# A LAGRANGIAN MOVING FINITE ELEMENT METHOD INCORPORATING MONITOR FUNCTIONS*

MIKE BAINES†, MATTHEW HUBBARD , AND PETER JIMACK‡

**Abstract.** We consider a Lagrangian Moving Finite Element method which has a mesh velocity based on conservation of the integral of a monitor function. The method arises naturally from the theory of fluids and the monitor function can be thought of as pseudo density and the corresponding velocity as that of a pseudo fluid. In this paper we describe a weak form of the method, which is illustrated on a two-dimensional nonlinear diffusion equation (the Porous Medium Equation) with a moving boundary.

**Key words.** finite element methods, moving meshes, moving boundaries, nonlinear diffusion

**AMS subject classifications.** 35K55, 65M60, 65M50

**1. Introduction.** In [1] a moving finite element method for PDEs using monitor functions is introduced which uses ideas from [4] and [5]. The method exploits local mass conservation to move the mesh and describes a Moving Finite Element method using a density monitor function.

Implicit in the approach is a link with Lagrangian fluid dynamics. Relating conservation to Lagrangian mass invariance and the mesh velocity to a fluid velocity, and using the tools described in [1], we discuss here the link with fluids in more detail.

The approach revolves around the use of a monitor function in place of the density, leading to a monitor velocity and monitor velocity potential. The moving finite element method uses a distributed form of the approach.

The application of the method to PDEs and systems of PDEs is described. Detailed computations are presented for the Porous Medium Equation (PME) in two dimensions.

**2. PDEs in a Moving Frame.** We consider PDEs and systems of PDEs of the form

$$\frac{\partial u}{\partial t} = Lu \tag{2.1}$$

where $u = u(\mathbf{x}, t)$ and $L$ is a multidimensional operator involving space derivatives only.

Instead of working in the fixed frame define an invertible mapping between fixed labelling coordinates $(\mathbf{a}, \tau)$ and (Lagrangian) moving frame coordinates $(\mathbf{x}, t)$ of the form

$$\mathbf{x} = \hat{\mathbf{x}}(\mathbf{a}, \tau); t = \tau$$

so that

$$u(\mathbf{x}, t) = u(\hat{\mathbf{x}}(\mathbf{a}, \tau), \tau) = \hat{u}(\mathbf{a}, \tau)$$

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say. Then the chain rule gives

$$\frac{\partial \hat{u}}{\partial \tau} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial \hat{x}}{\partial \tau}$$

where $\frac{\partial}{\partial t}$ means differentiation with respect to time $t$ with $\mathbf{x}$ frozen, so that $\frac{\partial u}{\partial t}$ is given by the PDE (2.1). Hence the PDE (2.1) can be written

$$\frac{\partial \hat{u}}{\partial \tau} - \frac{\partial u}{\partial x}\frac{\partial \hat{x}}{\partial \tau} = Lu \tag{2.2}$$

Two equations are required to determine the two unknowns $\frac{\partial \hat{u}}{\partial \tau}$ and $\frac{\partial \hat{x}}{\partial \tau}$ in (2.2). One is the PDE but the other must be obtained from a separate principle. In much moving mesh work the velocity $\frac{\partial \hat{x}}{\partial \tau}$ is found first and equation (2.2) then used to find $\frac{\partial \hat{u}}{\partial \tau}$. In this paper we seek a velocity $\frac{\partial \hat{x}}{\partial \tau}$ based on modifying the Lagrangian equation of fluid dynamics.

**3. Fluid Dynamics.** In classical fluid dynamics fixed **a**-space regions $\Omega_0$ have fixed volumes, i.e.

$$\int_{\Omega_0} d\mathbf{a} = \text{constant in time} \tag{3.1}$$

Using the above transformation to a (Lagrangian) moving frame this implies that the integral over the time-dependent transformed region $\Omega(t)$

$$\int_{\Omega(t)} \rho d\mathbf{x} \tag{3.2}$$

is constant in time, where $\rho$ is the Jacobian. The invariance (3.2) can be interpreted as conservation of mass and the Jacobian thus identified with the fluid density. The kinematic equation (3.2) is referred to as the Lagrangian conservation-of-mass equation.

