[r, r + N - 1]$. Hence $f^{(n)} = (f - P_m)^{(n)}$ has a zero in the same interval, using Rolle's theorem. Since r may be chosen arbitrarily large the algebraic function $f^{(n)}$ must vanish identically, and f is a polynomial. This proves Proposition 2.6. $\square$

3 Proof of Theorem 1.1

Let $E \subseteq \mathbb{N}$ and d, J, λ, f be as in the hypotheses. Label the elements of E as $1 \leq \alpha_1 < \alpha_2 < \dots$. Let R be a large positive integer such that

$$H = \frac{n}{J} \in \mathbb{N}, \quad \text{where } m = |E \cap [1, R]| \quad \text{and} \quad n = 2m. \quad (19)$$

Form the functions

$$q_{\mu, \nu}(z) = p_{\mu}(z)f(z)^{\nu}, \quad \mu = 0, 1, \dots, H - 1, \quad \nu = 0, 1, \dots, J - 1, \quad (20)$$

where p_{μ} is defined as in (8). This gives $HJ = n$ functions which we label $g_1, \dots, g_n$, where

$$g_k(z) = p_{\mu(k)}(z)f(z)^{\nu(k)}.$$

In order to prove Theorem 1.1, it suffices to show that the functions $g_1, \dots, g_n$ are linearly dependent over $\mathbb{C}$. Once such a relation

$$\sum_{k=1}^n B_k g_k(z) \equiv 0$$

is established with the B_k constants, not all zero, then it cannot be the case that there is an integer q such that $B_k \neq 0$ implies $\nu(k) = q$, because p_h has degree h in (8). Hence it follows that f is algebraic, and Proposition 2.6 shows that f is a polynomial.

In order to prove that the g_k are linearly dependent, observe first that

$$a_{j,k} = g_k(\alpha_j) \in \mathbb{Z},$$

using Lemma 2.5. Moreover, we have, for $j = 1, \dots, m$,

$$\begin{aligned} |a_{j,k}| &\leq e^H \left(\frac{R}{H} + 1 \right)^H (1 + M_\Omega(R, f))^{J-1} \\ &\leq e^H \left(\frac{R}{H} + 1 \right)^H e^{(J-1)\lambda R} = J(R) \leq A = [J(R)] + 1, \end{aligned} \quad (21)$$

by (9), (20) and the fact that R is large. Applying Lemma 2.1 with $N = 2$ yields integers $A_1, \dots, A_n$, not all zero, such that

$$\sum_{k=1}^n A_k g_k(\alpha_j) = 0 \quad (22)$$

for $j = 1, \dots, m$, and

$$|A_k| \leq 8nA, \quad (23)$$

since $n = 2m$. Set

$$F(z) = \sum_{k=1}^n A_k g_k(z). \quad (24)$$

Lemma 3.1. *Choose a real number M with*

$$4 \left(\frac{1 + \log(1 + J/2)}{J} \right) + 2(J-1)\lambda < M < \frac{d^2}{4}, \quad (25)$$

using (4). Provided R was chosen large enough we have

$$|E \cap [1, r]| \geq dr \quad \text{and} \quad \log^+ M_\Omega(r, F) \leq Mr \quad \text{for } r \geq R. \quad (26)$$

Proof. The first inequality of (26) holds provided R was chosen large enough, since E has lower density greater than d . Let c denote positive constants which do not depend on r or R . Then we have

$$\begin{aligned} M_{\Omega}(r, F) &\leq 8n^2 A e^H \left(\frac{r}{H} + 1 \right)^H (1 + M_{\Omega}(r, f))^{J-1} \\ &\leq c r^2 e^{2H} \left(\frac{r}{H} + 1 \right)^{2H} e^{2(J-1)\lambda r} \end{aligned}$$

using (9), (21), (23) and the fact that R is large. Now (19) gives

$$\frac{r}{H} \geq \frac{R}{H} = \frac{RJ}{n} \geq \frac{J}{2}.$$

Since the function

$$\frac{1 + \log(x + 1)}{x}$$

is decreasing for $x > 0$ this yields, for $r \geq R$,

$$\begin{aligned} \log^+ M_{\Omega}(r, F) &\leq 2r \left(\frac{H}{r} \right) \left(1 + \log \left(\frac{r}{H} + 1 \right) \right) + 2(J-1)\lambda r + O(\log r) \\ &\leq 4r \left(\frac{1 + \log(1 + J/2)}{J} \right) + 2(J-1)\lambda r + O(\log r) < Mr \end{aligned}$$

