# The Julia set in quasiregular dynamics

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# Introduction

- Quasiregular functions on $\mathbb{R}^n$ generalize analytic functions on $\mathbb{C}$.
- Can we develop an iterative theory for quasiregular maps analogous to complex dynamics?
- Today we'll first introduce quasiregular maps and then explore the Julia set of such functions.

# Quasiregular mappings

## Definition

A continuous function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is called quasiregular (qr) if $f \in W^1_{n,\mathrm{loc}}(\mathbb{R}^n)$ and there exists $K' \geq 1$ such that

$$\|Df(x)\|^n \leq K' J_f(x) \qquad \text{a.e.}$$

- The smallest $K'$ for which the above holds is called the *inner dilatation* $K_I(f)$.
- The *outer dilatation* $K_O(f)$ is defined similarly, and the *dilatation* is $K(f) = \max\{K_I, K_O\}$.
- If $K(f) \leq K$, then $f$ is called $K$-quasiregular.
- A composition of qr maps is itself qr and

$$K(f \circ g) \leq K(f)K(g).$$

# Properties of quasiregular maps

Quasiregular functions on $\mathbb{R}^n$ generalize analytic functions on $\mathbb{C}$.

**Theorem (Reshetnyak, 1967-68)**

*Non-constant quasiregular maps are discrete and open.*

Rickman proved a Picard theorem for quasiregular maps:

**Theorem (Rickman, 1980)**

*For $n \geq 2$ and $K \geq 1$ there exists a constant $q = q(n, K)$ with the following property:*
*every $K$-qr map $f : \mathbb{R}^n \to \mathbb{R}^n$ that omits $q$ values must be constant.*

# Examples of qr maps

- For $k \in \mathbb{N}$, the winding map $f\colon \mathbb{C} \to \mathbb{C}$ given by $f(re^{i\theta}) = re^{ik\theta}$ is quasiregular with $K_I(f) = K_O(f) = k$.

- The Zorich map $Z\colon \mathbb{R}^n \to \mathbb{R}^n \setminus \{0\}$ is a quasiregular analogue of the exponential function.
  It is periodic in $n-1$ directions and grows/decays exponentially in the other.

# Polynomial type vs transcendental type

## Definition

A qr map $f$ is said to be of *polynomial type* if $\lim_{x\to\infty} |f(x)| = \infty$. Otherwise, this limit does not exist and $f$ is *transcendental type*.

Equivalently, $f$ is of polynomial type iff $\deg f < \infty$, where

$$\deg f = \max_{y\in\mathbb{R}^n} \operatorname{card} f^{-1}(y).$$

## Definition

Can extend the definition of quasiregularity to functions $\overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ (analogous to rational maps of $\overline{\mathbb{C}}$).

A polynomial type qr map $f : \mathbb{R}^n \to \mathbb{R}^n$ extends to a qr map of $\overline{\mathbb{R}^n}$ by setting $f(\infty) = \infty$.

# Normal families and uniform quasiregularity

Miniowitz used Rickman's theorem to obtain an analogue of Montel's theorem:

**Theorem (Miniowitz, 1982)**

*Let $\mathcal{F}$ be a family of $K$-qr maps on a domain $D \subset \mathbb{R}^n$ and let $q = q(n, K)$ be Rickman's constant.*
*If $a_1, \ldots, a_q \in \mathbb{R}^n$ are distinct and every $f \in \mathcal{F}$ omits $a_1, \ldots, a_q$, then $\mathcal{F}$ is a normal family.*

- If every iterate $f^N$ is $K$-quasiregular <u>with the same $K$</u>, then $f$ is called *uniformly quasiregular* (uqr).
- For uqr maps many concepts of complex dynamics transfer nicely and the non-normality definition of the Julia set works well. (Hinkkanen, Martin, Mayer, Siebert.)

# General case?

In general, the dilatation $K(f^N) \to \infty$ as $N \to \infty$, ...

... so we can't apply Montel's theorem to the family $\{f^N\}$.

How do we study the dynamics of general quasiregular maps?