Differentiating (3.2) with respect to time,

$$\int_{\Omega(t)} \frac{\partial \rho}{\partial t} d\mathbf{x} + \oint_{\partial\Omega(t)} \rho \mathbf{v} \cdot d\mathbf{S} = 0 \tag{3.3}$$

where $\mathbf{v}$ is the fluid velocity seen from an Eulerian viewpoint. This leads to

$$\int_{\Omega(\mathbf{v})} \left( \frac{\partial \rho}{\partial \tau} + \nabla \cdot (\rho \mathbf{v}) \right) d\mathbf{x} = 0 \tag{3.4}$$

where, to underline the fact that $\Omega(t)$ moves with velocity $\mathbf{v}$, we have written $\Omega(t) = \Omega(\mathbf{v})$.

In most fluid dynamics models equation (3.2) is coupled with a kinetic equation and the two provide a system of PDEs for the unknowns $\rho$ and $\mathbf{v}$. However if the vorticity $curl\mathbf{v}$ is specified, the kinematic equation decouples from the kinetic equation and then, given $\rho$, constancy of (3.2) is a distinct equation for the fluid velocity $\mathbf{v}$.

**4. A Pseudo Density Monitor Function.** If the density $\rho$ is replaced by a general (non-negative) pseudo density monitor function, $M$ say, equation (3.2) determines a new velocity, $\mathbf{v}_M$ say, which we shall call the monitor velocity, satisfying the invariance

$$\int_{\Omega(\mathbf{v}_M)} M d\mathbf{x} = \text{constant in time}, \tag{4.1}$$

a property which is related to the Geometric Conservation Law ([5],[7]).

Differentiating (4.1) with respect to time, we obtain

$$\int_{\Omega(\mathbf{v}_M)} \left( \frac{\partial M}{\partial t} + \nabla \cdot (M\mathbf{v}_M) \right) d\mathbf{x} = 0 \tag{4.2}$$

(cf. (3.4)). As for fluids, if the vorticity $curl\mathbf{v}_M$ is specified, specification of $M$ gives an equation for the monitor velocity $\mathbf{v}_M$ (see next section).

**4.1. A Vorticity Condition.** In [5] the *curl* condition is taken to be of the form

$$curl(\mu\mathbf{v}_M) = curl(\mu\mathbf{v}_p) \tag{4.3}$$

where $\mu$ is a non-negative function and $\mathbf{v}_p$ is a prescribed velocity.

There then exists a monitor velocity potential $\phi_M$ such that

$$\mathbf{v}_M - \mathbf{v}_p = \mu^{-1}\nabla\phi_M \tag{4.4}$$

where from (4.2) $\phi_M$ satisfies the monitor potential equation

$$-\int_{\Omega(\mathbf{v}_M)} \nabla \cdot (M\mu^{-1}\nabla\phi_M) d\mathbf{x} = \int_{\Omega(\mathbf{v}_M)} \nabla \cdot (M\mathbf{v}_p) d\mathbf{x} + \int_{\Omega(\mathbf{v}_M)} \frac{\partial M}{\partial t} d\mathbf{x} \tag{4.5}$$

The pointwise form of equation (4.5) possesses a unique solution for $\phi$ if $\phi$ or the normal component of $\nabla\phi$ is given on the boundary.

**4.2. The Monitor Function.** The monitor $M$ can be any function of the variables of the problem for which the time derivative $\frac{\partial M}{\partial t}$ can be expressed in terms of space derivatives.

For example, if $M$ is the fluid density $\rho$

$$\frac{\partial M}{\partial t} = \frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho\mathbf{v})$$

giving from (4.2) the pointwise form

$$\nabla \cdot (\rho(\mathbf{v}_M - \mathbf{v})) = 0$$

Then, if $\mathbf{v}_M$ satisfies (4.3) with $\mathbf{v}_p = \mathbf{v}$, and the normal component of $\mathbf{v}_M - \mathbf{v}$ is zero at the boundary, it follows that $\mathbf{v}_M = \mathbf{v}$.

Generally, if $M$ is any function of the field variables the term $\frac{\partial M}{\partial t}$ in (4.5) can be expressed in terms of their space derivatives on the right hand side of (4.5) using a chain rule and the field equations (2.1).

**5. Summary.** We solve the coupled system for $M$, $\mathbf{v}_M$ and $\phi_M$ consisting of the invariance equation (4.1), the monitor velocity equation (4.4), and the the monitor velocity potential equation (4.5). The time derivative $\frac{\partial M}{\partial t}$ in (4.5) is expressed in terms of space derivatives of the field variables using (2.1). Equation (4.1) can be used to express $M$ in terms of $\mathbf{x}$ for substitution into (4.4) and (4.5) to yield a self-contained ODE for $\mathbf{x}$. (N.B. $\mathbf{v}_M = \dot{\mathbf{x}}$.)

