The Wiman conjecture

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1 Introduction

1.1 Fact

If $P(z) = \alpha \prod_{j=1}^n (z - a_j)$ is a real polynomial with only real zeros a_j then P' has real zeros.

This follows from Rolle's theorem, or from the formula

$$\frac{P'(z)}{P(z)} = \sum_{j=1}^n \frac{1}{z - a_j}, \tag{1}$$

which shows that $\mathrm{Im}(P'/P)$ and $\mathrm{Im}\, z$ have opposite signs.

1.2 The Laguerre-Pólya class

The Laguerre-Pólya class LP consists of all entire functions f with the following property: there exists a sequence (P_n) of real polynomials with only real zeros such that $P_n \to f$ l.u. on $\mathbb{C}$. This means that for every compact $K \subseteq \mathbb{C}$,

$$\lim_{n\to\infty} (\sup\{|P_n(z) - f(z)| : z \in K\}) = 0. \tag{2}$$

Examples include

$$\exp(-z^2) = \lim_{n\to\infty} (1 - z^2/n)^n,$$

but not, as we will see, $\exp(z^2)$.

1.3 Facts about LP

We have:

(a) If $f \in LP$ then f is real (i.e. $f(\mathbb{R}) \subseteq \mathbb{R}$), and if $f \not\equiv 0$ then f has only real zeros (by Hurwitz' theorem).

(b) If $f \in LP$ then $f' \in LP$ (by Weierstrass' theorem).

(c) If $f \in LP$ and f is transcendental then $f^{(k)}$ has only real zeros, for every $k \geq 0$ (by (a) and (b)).

Since $g(z) = \exp(z^2)$ has

$$g''(z)/g(z) = 4z^2 + 2,$$

we see that $g \notin LP$.

1.4 The conjectures of Pólya and Wiman

Wiman conjectured around 1911 that if f is a real entire function and f and f'' have only real zeros then $f \in LP$. A weaker conjecture of Pólya asked whether $f \in LP$ whenever f is a real entire function such that $f^{(k)}$ has only real zeros, for every $k \geq 0$.

1.5 The class U_0

This consists of all entire functions f having a product representation

$$f(z) = e^{-az^2+bz+c} z^n \prod (1 - z/a_j) e^{z/a_j}, \tag{3}$$

in which a, b, c, a_j are real, with $a \geq 0, a_j \neq 0$. The product may have finitely many or infinitely many terms, but in the latter case the zeros a_j of f in $\mathbb{C} \setminus \{0\}$ (in which repetition is allowed) are assumed to satisfy $\sum 1/a_j^2 < \infty$.

Since

$$e^{-az^2} = \lim_{n\to\infty} (1 - az^2/n)^n, \quad e^{bz} = \lim_{n\to\infty} (1 + bz/n)^n,$$

we see that $U_0 \subseteq LP$.

The *Laguerre-Pólya theorem* asserts that $LP = U_0$. To prove this we need to set up some machinery.

2 The Krein class

The Krein class consists of all real meromorphic functions f in the plane (here real means that $f(\mathbb{R}) \subseteq \mathbb{R} \cup \{\infty\}$) with the property that $f(H) \subseteq H$, where H is the upper half-plane $H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. It follows that such f also map the lower half-plane into itself.

If $f \in LP$ and f'/f is constant then obviously $f \in U_0$. Suppose now that $f \in LP$ and f'/f is non-constant. If $P_n \to f$ and a is not a zero of f then $P_n'(a)/P_n(a) \to f'(a)/f(a)$, and we deduce from (1) that $-f'/f$ is in the Krein class. Note that the example

$$P(z) = z(z - \varepsilon), \quad f(z) = z^2, \quad P_n'(-\varepsilon/2) = 0,$$

shows that we cannot in general conclude that $P_n'/P_n \to f'/f$ l.u. in $\mathbb{C}$, not even with respect to the spherical metric.

2.1 Properties of the Krein class

Let F be in the Krein class. Then all zeros and poles of F are real. Further, all zeros and poles are simple, poles have negative residues, and zeros b have $F'(b) > 0$.

Consideration of the graph of F shows that between distinct poles of F lies a zero of F, and in any interval (c, d) not containing poles of F there is at most one zero of F.

To develop further properties of functions in the Krein class, it simplifies matters if we assume that the set of poles a_k of F is unbounded above and below, that $a_0 = 0$ and $a_k < a_{k+1}$.

2.2 Theorem

Let F be real meromorphic with real poles, its set of poles a_k unbounded above and below, and such that $a_k < a_{k+1}, a_0 = 0$. Then F is in the Krein class if and only if

$$F(z) = c\frac{b_0 - z}{a_0 - z}\prod_{k\neq 0}\frac{1 - z/b_k}{1 - z/a_k}, \tag{4}$$

in which $c > 0$ and $a_k < b_k < a_{k+1}$.

Proof. First we note that with a_k, b_k as given the sum $\sum_{k>0}(1/a_k - 1/b_k)$ converges by the alternating series test. If $|z| \leq M < a_k/2$ then

$$\left|\frac{1 - z/b_k}{1 - z/a_k} - 1\right| \leq 2M(1/a_k - 1/b_k).$$

Thus the inequality

$$|\log(1 + z)| \leq 2|z|, \quad |z| \leq \frac{1}{2}$$

implies that

$$\sum_{a_k > 2M}\log\left(\frac{1 - z/b_k}{1 - z/a_k}\right)$$

converges absolutely and uniformly for $|z| \leq M$. Applying similar reasoning for $b_k < -2M$ we see that the product does converge.