provided R was chosen large enough. $\square$

The function F satisfies $F(z) \in \mathbb{Z}$ for all $z \in E$, and $F(z) = 0$ for all $z \in E \cap [1, R]$ by (22) and (24). It then follows from (25), (26) and Lemma 2.3, with $S = R$ and $G = E$, that $F(z) = 0$ for all $z \in E$. But (25) also gives

$$4M < d^2 < d, \quad M < \pi d,$$

and so (26) and Lemma 2.4, applied to the function $F(-iz)$, show that $F(z)$ vanishes identically, which completes the proof of Theorem 1.1. $\square$

References

- [1] J.-P. Bézivin, Sur les points où une fonction analytique prend des valeurs entières, *Ann. Inst. Fourier (Grenoble)* 40 (1990), 785-809.
- [2] J.-P. Bézivin, Suites d'entiers et fonctions entières arithmétiques, *Ann. Fac. Sci. Toulouse Math. (6)* 3 (1994), 313-334.

- [3] R.P. Boas Jr., *Entire functions*, (1954) Academic Press Inc., New York.
- [4] R.C. Buck, Integral valued entire functions, *Duke Math. J.* 15 (1948), 879–891.
- [5] W.H.J. Fuchs, A generalization of Carlson’s theorem, *J. London Math. Soc.* **21** (1946), 106–110.
- [6] A.O. Gelfond, *Transcendental and Algebraic Numbers* (1960) Dover Publications, New York.
- [7] A.A. Gol’dberg and I.V. Ostrovskii, *Distribution of values of meromorphic functions*, (1970) Nauka, Moscow.
- [8] J.K. Langley, On second order linear differential polynomials, *Results Math.* 26 (1994), 51–82.
- [9] J.K. Langley, Two results related to a question of Hinkkanen, *Kodai Math. J.* 19 (1996), 52–61.
- [10] J.K. Langley, Quasiconformal modifications and Bank-Laine functions, *Arch. Math. (Basel)* 71 (1998), 233–239.
- [11] J.K. Langley, Integer points of meromorphic functions, *Comput. Methods Funct. Theory* 5 (2005), 253–262.
- [12] J.K. Langley, Integer points of entire functions, *Bulletin London Math. Soc.* 38 (2006), 239–249.
- [13] J.K. Langley, Integer-valued analytic functions in a half-plane, *Comput. Methods Funct. Theory* 7 (2007), 433–442.
- [14] G. Pólya, Über ganze ganzwertige Funktionen, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen* (1921), 1–10.
- [15] Ch. Pommerenke, *Boundary behaviour of conformal maps*, (1992) Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 299. Springer-Verlag, Berlin.
- [16] R.M. Robinson, Integer-valued entire functions, *Trans. Amer. Math. Soc.* 153 (1971), 451–468.
- [17] M. Waldschmidt, Integer valued entire function on Cartesian products, *Number Theory in Progress* Vol. 1 (Zakopane-Koscielisko 1997), 553–576, de Gruyter, Berlin, 1999.
- [18] R.V. Walliser, On entire functions assuming integer values in a geometric sequence, *Theorie des Nombres* (Quebec 1987), 981–989, de Gruyter, Berlin, 1989.
- [19] M. Welter, Ensembles régulièrement lacunaires d’entier et fonctions entières arithmétiques, *J. Number Theory* 109 (2004), 163–181.
- [20] M. Welter, A new class of integer-valued entire functions, *J. reine Angew. Math.* 583 (2005), 175–192.

- [21] J.M. Whittaker, *Interpolatory function theory*, (1935) Cambridge Tract No.33, Cambridge University Press.
- [22] K. Yoshino, Pólya's theorem for nonentire functions, *Rend. Circ. Mat. Palermo (2)* (1984) Suppl. No. 3, 385-395.
- [23] K. Yoshino, On Carlson's theorem for holomorphic functions, *Algebraic Analysis* Vol. II, 943-950, Academic Press, Boston MA, 1988.
- [24] K. Yoshino, Arithmetic holomorphic functions of exponential type on the product of halfplanes, *Tokyo Math. J.* 18 (1995), no. 1, 147-150.

School of Mathematical Sciences, University of Nottingham, NG7 2RD.