Equation (4.1) may also be used as a recovery equation for $u$ or, if that is not feasible, we can use (5.2) or (2.2).

**5.1. Application to Scalar PDEs.** For a scalar PDE of the form (2.1) the method is self-contained. As in [3] we can use (4.1) to express the monitor $M$ in terms of the (Lagrangian) coordinates $\mathbf{x}$ and, substituting into (4.3) or (4.5), we obtain an equation for $\phi$ in terms of $\mathbf{x}$ only. Moreover, writing $\mathbf{v}_M$ in Lagrangian form as $\dot{\mathbf{x}}$, equation (4.4) becomes

$$\dot{\mathbf{x}} = \mathbf{v}_p + \mu^{-1}\nabla\phi_M \tag{5.1}$$

which, coupled with the solution of (4.5), gives an ODE system for $\mathbf{x}$. After $\mathbf{x}$ has been obtained, back substitution into the (algebraic) invariance equation (4.1) allows the reconstruction of $M$ and hence $u$.

**5.2. Application to Systems.** The same procedure can be used for systems of equations (including the equations of fluid mechanics), except for the final reconstruction step. However, if the monitor function $M$ contains only one field variable, then that field variable can be reconstructed from (4.1) in the same way. More generally, the field variables can be obtained directly from the PDE in the moving frame (2.2). For example, in the special case where $Lu$ is the divergence of a flux function $\mathbf{F}$ we can use an ALE (Arbitrary Lagrange Euler) equation in the flux balance form

$$\frac{d}{dt}\int_{\Omega(\mathbf{v}_M)} u d\mathbf{x} = -\oint_{\partial\Omega(\mathbf{v}_M)} \{\mathbf{F} - u\mathbf{v}_M\}\cdot d\mathbf{S} \tag{5.2}$$

**5.3. Finite Volumes.** The integral forms (4.1) and (4.5) hold for *finite* volumes $\Omega_i(\mathbf{v}_M)$, provided only that the corresponding volumes $\Omega_i$ in the fixed frame remain fixed in time. A finite volume form of (4.4) (with $\mathbf{v}_M = \dot{\mathbf{x}}$) is

$$\int_{\Omega(\dot{\mathbf{x}})} (\dot{\mathbf{x}} - \mathbf{v}_p) d\mathbf{x} = \int_{\Omega(\dot{\mathbf{x}})} \mu^{-1}\nabla\phi_M d\mathbf{x} \tag{5.3}$$

Taking finite dimensional approximations to the dependent variables the Finite Volume equations may be solved as in the continuous case by using (4.1) to express $M$ in terms of $\mathbf{x}$ and then forming and integrating the ODE system for $\mathbf{x}$ inherent in (5.3) through the solution of (4.5) for $\phi_M$.

The variable $\mathbf{x}$ may be used via (4.1) to reconstruct the function $M$ and therefore the solution $u$. Alternatively, $u$ can be obtained through the ALE equations (5.2) or the PDE in the moving frame (2.2).

The procedure with finite volumes works well in 1-D (see [3],[2]) but if the number of dimensions is greater than one a finite element approach is more effective.

**6. A Distributed Monitor Density.** To construct a finite element method we require weighted weak forms. Consider therefore the original fixed frame invariance (3.1) in the weighted form

$$\int_{\Omega_0} \hat{w} d\mathbf{a} = \text{constant in time} \tag{6.1}$$

where $\hat{w}$ is a non-negative weight function (thus conserving partial particle counts).

Transforming to the Lagrangian frame (moving under the velocity $\mathbf{v}_M$) as in section 2, we obtain the weighted invariance (cf. (3.2))

$$\int_{\Omega(\mathbf{v}_M)} wM d\mathbf{x} = \text{constant in time} \tag{6.2}$$

where $M$ is the Jacobian $\left|\frac{\partial \mathbf{a}}{\partial \mathbf{x}}\right|$ and $w$ is the image of the weight function $\hat{w}$ under the transformation.