Now for F as in (4) we have $b_k/a_k > 0$ for $k \neq 0$. Hence for $z \in H$ we have

$$\arg F(z) = \sum(\arg(b_k - z) - \arg(a_k - z)) = \sum\theta_k \in (0, \pi),$$

where θ_k is the angle between the lines from z to b_k, a_k respectively. This proves that F is in the Krein class.

Conversely, given F satisfying the hypotheses and in the Krein class, let ψ be the product in (4). Then $F = \psi e^g$, with g entire. But then $|\mathrm{Im}\, g(z)| = |\arg F(z)/\psi(z)| \leq 2\pi$ for $\mathrm{Im}\, z \neq 0$, from which we see that g is constant.

2.3 Theorem

Let F be as in (4), with $a_k < b_k < a_{k+1}$, $a_0 = 0$ and the set of a_k unbounded above and below. Then F has a convergent partial fractions representation

$$F(z) = Az + B - \frac{d}{z} + \sum_{k\neq 0} A_k\left(\frac{1}{a_k - z} - \frac{1}{a_k}\right), \tag{5}$$

in which $B \in \mathbb{R}$ and $A \geq 0$, $d \geq 0$, $A_k \geq 0$ and

$$\sum_{k\neq 0}\frac{A_k}{a_k^2} < \infty. \tag{6}$$

Note that if $F = -f'/f$, where f is a real entire function with real zeros, then d and the A_k are positive integers, and integrating (5) gives the product representation (3). This therefore completes the proof of the Laguerre-Pólya theorem, at least in the case where the set of zeros of f is unbounded above and below. The general case is proved the same way, but with modifications to the product F.

Proof of the theorem. We have, assuming WLOG that $c = 1$,

$$F(z) = \lim_{n\to\infty} F_n(z), \quad F_n(z) = \frac{b_0 - z}{a_0 - z} \prod_{0<|k|\le n} \frac{1 - z/b_k}{1 - z/a_k}. \tag{7}$$

Each F_n is a rational function, with $F_n(\infty) \in \mathbb{R}$, and the same proof as in Theorem 2.2 gives $F_n(H) \subseteq H$. Thus F_n has negative residues, and we can write

$$F_n(z) = c_n - \frac{d_n}{z} + \sum_{0<|k|\le n} A_{k,n}\left(\frac{1}{a_k - z} - \frac{1}{a_k}\right), \tag{8}$$

with c_n real, and $d_n \ge 0$, $A_{k,n} \ge 0$. As $n \to \infty$, we have

$$d_n \to d = -\text{Res}(F, 0), \quad A_{k,n} \to A_k = -\text{Res}(F, a_k). \tag{9}$$

Also, if ε is small and positive then integrating $F_n(z)z^{-2}$ and $F(z)z^{-2}$ around $|z| = \varepsilon$ shows that

$$\lim_{n\to\infty} \sum_{0<|k|\le n} \frac{A_{k,n}}{a_k^2} = \text{Res}(F(z)z^{-2}, 0) \in \mathbb{C}.$$

Hence there exists $M > 0$ such that

$$\sum_{0<|k|\le n} \frac{A_{k,n}}{a_k^2} \le M, \quad n \in \mathbb{N}. \tag{10}$$

Fix N. Then since $A_{k,n} \ge 0$ we have, as $n \to \infty$, by (10),

$$M \ge \sum_{0<|k|\le n} \frac{A_{k,n}}{a_k^2} \ge \sum_{0<|k|\le N} \frac{A_{k,n}}{a_k^2} \to \sum_{0<|k|\le N} \frac{A_k}{a_k^2},$$

using (9). Letting $N \to \infty$ we deduce (6). This now means that we can write

$$F(z) = P(z) - \frac{d}{z} + \sum_{k\ne 0} A_k\left(\frac{1}{a_k - z} - \frac{1}{a_k}\right), \tag{11}$$

with P entire, and we need to show that $P(z) = Az + B$, $A \ge 0$, $B \in \mathbb{R}$. Our first step is to show that $\text{Im}\, P(z) \ge 0$ for $z \in H$.

To this end, let $z \in H$ and write

$$P(z) = F(z) + \frac{d}{z} - \sum_{k\ne 0} A_k\left(\frac{1}{a_k - z} - \frac{1}{a_k}\right) = \lim_{N\to\infty} Q_N(z),$$

where

$$Q_N(z) = F(z) + \frac{d}{z} - \sum_{0<|k|\leq N} A_k \left(\frac{1}{a_k - z} - \frac{1}{a_k} \right).$$

But

$$Q_N(z) = \lim_{n\to\infty} q_n(z),$$

in which

$$q_n(z) = F_n(z) + \frac{d_n}{z} - \sum_{0<|k|\leq N} A_{k,n} \left(\frac{1}{a_k - z} - \frac{1}{a_k} \right) = c_n + \sum_{N<|k|\leq n} A_{k,n} \left(\frac{1}{a_k - z} - \frac{1}{a_k} \right),$$

using (8)and (9). Clearly $q_n(z) \in H$. Letting $n \to \infty$ we see that $Q_N(z)$ has non-negative imaginary part, and letting $N \to \infty$ the same is true of $P(z)$.

It follows from the argument principle that P has at most one zero, necessarily real, and we can write either

$$P(z) = e^{Q(z)}, \quad P(z) = (z - x_0)e^{Q(z)},$$

with Q entire. As in the proof of Theorem 2.2 we see that $\operatorname{Im} Q(z)$ is bounded, and so Q is constant.