# Two dimensions, finite degree

Sun and Yang considered qr maps $f\colon \overline{\mathbb{C}} \to \overline{\mathbb{C}}$. They suggested using the familiar “blowing-up” property of Julia sets as a definition:

$$J(f) := \{z \in \overline{\mathbb{C}} : \text{for every neighbourhood } U \text{ of } z, \\ \overline{\mathbb{C}} \setminus O^+(U) \text{ contains at most 2 points}\}.$$

**Theorem (Sun-Yang, 1999-2001)**

*If $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is qr and $\deg f > K_I(f)$, then $J(f) \neq \emptyset$ and …*
*… $J(f)$ has many properties expected of a Julia set.*

To take things further, “at most 2 points” needs modifying.
We need a different notion of small sets.

- In the rest of this talk, we'll focus on quasiregular maps $\mathbb{R}^n \to \mathbb{R}^n$ of transcendental type.
- This work builds upon similar results of W. Bergweiler for qr maps $\overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ for which the degree exceeds the inner dilatation.

# Capacity

For an open set $A \subset \mathbb{R}^n$ and a compact subset $C \subset A$, the pair $(A, C)$ is called a *condenser*. Its *(conformal) capacity* is defined by

$$\operatorname{cap}(A, C) = \inf_u \int_A |\nabla u|^n \, dm,$$

where the inf is over non-negative $u \in C_0^\infty(A)$ with $u(x) \geq 1$ for $x \in C$.

Equivalently, if $\Gamma$ is the family of paths in $A$ that join $C$ to $\partial A$, then

$$\begin{aligned} \operatorname{cap}(A, C) &= M(\Gamma), && \text{modulus of path family} \\ &= \lambda(\Gamma)^{1-n}, && \text{where } \lambda \text{ is extremal length.} \end{aligned}$$

# Sets of zero capacity

- If $\operatorname{cap}(A, C) = 0$ then $\operatorname{cap}(A', C) = 0$ for every open set $A$ containing $C$.
  In this case, we say that $C$ is of *capacity zero* and write $\operatorname{cap} C = 0$.

- Otherwise we say $C$ has *positive capacity*, $\operatorname{cap} C > 0$.

- For an unbounded closed set $C$, we say $\operatorname{cap} C = 0$ if every compact subset has capacity zero.

- For example, countable sets have capacity zero.

- Capacity zero $\Rightarrow$ Hausdorff dimension zero.

# Julia set definition

For a quasiregular $f : \mathbb{R}^n \to \mathbb{R}^n$ of transcendental type, we define the Julia set as

$$J(f) := \left\{x \in \mathbb{R}^n : \text{for every nhd } U \text{ of } x,\ \operatorname{cap}\left(\mathbb{R}^n \setminus O^+(U)\right) = 0\right\}.$$

It follows immediately that $J(f)$ is closed and completely invariant.

**Theorem**

*For quasiregular $f$ of trans type, the Julia set $J(f) \neq \emptyset$.*
*In fact, $J(f)$ is infinite.*

**Theorem**

*For a trans entire function $f : \mathbb{C} \to \mathbb{C}$, the definition of $J(f)$ given above agrees with the usual one.*

# Examples

- Certain quasiregular sine function analogues $S\colon \mathbb{R}^n \to \mathbb{R}^n$ have the property that $O^+(U) = \mathbb{R}^n$ for all non-empty open $U$.
  Thus $J(S) = \mathbb{R}^n$.

- Let $Z\colon \mathbb{R}^3 \to \mathbb{R}^3 \setminus \{0\}$ be a Zorich map (qr version of exp).
  Let $a > 0$ be large and let

  $$f_a(x) = Z(x) - (0, 0, a).$$

  Then there exists a unique attracting fixed point $\xi$ of $f_a$,

  $$J(f_a) = \mathbb{R}^3 \setminus \mathcal{A}(\xi)$$

  and $J(f_a)$ is a Cantor bouquet.

# Results about Julia sets

**Theorem**

*Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be quasiregular of trans type with an attracting fixed point $\xi$. Then*

$$J(f) \cap \mathcal{A}(\xi) = \emptyset \quad \textit{and} \quad J(f) \subset \partial\mathcal{A}(\xi).$$

Unlike the analytic case, explicit examples show that for quasiregular maps $J(f)$ can be a proper subset of $\partial\mathcal{A}(\xi)$.

Define the *escaping set* $I(f) = \{x \in \mathbb{R}^n : f^k(x) \to \infty\}$.