Differentiating the invariance equation (6.2) leads to the weighted monitor velocity equation (cf. (4.2))

$$\frac{d}{dt}\int_{\Omega(\mathbf{v}_M)} wM d\mathbf{x} = \int_{\Omega(\mathbf{v}_M)} \left(\frac{\partial}{\partial t}(wM) + \nabla\cdot(wM\mathbf{v}_M)\right) d\mathbf{x} = 0. \tag{6.3}$$

¿From (6.1)

$$0 = \frac{d}{dt}\int_{\Omega_0} \hat{w} d\mathbf{a} = \int_{\Omega_0} \frac{d\hat{w}}{dt} d\mathbf{a} = \int_{\Omega_0} \left(\frac{\partial w}{\partial t} + \mathbf{v}_M\cdot\nabla w\right) d\mathbf{a} \tag{6.4}$$

so that, pointwise,

$$\frac{\partial w}{\partial t} + \mathbf{v}_M\cdot\nabla w = 0 \tag{6.5}$$

Using (6.5), equation (6.3) reduces to

$$\int_{\Omega(\mathbf{v}_M)} w\left(\frac{\partial M}{\partial t} + \nabla\cdot(M\mathbf{v}_M)\right) d\mathbf{x} = 0. \tag{6.6}$$

As before, assuming a *curl* condition of the form (4.3) there exists a monitor velocity potential $\phi_M$ satisfying (4.4), giving from (6.6) a weighted monitor potential equation (cf. (4.5))

$$-\int_{\Omega(\mathbf{v}_M)} w\nabla\cdot(M\mu^{-1}\nabla\phi_M) d\mathbf{x} = \int_{\Omega(\mathbf{v}_M)} w\left(\nabla\cdot(M\mathbf{v}_p) + \frac{\partial M}{\partial t}\right) d\mathbf{x} \tag{6.7}$$

Finally, using integration by parts, the left hand side of equation (6.7) may be written as

$$-\oint_{\partial\Omega(\mathbf{v}_M)} wM\mu^{-1}\nabla\phi_M\cdot d\mathbf{S} + \int_{\Omega(\mathbf{v}_M)} M\mu^{-1}\nabla w\cdot\nabla\phi_M d\mathbf{x} \tag{6.8}$$

The time derivative $\frac{\partial M}{\partial t}$ in (6.7) can be expressed in terms of space derivatives of the field variables using the weak form

$$\int_{\Omega(\mathbf{v}_M)} wM'(u)\frac{\partial u}{\partial t} d\mathbf{x} = \int_{\Omega(\mathbf{v}_M)} wM'(u)Lu d\mathbf{x} \tag{6.9}$$

of the PDE (2.1), giving

$$-\oint_{\partial\Omega(\mathbf{v}_M)} wM\mu^{-1}\nabla\phi_M\cdot d\mathbf{S} + \int_{\Omega(\mathbf{v}_M)} M\mu^{-1}\nabla w\cdot\nabla\phi_M d\mathbf{x}$$

$$= \int_{\Omega(\mathbf{v}_M)} w\nabla\cdot(M\mathbf{v}_p)d\mathbf{x} + \int_{\Omega(\mathbf{v}_M)} wM'(u)Lud\mathbf{x} \tag{6.10}$$

In the case where $Lu = \nabla\cdot\mathbf{F}(u)$ integration by parts on the last term gives

$$\oint_{\partial\Omega(\mathbf{v}_M)} wM'(u)\mathbf{F}(u)\cdot d\mathbf{S} - \int_{\Omega(\mathbf{v}_M)} \nabla(wM'(u))\cdot\mathbf{F}(u)d\mathbf{x} \tag{6.11}$$

To recover the velocity $\mathbf{v}_M$ a weak form of equation (4.4) is required which, with $\mathbf{v}_M = \dot{\mathbf{x}}$, we take in the form (cf. (5.3))

$$\int_{\Omega(\mathbf{v}_M)} w(\dot{\mathbf{x}} - \mathbf{v}_p)d\mathbf{x} = \int_{\Omega(\mathbf{v}_M)} \mu^{-1}w\nabla\phi_M d\mathbf{x} \tag{6.12}$$

**6.1. Summary.** The procedure in the distributed case is to solve a coupled system for $M$, $\phi_M$ and $\mathbf{v}_M$ consisting of the invariance equation (6.2), the monitor velocity potential equation (6.10) and the monitor velocity equation (6.12).

As in the continuous and finite volume cases, equation (6.2) can be used to express $M$ in terms of $\mathbf{x}$ and an ODE system formed for $\mathbf{x}$ from (6.12) using the solution of (6.10) for $\phi_M$. The new mesh is obtained by integrating the ODE system. The function $M$ may be reconstructed via equation (6.2) and thereby the solution $u$. If that is not feasible (for example for systems), an alternative is to obtain $u$ through the ALE equations (5.2) or the PDE in the moving frame (2.2).