3 Proving the Wiman conjecture, first part

We recall that Wiman's conjecture was the following: if f is a real entire function and f and f'' have only real zeros then $f \in LP$. We list here a number of the main highlights in the history of this conjecture.

(i) Levin and Ostrovskii 1960: *if f is real entire and f and f'' have only real zeros then*

$$\log^+ \log^+ M(r, f) = O(r \log r), \quad r \to \infty.$$

(ii) Hellerstein and Williamson 1977: *if f is real entire and f, f', f'' all have only real zeros then $f \in LP$* (thus proving Pólya's conjecture).

(iii) Sheil-Small 1989: *the Wiman conjecture is true for functions of finite order ρ i.e.*

$$\rho = \rho(f) = \limsup_{r\to\infty} \frac{\log^+ \log^+ M(r, f)}{\log r} < \infty.$$

(iv) Edwards and Hellerstein 2002: *if f is a real entire function of finite order and f and $f^{(k)}$ have only real zeros, for some $k \geq 2$, then $f \in LP$.*

(v) Bergweiler, Eremenko and Langley 2002: *if f is a real entire function of infinite order then ff'' has infinitely many non-real zeros.* Together with (iii) this proves the Wiman conjecture in full.

The result (v) is false without the assumption that f is real: for example $f(z) = \exp(e^{iz})$

is such that f and f'' have only real zeros, but $f \notin U_0 = LP$. Also the example $\exp(\sin z)$ shows that we can't replace f'' by f' in (v). It is not known whether f'' can be replaced by $f^{(k)}, k \geq 3$, in (v).

The main ideas used come from (i) and (iii) and we will sketch some of these. We assume henceforth that f is a real entire function such that f and f'' have only real zeros.

3.1 The Levin-Ostrovskii representation

A method due to Levin and Ostrovskii gives a factorisation for $L = f'/f$ of form

$$L(z) = \phi(z)\psi(z). \tag{12}$$

where ϕ is a real entire function and either $\psi \equiv 1$ or ψ is in the Krein class. To achieve this the function ψ is defined as follows.

First, if a is the greatest real zero of f let

$$p_a(z) = \frac{1}{a - z}.$$

Here if $z \in H$ then $\arg p_a(z)$ is the angle θ_a between the line from z to a and the horizontal line extending rightwards from z.

Now suppose that a is a zero of f, but not the greatest zero. Let a' be the least zero of f in (a, ∞) and choose a zero b of f' lying in (a, a'). Then set

$$p_a(z) = \frac{b - z}{a - z} \quad (ab \leq 0), \quad p_a(z) = \frac{1 - z/b}{1 - z/a} \quad (ab > 0).$$

Note that if $ab > 0$ then

$$\frac{b - z}{a - z}, \quad \frac{1 - z/b}{1 - z/a}$$

have the same argument for all $z \in H$, and that there is at most one zero a for which $ab \leq 0$. Now for $z \in H$ we find that $\arg p_a(z)$ is the (positive) angle between the line from z to a and the line from z to b.

The function ψ is then the product of all the factors p_a over all zeros a of f, and is 1 if f has no zeros. When f has infinitely many zeros the product converges by the alternating series test. If f has at least one zero, then the sum of the terms $\arg p_a(z)$ is in $(0, \pi)$ for $z \in H$, and so ψ is in the Krein class.

3.2 A simplifying assumption

For simplicity assume that the set $\{a_k\}$ of zeros of f is unbounded above and below (the arguments in the contrary case are similar but in some parts more complicated), labelled such that that $a_k < a_{k+1}, a_0 = 0$. Note that each a_k is a pole of L, the residue a positive integer. Then ψ is in the Krein class, mapping H into H, and its poles a_k have negative residues. Further, ϕ is entire, and both ϕ and ψ are real.

If ϕ is constant then it must be real and negative (look at the residues of L and ψ), and this implies that $-f'/f$ is in the Krein class and so $f \in U_0 = LP$. We assume henceforth that ϕ is non-constant.

3.3 An inequality for ψ

For $w = re^{i\theta}$, with $r \geq 1$ and $0 < \theta < \pi$ we have

$$\frac{|\psi(i)|\sin\theta}{4r} \leq |\psi(w)| \leq \frac{4|\psi(i)|r}{\sin\theta}. \tag{13}$$

This is proved via Schwarz' lemma. For $b, w \in H$ the transformation

$$z = \frac{w-b}{w-\overline{b}}$$

maps H onto $B(0,1)$ and b to 0. Set

$$z = \frac{w-i}{w+i}, \quad h(z) = \frac{\psi(w)-\psi(i)}{\psi(w)-\overline{\psi(i)}}.$$

Then h maps $B(0,1)$ into itself, and 0 to 0. Thus $|h(z)| \leq |z|$, which gives

$$|\psi(w)| = \left|\frac{\psi(i)-h(z)\overline{\psi(i)}}{1-h(z)}\right| \leq |\psi(i)|\frac{1+|z|}{1-|z|} \leq \frac{4|\psi(i)|}{1-|z|^2}.$$

But, for $r \geq 1$,

$$1-|z|^2 = \frac{2(i\overline{w}-iw)}{|w|^2+1+i\overline{w}-iw} \geq \frac{4r\sin\theta}{4r^2}.$$

This gives the right-hand inequality of (13), and to get the left-hand one we just look at $-1/\psi$.

The inequality (13) shows that, away from the real axis, ψ is neither too large nor too small, so that the modulus of f'/f is controlled there by that of ϕ.