**Theorem (Bergweiler, Fletcher, Langley, Meyer)**

*If $f$ is qr of trans type, then $I(f)$ has an unbounded component.*

We also consider the set of points with bounded orbit

$$BO(f) = \left\{x \in \mathbb{R}^n : \left(f^k(x)\right) \text{ is bounded}\right\}.$$

**Theorem**

*If $f$ is qr of trans type, then* $\operatorname{cap} BO(f) > 0$.

Using the above, complete invariance and $I(f) \cap BO(f) = \emptyset$, we get

**Theorem**

*If $f$ is qr of trans type, then $J(f) \subset \partial I(f) \cap \partial BO(f)$.*

In the analytic case we have $J(f) = \partial I(f) = \partial BO(f)$, but for qr maps the above inclusion may be strict.

# Faster is better?

- We've seen that $J(f) \subset \partial I(f)$ — it may be a proper subset.
- Similar proof gives that $J(f) \subset \partial A(f)$, where $A(f)$ is the fast escaping set.
  See AF for A(f)!

**Theorem (Bergweiler, Fletcher, N.)**

*Let $f$ be trans type quasiregular of positive lower order. Then*

$$J(f) = \partial A(f).$$

# Two familiar definitions

- Define, as usual, the *backward orbit* of a point $x \in \mathbb{R}^n$

$$O^-(x) = \{y \in \mathbb{R}^n : f^k(y) = x, \text{ some } k \geq 0\}.$$

- The *exceptional set* $E(f)$ is the set of points with finite backward orbit under $f$.

- For quasiregular $f$ of trans type,
Rickman's Picard Theorem $\Rightarrow E(f)$ contains at most $q - 1$ points.

# Typical Julia set properties

For a wide range of qr trans type maps, we can prove more about $J(f)$:

(J1) $J(f)$ is perfect,

(J2) $J(f^p) = J(f)$ for all $p \in \mathbb{N}$,

(J3) $J(f) \subset \overline{O^-(x)}$ for all $x \in \mathbb{R}^n \backslash E(f)$,

(J4) $J(f) = \overline{O^-(x)}$ for all $x \in J(f) \backslash E(f)$,

(J5) $\mathbb{R}^n \backslash O^+(U) \subset E(f)$ for every open set $U$ intersecting $J(f)$.

Note that (J5) implies that

$$J(f) = \{x \in \mathbb{R}^n : \text{for every nhd } U \text{ of } x, \\ \mathbb{R}^n \backslash O^+(U) \text{ contains at most } q-1 \text{ points}\}.$$

### Theorem

*A quasiregular trans type map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ will satisfy the properties*

(J1) $J(f)$ *is perfect,*

(J2) $J(f^p) = J(f)$ *for all* $p \in \mathbb{N}$,

(J3) $J(f) \subset \overline{O^-(x)}$ *for all* $x \in \mathbb{R}^n \backslash E(f)$,

(J4) $J(f) = \overline{O^-(x)}$ *for all* $x \in J(f) \backslash E(f)$,

(J5) $\mathbb{R}^n \backslash O^+(U) \subset E(f)$ *for every open set* $U$ *intersecting* $J(f)$,

*if any one of the following conditions holds:*

(a) $n = 2$ *(i.e.* $f\colon \mathbb{C} \to \mathbb{C}$*);*

(b) $f$ *is locally Lipschitz continuous;*

(c) *the local index of* $f$ *at* $x$ *is bounded above for all* $x \in \mathbb{R}^n$;

(d) $f$ *does not have (a version of) the "pits effect".*

- $f$ has the pits effect if $|f(x)|$ is 'large' except in 'small' domains.
- Example: $f$ bdd on path to $\infty \Rightarrow f$ doesn't have the pits effect.

# A conjecture

The structure of the proof is roughly

$$\begin{array}{c} f \text{ satisfies condition} \\ \text{(a), (b), (c) or (d)} \end{array} \Rightarrow \begin{array}{c} \operatorname{cap} \overline{O^-(x)} > 0, \\ \text{for all } x \notin E(f) \end{array} \Rightarrow \text{ properties (J1)–(J5),}$$

where the first implication is the tricky part.

We'd like (J1)–(J5) to hold for all quasiregular maps of trans type.
This would follow from:

## Conjecture

If $f$ is quasiregular of transcendental type, then

$$\operatorname{cap} \overline{O^-(x)} > 0, \quad \text{for all } x \notin E(f).$$