**6.2. A Moving Finite Element method.** A Moving Finite Element method may be constructed using the weak forms (6.2), (6.10) and (6.12). For full details see [1]. Linear elements are used for $u$, $\mathbf{v}_M$, and $\phi_M$, here denoted by $U$, $\dot{\mathbf{X}}$, and $\Phi$, on a (moving) triangulation of the region. The weight function $\widehat{w}$ is taken to be the usual piecewise linear basis function $\hat{W}_i$ (in the reference space). Since $\mathbf{v}_M$ is piecewise linear and $W_i$ is the advected form of $\hat{W}_i$ (cf. (6.6)), the corresponding functions $W_i$ are the usual linear basis functions on the moving mesh. The support $\Omega(\mathbf{v}_M)$ of the integrals in the $i$'th equation is the patch of elements $\Pi_i(\dot{\mathbf{X}})$ surrounding the node.

**6.3. Matrix Forms.** Expanding $M$ in the space of functions $W_i$ as

$$M = \sum M_i W_i \tag{6.13}$$

equation (6.2) leads to the matrix equation

$$A(\vec{\mathbf{X}})\vec{M} = \vec{C} \tag{6.14}$$

where $\vec{M}$ is the vector of coefficients $M_i$ and $\vec{C}$ is a vector of constants determined by the initial distribution of $M$ in (6.2). Here

$$A(\vec{\mathbf{X}}) = \{A_{ij}\} \quad \text{with} \quad A_{ij} = \int_{\Pi_i(\dot{\mathbf{X}})} W_i W_j d\Omega \tag{6.15}$$

is a mass matrix.

Also expanding $\Phi$ in the space of functions $W_i$ as

$$\Phi = \sum_i \Phi_i W_i \tag{6.16}$$

equation (6.10) leads in the standard way to the matrix equation

$$K(\vec{\mathbf{X}}, M)\vec{\Phi} = \vec{f} \tag{6.17}$$

where $\vec{\Phi}$ is the vector of coefficients $\Phi_i$ and $\vec{\mathbf{X}}$ is a vector of the nodal positions $\mathbf{X}_i$. Here (apart from boundary terms)

$$K(\vec{\mathbf{X}}, M) = \{K_{ij}\} \quad \text{with} \quad K_{ij} = \int_{\Pi_i(\dot{\mathbf{X}})} \mu^{-1} M \nabla W_i . \nabla W_j d\Omega \tag{6.18}$$

is a weighted stiffness matrix and

$$\vec{f} = \{f_j\} \quad \text{with} \quad f_j = \int_{\Pi_j(\dot{\mathbf{X}})} W_j \left(\nabla \cdot (M\mathbf{V_p}) + M'(U)LU\right) d\Omega \tag{6.19}$$

where $\mathbf{V}_p$ is the discretisation of $\mathbf{v}_p$.

Finally, equation (6.12) determines the $\dot{\mathbf{X}}_i$ from the $\Phi$. Since $\Phi$ and $\dot{\mathbf{X}}$ are piecewise linear functions we have the matrix equation

$$A(\vec{\mathbf{X}})\frac{d}{dt}\vec{\mathbf{X}} = \vec{b} \tag{6.20}$$

where

$$\vec{b} = \{b_i\} \quad \text{with} \quad b_i = \int_{\Pi_i(\dot{\mathbf{X}})} W_i(\mathbf{V}_p + \mu^{-1}\nabla\Phi) d\Omega \tag{6.21}$$

The boundary conditions on (6.14) are not imposed strongly, otherwise the resulting equations are inconsistent with mass conservation and the velocity potential equation (6.17). Note that if $M = 0$ on the outer boundary then since $W_i = 0$ on the internal boundaries all the boundary terms in (6.10) vanish.

Once the $\mathbf{X}$'s are known another application of (6.14) allows us to recover the function $M$. Then, by inverting this function $U$ may be obtained.

**6.4. ALGORITHM.** We solve the ODE system

$$\frac{d}{dt}\vec{\mathbf{X}} = \vec{\mathbf{F}}(\vec{\mathbf{X}}) \tag{6.22}$$

using the sequence

- Given $\vec{\mathbf{X}}$ recover $M$ from (6.14)
- Given $M$ calculate $\vec{\Phi}$ from (6.17)
- Calculate $\vec{\mathbf{F}}(\vec{\mathbf{X}})$ as $A(\vec{\mathbf{X}})^{-1}\vec{b}$ (see (6.20))
- Return

No iteration is needed. Once $M$ is known the solution $U$ may be recovered by inverting $M(U)$.

**7. The Porous Medium Equation.** We demonstrate the method for a particular PDE, the Porous Medium Equation,

$$\frac{\partial u}{\partial t} = \nabla \cdot (u^m \nabla u), \tag{7.1}$$

which is a well-known model equation for gas flows in porous media, spreading liquids etc. It admits compact support solutions with a moving boundary. for which comparison results also exist [6].