3.4 A result of Hayman

In 1959 Hayman proved the following, which is widely known as *Hayman's alternative*: if g is meromorphic in the plane and g omits a finite value a, while $g^{(k)}$, for some $k \in \mathbb{N}$, omits a finite, non-zero value b, then g is constant. The proof may be found in *Meromorphic Functions*, p.60, and is based on Nevanlinna theory.

This Picard-type result is striking in that only two values a, b are considered, rather than the three values which are required in Picard's theorem. Note that the condition $b \neq 0$ is necessary because of the example e^z, while $k \geq 1$ is required because of $1/(1+e^z)$.

Hayman's alternative has the following consequence: suppose that G is an entire function, and G and G'' have no zeros. Let $g = G/G'$. Then g is meromorphic, without zeros, and

$$g' = 1 - \frac{GG''}{G'^2} \neq 1.$$

Thus g is constant and $G(z) = e^{az+b}$. Note however that Wiman's conjecture pre-dates this result of Hayman.

We need to apply two local analogues of Hayman's alternative. The first is for an arbitrary plane domain, the second a more specific result for a half-plane.

3.5 Normal families

Let D be a domain in $\mathbb{C}$, and let G be a family of functions meromorphic on D. Then G is said to be *normal* if the following is true: for every sequence (g_n) in G, there exists a subsequence (h_n), say, such that (h_n) converges locally uniformly on D, with respect to the spherical metric, to a limit function h which is either meromorphic or $\equiv \infty$. For example, the family of functions nz, $n \in \mathbb{N}$, is normal on $\mathbb{C} \setminus \{0\}$, with limit function ∞, but not normal on $\mathbb{C}$.

Normality is a local property: if each $z_0 \in D$ has a neighbourhood $B(z_0, r_0)$ on which G is normal, then G is normal on D.

Many value distribution results in the plane have normal family analogues in domains. The classic instance is Montel's theorem: *the family of all analytic $g : D \to \mathbb{C} \setminus \{0, 1\}$ is normal.* The result corresponding to Hayman's alternative was proved by Gu Yong-Xing (1978):

Theorem. *Let D be a plane domain, let $k \in \mathbb{N}$, $a \in \mathbb{C}$ and $b \in \mathbb{C} \setminus \{0\}$. Let G be the family of all g meromorphic on D such that $g \neq a$ and $g^{(k)} \neq b$ on D. Then G is normal.*

Gu's proof was extremely complicated, but a much simpler proof is based on the recent re-scaling lemma of Pang-Zalcman. The relevant version of this is the following.

Suppose that G is a family of functions meromorphic on $B(0, 1)$, and that G is not normal. Suppose also that each $g \in G$ is zero-free on $B(0, 1)$, and that $k \in \mathbb{Z}$, $k \geq 0$. Then we can find:
(i) $r \in (0, 1)$;
(ii) z_n with $|z_n| \leq r$;
(iii) $\rho_n \to 0$;
(iv) $g_n \in G$ such that the functions

$$h_n(z) = \rho_n^{-k} g_n(z_n + \rho_n z) \tag{14}$$

converge LU on $\mathbb{C}$ to a non-constant meromorphic function h.

Suppose now that G is a family of functions meromorphic on $B(A, R)$, with $g(z) \neq a, g^{(k)}(z) \neq b$ on $B(A, R)$, for each $g \in G$, and that G is not normal. Here $b \neq 0$ and $k \in \mathbb{N}$. We may assume that $a = 0$ and that $A = 0, R = 1$ (otherwise consider $h(z) = g(A+Rz)-a$).

The Pang-Zalcman lemma gives us the functions $h_n(z)$, converging LU to h, and $h_n(z) \neq 0$. Also $h_n^{(k)}(z) = g_n^{(k)}(z_n + \rho_n z) \neq b$, and so h omits 0, and $h^{(k)}$ is either $\equiv b$, or omits b. Both of these conclusions follow from Hurwitz' theorem.

But $h^{(k)} \equiv b$ is impossible, since h omits 0, and we have a function h whose existence contradicts Hayman's alternative. This proves Gu's theorem.

We need next to look at consequences of the condition $g \neq 0, g' \neq 1$ in a half-plane.

3.6 The Tsuji characteristic

Nevanlinna's characteristic function is defined by

$$T(r, g) = m(r, g) + N(r, g) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |g(re^{i\theta})| d\theta + N(r, g),$$

in which $\log^+ x = \max\{\log x, 0\}$ and

$$N(r,g) = n(0,g)\log r + \int_0^r \frac{n(t,g)-n(0,g)}{t}dt,$$

where $n(t,g)$ is the number of poles of g in $|z| \leq t$, counting multiplicity. This theory work for g meromorphic in $|z| < R$, $0 < R \leq \infty$.

An analogous characteristic for functions in a half-plane was developed by Tsuji and by Levin and Ostrovskii. Let g be a meromorphic function in a domain containing the closed upper half-plane $\overline{H} = \{z \in \mathbb{C} : \mathrm{Im}\, z \geq 0\}$. For $t \geq 1$ let $\mathfrak{n}(t,g)$ be the number of poles of g, counting multiplicity, in $\{z : |z - it/2| \leq t/2, |z| \geq 1\}$, and set

$$\mathfrak{N}(r,g) = \int_1^r \frac{\mathfrak{n}(t,g)}{t^2}dt, \quad r \geq 1.$$

The Tsuji characteristic is defined as

$$\mathfrak{T}(r,g) = \mathfrak{m}(r,g) + \mathfrak{N}(r,g),$$

where

$$\mathfrak{m}(r,g) = \frac{1}{2\pi}\int_{\sin^{-1}(1/r)}^{\pi-\sin^{-1}(1/r)} \frac{\log^+|g(r\sin\theta e^{i\theta})|}{r\sin^2\theta}d\theta.$$