Following Budd et al [1] we observe that the radially symmetric PME

$$\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(u^m \frac{\partial u}{\partial r}\right) \tag{7.2}$$

with global mass conservation is invariant under the scalings

$$t \to \lambda t, \qquad r \to \lambda^{\frac{1}{2m+2}} r, \qquad u \to \lambda^{-\frac{1}{m+1}} u \tag{7.3}$$

This invariance allows self-similar solutions to be constructed and suggests that the monitor function should be chosen to be $M = u$, since the integral $\int urdr$ is *locally* invariant. With this choice of $M$ the boundary integrals in (6.10) vanish.

A radially symmetric self-similar solution of the radially symmetric PME is

$$u = \begin{cases} \left(\frac{t_0}{t}\right)^{\frac{1}{m+1}} \left(1 - \left(\frac{r}{Kt^{\frac{1}{2m+2}}}\right)^2\right)^{\frac{1}{m}} & r \le Kt^{\frac{1}{2m+2}} \\ 0 & r > Kt^{\frac{1}{2m+2}} \end{cases} \tag{7.4}$$

where $t_0, K$ are constants. The function is zero at the moving boundary and is sketched in cross section in Figure 7.1 for $m = 4$. For $m = 1$ the slope at the boundary is finite while for $m > 1$ it is infinite. The global mass is conserved. We shall use this solution to test the 2-D algorithm.

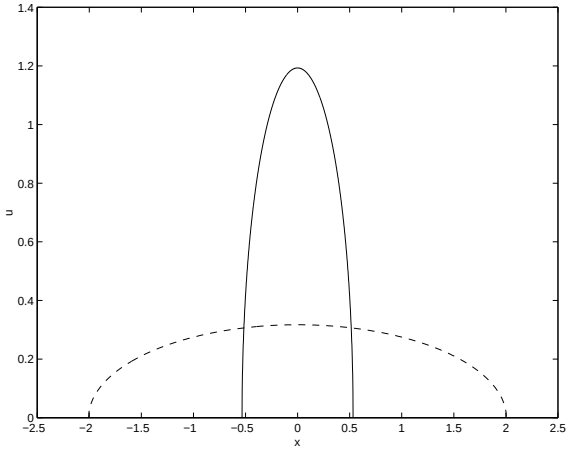


FIG. 7.1. *Behaviour of the radially symmetric self-similar solution of the PME (in cross-section) for* $m = 4$.

**8. Numerical Results.** The finite element equations ((6.2), (6.10) and (6.12)) have been solved, as in section 5.3, with $M = u$ and $\frac{\partial M}{\partial t}$ given by the PME (7.1).

We take $\mu = 1$ and $\mathbf{v}_p = 0$ so that $curl\mathbf{v}_M$=0. $K$ is chosen to give an initial radius of 0.5.

The evolution of the three cases when $m = 1, 2, 4$ is illustrated in Figure 7.2, which contains slices through the origin of the approximate and exact solutions, alongside three-dimensional views of the numerical solution at $t = 2.0$ obtained on a genuinely unstructured, but still uniform, 2349 node, 4539 cell, mesh. The approximations

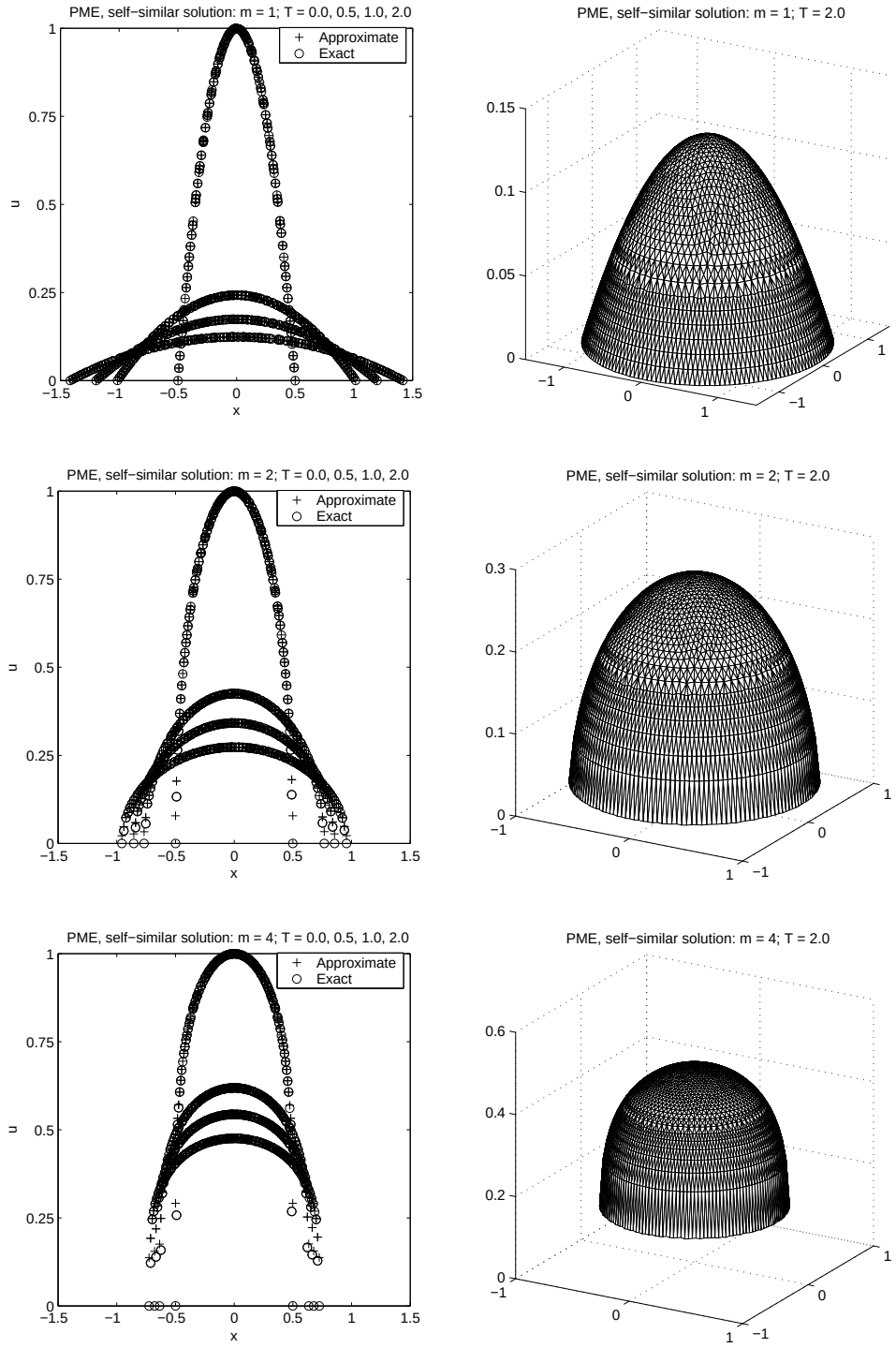


FIG. 7.2. *Slices through the origin of the radially symmetric self-similar solutions to the PME (7.4) at various times, comparing exact with approximate (left), along with the approximation at* $t = 2.0$ *(right), for* $m = 1$ *(top),* $2$ *(middle) and* $4$ *(bottom).*

are accurate, but the figures clearly show that $u = 0$ is enforced only weakly at the boundary.

Clustering nodes towards the boundary improves the quality of the solution, particularly in the $m = 2, 4$ cases, although it begins to deteriorate if the clustering is too pronounced, possibly because the simple node positioning algorithm used to create the initial meshes pulls nodes away from regions where the second derivatives of the solution are high.

The scaled variables

$$t^{1/(m+1)}u \qquad \text{and} \qquad t^{-1/(2m+2)}r$$

remain constant, as predicted by the theory (see (7.3)). Mass is always conserved to machine accuracy, which is crucial to the domain maintaining its circular shape over long time periods.

**8.1. Errors.** Errors in the solution are found to be $O((\Delta x)^2)$ in $l1$ for $m = 1$. The two other exponents, where the gradient of the exact solution is infinite at the boundary of the domain, give lower accuracy. Further details can be found in [1].

At any mesh point, the scaled variables, $\hat{u} = t^{1/(m+1)})u$ and $\hat{r} = t^{-1/(2(m+1))}r$ remain constant, as predicted by the theory [4].

A second measure of the accuracy of the solution is given by the error in the position of the boundary of the domain. The normalised maximum and minimum radii of the computational domain converge towards each other, and eventually to the exact radius (see [1]).

It is also possible to impose a background velocity field $\mathbf{v}_p$ on the mesh movement equations via the extra vorticity term in (4.3). With a background mesh velocity of $\mathbf{v}_p = 2.5r(-y, x)^{\mathrm{T}}$ the solution remains close to that obtained on a fixed mesh until tangling becomes inevitable (see [1]).