For non-constant g and any $a \in \mathbb{C}$ the first fundamental theorem then reads

$$\mathfrak{T}(r,g) = \mathfrak{T}(r, 1/(g-a)) + O(1), \quad r \to \infty, \tag{15}$$

and the lemma on the logarithmic derivative gives

$$\mathfrak{m}(r, g'/g) = O(\log r + \log^+ \mathfrak{T}(r,g)) \tag{16}$$

as $r \to \infty$ outside a set of finite measure. Further, $\mathfrak{T}(r,g)$ differs from a non-decreasing function by a bounded additive term. Standard inequalities give

$$\mathfrak{T}(r, g_1+g_2) \leq \mathfrak{T}(r,g_1) + \mathfrak{T}(r,g_2) + \log 2, \quad \mathfrak{T}(r, g_1g_2) \leq \mathfrak{T}(r,g_1) + \mathfrak{T}(r,g_2), \tag{17}$$

whenever g_1, g_2 are meromorphic in $\overline{H}$. Using the obvious fact that $\mathfrak{T}(r, 1/z) = 0$ for $r \geq 1$ we easily derive from (15) and (17) that $\mathfrak{T}(r,g)$ is bounded if g is a rational function. Note however that $\mathfrak{T}(r, e^{-iz})$ is also bounded.

A key role will be played by the following two results of Levin and Ostrovskii. The first is the analogue for the half-plane of Hayman's alternative.

3.7 Lemma

Let $k \in \mathbb{N}$ and let g be meromorphic in $\overline{H}$, with $g \neq 0$ and $g^{(k)} \neq 1$. Then $\mathfrak{T}(r,g) = O(\log r)$ as $r \to \infty$.

This is proved by following Hayman's proof exactly as in [MF]. The second result is obtained by a change of variables in a double integral.

3.8 Lemma

Let $Q(z)$ be meromorphic in $\overline{H}$, and for $r \geq 1$ set

$$m_{0\pi}(r,Q) = \frac{1}{2\pi}\int_0^\pi \log^+ |Q(re^{i\theta})|d\theta. \tag{18}$$

Then for $R \geq 1$ we have

$$\int_R^\infty \frac{m_{0\pi}(r,Q)}{r^3}dr \leq \int_R^\infty \frac{\mathfrak{m}(r,Q)}{r^2}dr. \tag{19}$$

We sketch the proof. Fix $R \geq 1$, and let Ω_R be the region $\operatorname{Im} z \geq 0, |z - iR/2| \geq R/2, |z| \geq 1$. Also, let K_R be the region $\operatorname{Im} z \geq 0, |z| \geq R$. Then $K_R \subseteq \Omega_R$, and

$$\frac{1}{2\pi}\int\int_{\Omega_R} \frac{\log^+|Q(z)|}{|z|^4}dxdy \geq \frac{1}{2\pi}\int\int_{K_R} \frac{\log^+|Q(z)|}{|z|^4}dxdy =$$

$$= \frac{1}{2\pi}\int_R^\infty r^{-3}\left(\int_0^\pi \log^+|Q(re^{i\theta})|d\theta\right)dr = \int_R^\infty r^{-3}m_{0\pi}(r,Q)dr.$$

For $\rho \geq R$ consider now the circle C_ρ given by $|z - i\rho/2| = \rho/2$, which has diameter ρ and is tangent to the real axis at 0. For $z \neq 0$ on this circle let $\theta = \arg z \in (0,\pi)$. The sine rule gives

$$\frac{|z|}{\sin 2\theta} = \frac{\rho}{2\sin(\pi/2 - \theta)}$$

and so $z = \rho \sin\theta e^{i\theta}$. The part C'_ρ of C_ρ with $|z| \geq 1$ corresponds to $\sin^{-1}(1/\rho) \leq \theta \leq \pi - \sin^{-1}(1/\rho)$, and the union of these C'_ρ for $\rho \geq R$ is Ω_R. The transformation $r = \rho\sin\theta, \theta = \theta$ has Jacobian $\sin\theta$, and so

$$\frac{1}{2\pi}\int\int_{\Omega_R} \frac{\log^+|Q(re^{i\theta})|}{r^4}rdrd\theta = \frac{1}{2\pi}\int_R^\infty \int_{\sin^{-1}(1/\rho)}^{\pi-\sin^{-1}(1/\rho)} \frac{\log^+|Q(\rho\sin\theta e^{i\theta})|}{\rho^4\sin^4\theta}\rho\sin^2\theta d\rho d\theta =$$

$$= \int_R^\infty \rho^{-2}\mathfrak{m}(\rho,Q)d\rho.$$

4 Proof of the conjecture, second part

4.1 Lemma

Let $L = f'/f = \psi\phi$ be as before. Then the Tsuji characteristic of L satisfies $\mathfrak{T}(r,L) = O(\log r)$ as $r \to \infty$.

Proof. Let $g_1 = 1/L$. Then

$$g_1' = -L'/L^2.$$

Since L has only real poles and since $f''/f = L + L'/L$ has by assumption only real zeros it follows that g_1 and $g_1' - 1$ have no zeros in H. Lemma 3.7 now gives $\mathfrak{T}(r,g_1) = O(\log r)$.

4.2 Lemma

The entire function ϕ has order at most 1.

Proof. Remember that ϕ is assumed non-constant. Lemmas 3.8 and 4.1 give

$$\int_R^\infty \frac{m_{0\pi}(r,L)}{r^3} dr \leq \int_R^\infty \frac{\mathfrak{m}(r,L)}{r^2} dr = O(R^{-1}\log R), \quad R \to \infty.$$

But (13) gives $m_{0\pi}(r, 1/\psi) = O(\log r)$ and so we obtain

$$\int_R^\infty \frac{m_{0\pi}(r,\phi)}{r^3} dr = O(R^{-1}\log R), \quad R \to \infty.$$

The function ϕ is entire and real on the real axis and so

$$T(r,\phi) = m(r,\phi) = 2m_{0\pi}(r,\phi).$$

Since $T(r,\phi)$ is a non-decreasing function of r we deduce that

$$T(R,\phi) = O(R\log R), \quad R \to \infty,$$

which proves the lemma.

4.3 Subharmonic functions

By a continuous subharmonic function on a domain D we mean a continuous $u : D \to [-\infty, \infty)$ which satisfies, for every $z_0 \in D$ and every sufficiently small $r > 0$,

$$u(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it}) dt$$

(sub-mean-value property). If f is analytic then $\log|f|$ is subharmonic, and if u_1, u_2 are s.h. then so is $u(z) = \max\{u_1(z), u_2(z)\}$. Subharmonic functions satisfy the maximum principle, and a general principle is that the more slowly a s.h. function's maximum grows, the "thicker" the set must be on which the function is larger. This is made precise as follows:

4.4 Carleman-Tsuji inequality

Let u be a non-constant continuous subharmonic function in the plane. For $r > 0$ let $B(r,u) = \max\{u(z) : |z| = r\}$, and let $\theta(r)$ be the angular measure of that subset of the circle $C(0,r) = \{z \in \mathbb{C} : |z| = r\}$ on which $u(z) > 0$. Define $\theta^*(r)$ by $\theta^*(r) = \theta(r)$, except that $\theta^*(r) = \infty$ if $u(z) > 0$ on the whole circle $C(0,r)$. Then if $r > 2r_0$ and $B(r_0,u) > 1$ we have

$$\log \|u^+(4re^{i\theta})\| \geq \log B(2r,u) - c_1 \geq \int_{2r_0}^r \frac{\pi dt}{t\theta^*(t)} - c_2,$$

in which c_1 and c_2 are absolute constants, and

$$\|u^+(re^{i\theta})\| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \max\{u(re^{i\theta}),\, 0\}\, d\theta.$$

The first inequality follows from Poisson's formula, and the second requires harmonic measure. Note that in the case that $u = \log|f|$ where f is an entire function, $\|u^+(re^{i\theta})\|$ coincides with the Nevanlinna characteristic $T(r,f)$.

4.5 Lemma

Let $\delta_1 > 0$ and $K > 1$. Then we have

$$|wL(w)| > K, \quad |w| = r, \quad \delta_1 \leq \arg w \leq \pi - \delta_1, \tag{20}$$

for all r outside a set E_1 which has zero logarithmic density i.e.

$$\int_{[1,r]\cap E_1} dt/t = o(\log r), \quad r \to \infty.$$

Proof. Choose δ_2 with $0 < \delta_2 < \delta_1$. Let

$$\Omega_0 = \{z \in \mathbb{C} : \frac{1}{2} < |z| < 2, \quad \frac{\delta_2}{2} < \arg z < \pi - \frac{\delta_2}{2}\}.$$

For $r > 0$ let $g_r(z) = 1/(rL(rz))$. Then $g_r(z) \neq 0$ on Ω_0, since all poles of L are real. Further,

$$g_r'(z) = -L'(rz)/L(rz)^2.$$

Since L has no poles in H and $f''/f' = L + L'/L$ has no zeros in H the equation $g_r'(z) = 1$ has no solutions in Ω_0. Thus the functions $g_r(z)$ form a normal family on Ω_0, by Gu's theorem.

Suppose that $|w_0| = r \geq r_0$, and $\delta_1 \leq \arg w_0 \leq \pi - \delta_1$, and that

$$|w_0 L(w_0)| \leq K. \tag{21}$$

Then

$$|g_r(z_0)| \geq 1/K, \quad z_0 = \frac{w_0}{r},$$

and so since the g_r are zero-free and form a normal family we have

$$|g_r(z)| \geq 1/K_1, \quad |z| = 1, \quad \delta_2 \leq \arg z \leq \pi - \delta_2, \tag{22}$$

for some positive constant $K_1 = K_1(r_0, \delta_1, \delta_2, K)$, independent of r. By (22) we have, for $|w| = r,\ \delta_2 \leq \arg w \leq \pi - \delta_2$, the estimates

$$\begin{aligned} |wL(w)| &= |w\psi(w)\phi(w)| \leq K_1, \\ |\phi(w)| &\leq K_2 = \frac{5K_1}{|\psi(i)| \sin \delta_2}. \end{aligned} \tag{23}$$

Thus (21) implies (23). Since ϕ has order 1, the set on which $\log|\phi(w)/K_2| > 0$ must be relatively "thick", and this gives the conclusion of the lemma, the detailed proof being as follows. For $t \geq r_0$ let

$$E_2(t) = \{w \in \mathbb{C} : |w| = t, |\phi(w)| > K_2\}.$$

Further, let $\theta(t)$ be the angular measure of $E_2(t)$, and as in Lemma 4.4 let $\theta^*(t) = \theta(t)$, except that $\theta^*(t) = \infty$ if $E_2(t) = C(0, t)$. Let

$$E_3 = \{t \in [r_0, \infty) : \theta(t) \leq 4\delta_2\}.$$

Since (21) implies (23), we have (20) for $t \in [r_0, \infty) \setminus E_3$. Applying Lemma 4.4 we obtain, since ϕ has order at most 1 by Lemma 4.2,

$$(1 + o(1)) \log r \geq \int_{r_0}^{r} \frac{\pi dt}{t\theta^*(t)} \geq \int_{[r_0,r]\cap E_3} \frac{\pi dt}{4\delta_2 t},$$

from which it follows that E_3 has upper logarithmic density at most $4\delta_2/\pi$. Since δ_2 may be chosen arbitrarily small, the lemma is proved.

The estimates (13) and (20) and the fact that ϕ is real now give

$$|\phi(z)| > \frac{K \sin \delta_1}{5|\psi(i)|r^2}, \quad \delta_1 \leq |\arg z| \leq \pi - \delta_1,$$

for $|z| = r$ in a set of logarithmic density 1. Since ϕ has order at most 1 by Lemma 4.2, but is non-constant, we deduce:

4.6 Lemma

The function ϕ has at least one zero.

Let

$$F(z) = z - \frac{1}{L(z)}, \quad F'(z) = 1 + \frac{L'(z)}{L(z)^2}. \tag{24}$$

Since L has only real poles and $L + L'/L$ has only real zeros we obtain at once:

4.7 Lemma

The function F has no critical points over $\mathbb{C} \setminus \mathbb{R}$, i.e. zeros z of F' with $F(z)$ non-real.

Let

$$W = \{z \in H : F(z) \in H\}, \quad Y = \{z \in H : L(z) \in H\}.$$

Then $Y \subseteq W$, by (24), so that each component C of Y is contained in a component A of W.

4.8 Lemma

All components C of Y are unbounded and satisfy

$$\limsup_{z\to\infty,\, z\in C} \operatorname{Im} L(z) > 0. \tag{25}$$

Proof. We first show that L has no poles in the closure of Y. To see this, let x_0 be a pole of L. Then x_0 is real, and is a simple pole of L with positive residue. Hence $\lim_{y\to 0+} \operatorname{Im} L(x_0 + iy) = -\infty$ and since L is univalent on an open disc $N_0 = B(x_0, R_0)$ it follows that $\operatorname{Im} L(z) < 0$ on $N_0 \cap H$. Thus $N_0 \cap Y = \emptyset$.

Suppose now that C is a component of Y. Then $\operatorname{Im} L(z)$ is harmonic and positive in C, and vanishes on ∂C. Thus C satisfies both conclusions of the lemma by the maximum principle.

4.9 Lemma

There exists at least one zero $\eta \in H \cup \mathbb{R}$ of L which satisfies at least one of the following conditions: (I) $\eta \in H$; (II) $L'(\eta) = 0$; (III) $\eta \in \mathbb{R}$ *and* $L'(\eta) > 0$.

Proof. By Lemma 4.6 ϕ has at least one zero; this must be a zero of L. We assume that there are no zeros of L satisfying (I) or (II) and will deduce that there is at least one with the property (III).

Since we assumed that the set of zeros $\{a_k\}$ of f is unbounded above and below, there must be an interval (a_k, a_{k+1}) containing at least one zero x_k of ϕ. But a_k and a_{k+1} are poles of ψ, with negative residues, and so there must be a zero y_k of ψ in (a_k, a_{k+1}), and we may assume that $y_k \neq x_k$, since L has by assumption no multiple zeros. But then the graph of L must cut the real axis at least twice in (a_k, a_{k+1}), and so there exists a zero η of L in (a_k, a_{k+1}) with $L'(\eta) > 0$. Thus we obtain (III).

4.10 Lemma

There exists $\alpha \in H$ with the property that $F(z) \to \alpha$ as $z \to \infty$ along a path γ_α in H.

Proof. Assume that there is no $\alpha \in H$ such that $F(z)$ tends to α along a path tending to infinity in H.

Take a component A of W. Then A is conformally equivalent to H under F. This is because every branch of F^{-1} with values in A can be analytically continued along every path in H, and so F^{-1} extends by the monodromy theorem to an analytic function on H, which implies in particular that F maps A univalently onto H.

Recall next from Lemma 4.9 that there is a zero η of L satisfying at least one of the conditions (I), (II) or (III) of Lemma 4.9. Then η belongs to the boundary of a component C of the set Y. As $F(\eta) = \infty$, by (24), and $Y \subseteq W$, we have $\eta \in \partial C \cap \partial A$, where A is a component of the set W containing C. Using again the fact that $F(\eta) = \infty$, it follows that for an arbitrarily small neighbourhood N of η, all values w of positive imaginary part and sufficiently large modulus are taken by F in $A \cap N$. As F is univalent on A we deduce that $F(z)$ is bounded as $z \to \infty$ in A. Now (24) gives $L(z) \to 0$ as $z \to \infty$ in A, and hence as $z \to \infty$ in C. This contradicts (25).

Now set

$$g(z) = z^2 L(z) - z = \frac{zF(z)}{z - F(z)}, \quad h(z) = \frac{1}{F(z) - \alpha}, \tag{26}$$

in which α is as in Lemma 4.10. Then g has no poles in H and (17), (24) and Lemma 4.1 give

$$\mathfrak{T}(r, g) + \mathfrak{T}(r, h) = O(\log r), \quad r \to \infty.$$

Hence Lemma 3.8 leads to

$$\int_1^\infty \frac{m_{0\pi}(r, g)}{r^3} dr + \int_1^\infty \frac{m_{0\pi}(r, h)}{r^3} dr < \infty, \tag{27}$$

in which $m_{0\pi}(r, g)$ and $m_{0\pi}(r, h)$ are as defined in (18).

4.11 Lemma

The function F has at most four finite non-real asymptotic values.

Proof. Assume the contrary. Since $F(z)$ is real on the real axis we may take distinct finite non-real $\alpha_0, \ldots, \alpha_n,\ n \geq 2$, such that $F(z) \to \alpha_j$ as $z \to \infty$ along a simple path $\gamma_j : [0, \infty) \to H \cup \{0\}$. Here we assume that $\gamma_j(0) = 0$, that $\gamma_j(t) \in H$ for $t > 0$, and that $\gamma_j(t) \to \infty$ as $t \to \infty$. We may further assume that $\gamma_j(t) \neq \gamma_{j'}(t')$ for $t > 0$, $t' > 0$, $j \neq j'$.

Re-labelling if necessary, we obtain n pairwise disjoint simply connected domains $D_1, \ldots, D_n$ in H, with D_j bounded by γ_{j-1} and γ_j, and for $t > 0$ we let $\theta_j(t)$ be the angular measure of the intersection of D_j with the circle $C(0, t)$. By (26), the function $g(z)$ tends to α_j as $z \to \infty$ on γ_j. Thus $g(z)$ is unbounded on each D_j but bounded on the finite boundary ∂D_j of each D_j.

Let c be large and positive, and for each j define

$$u_j(z) = \log^+ |g(z)/c|, \quad z \in D_j. \tag{28}$$

Set $u_j(z) = 0$ for $z \notin D_j$. Then u_j is continuous, and subharmonic in the plane since g is analytic in $H \cup \{0\}$.

Lemma 4.4 gives, for some $R > 0$ and for each j,

$$\int_R^r \frac{\pi dt}{t\theta_j(t)} \leq \log \|u_j(4re^{i\theta})\| + O(1)$$

as $r \to \infty$. Since u_j vanishes outside D_j we deduce using (28) that

$$\int_R^r \frac{\pi dt}{t\theta_j(t)} \leq \log m_{0\pi}(4r, g) + O(1), \quad r \to \infty, \tag{29}$$

for all $j \in \{1, \ldots, n\}$. The key point now is that because the D_j are disjoint at least one of them must be "thin". In fact, the Cauchy-Schwarz inequality gives

$$n^2 \leq \sum_{j=1}^n \theta_j(t) \sum_{j=1}^n \frac{1}{\theta_j(t)} \leq \sum_{j=1}^n \frac{\pi}{\theta_j(t)}$$

which on combination with (29) leads to, for some positive constant c_3,

$$n \log r \leq \log m_{0\pi}(4r, g) + O(1), \quad m_{0\pi}(r, g) \geq c_3 r^n, \quad r \to \infty.$$

Since $n \geq 2$ this contradicts (27), and Lemma 4.11 is proved.

From Lemmas 4.7 and 4.11 we deduce that the inverse function F^{-1} has finitely many non-real singular values. Using Lemma 4.10, take $\alpha \in H$ such that $F(z) \to \alpha$ along a path γ_α tending to infinity in H, and take ε_0 with $0 < \varepsilon_0 < \operatorname{Im} \alpha$ such that F has no critical or asymptotic values in $0 < |w - \alpha| \leq \varepsilon_0$. Take a component C_0 of the set $\{z \in \mathbb{C} : |F(z) - \alpha| < \varepsilon_0\}$ containing an unbounded subpath of γ_α. Then by a standard argument involving a logarithmic change of variables the inverse function F^{-1} has a logarithmic singularity over α, the component C_0 is simply connected, and $F(z) \neq \alpha$ on C_0. Further, the boundary of C_0

consists of a single simple curve going to infinity in both directions. Thus we may define a continuous, non-negative, non-constant subharmonic function in the plane by

$$u(z) = \log \left| \frac{\varepsilon_0}{F(z) - \alpha} \right| = \log |\varepsilon_0 h(z)| \quad (z \in C_0), \quad u(z) = 0 \quad (z \notin C_0), \tag{30}$$

using (26).

The next lemma follows from (24) and (30).

4.12 Lemma

For large z with $|zL(z)| > 3$ we have $|F(z) - \alpha| > |z|/2$ and $u(z) = 0$.

4.13 Lemma

We have

$$\lim_{r \to \infty} \frac{\log \|u(re^{i\theta})\|}{\log r} = \infty. \tag{31}$$

Proof. Apply Lemma 4.5, with $K = 3$ and δ_1 small and positive. By Lemma 4.12 we have $u(z) = 0$ if $\delta_1 \leq |\arg z| \leq \pi - \delta_1$ and $|z|$ is large but not in E_1. For large t let $\sigma(t)$ be the angular measure of that subset of $C(0, t)$ on which $u(z) > 0$. Since u vanishes on the real axis Lemma 4.4 and Lemma 4.5 give, for some $R > 0$,

$$\log \|u(4re^{i\theta})\| + O(1) \geq \int_R^r \frac{\pi dt}{t\sigma(t)} \geq \int_{[R,r] \setminus E_1} \frac{\pi dt}{4\delta_1 t} \geq \frac{\pi}{4\delta_1}(1 - o(1)) \log r$$

as $r \to \infty$. Since δ_1 may be chosen arbitrarily small the lemma follows.

Now (30) gives

$$\|u(re^{i\theta})\| \leq m_{0\pi}(r, h) + O(1),$$

from which we deduce using (31) that

$$\lim_{r \to \infty} \frac{\log m_{0\pi}(r, h)}{\log r} = \infty.$$

This obviously contradicts (27), and the proof is complete.