**8.2. Comparison results.** The new scheme is not restricted to modelling self-similar solutions. Within certain bounds, the evolution of any initial conditions can be predicted.

We have investigated a comparison property of the approximate solutions which reflects the same property of the exact solution of the PME (7.2) (viz. the comparison theorems in [6]), i.e. given three sets of initial conditions,

$$u_1(x, y, t_0) \le u_2(x, y, t_0) \le u_3(x, y, t_0) \quad \forall\, (x, y) \in \Omega\,, \tag{8.1}$$

then

$$u_1(x, y, t) \le u_2(x, y, t) \le u_3(x, y, t) \quad \forall\, (x, y) \in \Omega,\ t \ge t_0\,. \tag{8.2}$$

This property also appears to hold for the approximate solution derived here, as can be seen in Figures 8.1 and 8.2 which show two experiments in which the initial conditions are perturbed. In Figure 8.1 a random perturbation is applied to the initial solution and its evolution compared with two radially symmetric solutions scaled according to the minimum and maximum perturbations. In Figure 8.2 a sinusoidal perturbation is applied to the initial position of the boundary and its evolution is found to be sandwiched in a similar manner. In both cases the initial random perturbations are smoothed out very rapidly.

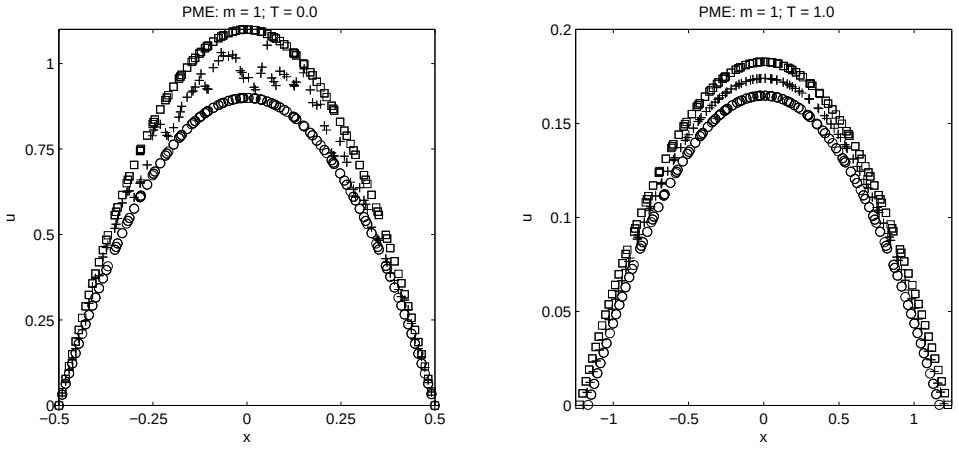


FIG. 8.1. *Slices of the initial conditions (left) and approximate solutions at $t = 1$ (right) taken through the origin, illustrating the 'sandwiching' of a randomly perturbed solution to the PME with $m = 1$.*

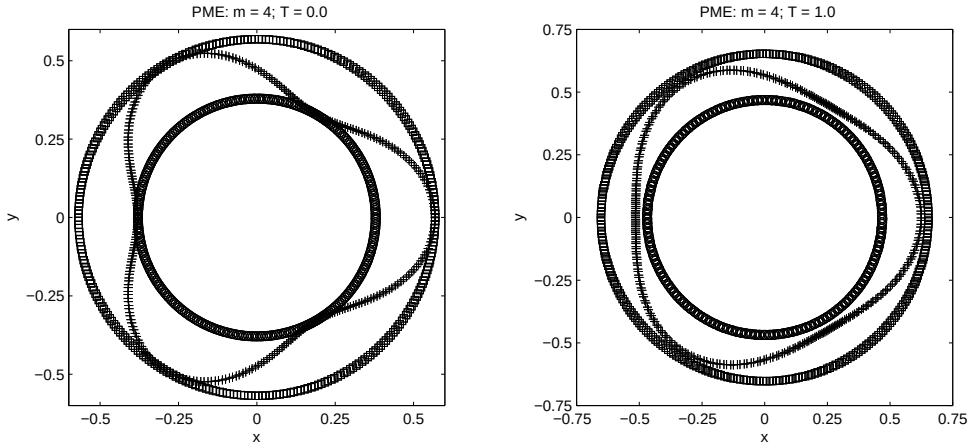


FIG. 8.2. *Slices of the initial conditions (left) and approximate solutions at $t = 1$ (right) taken through the origin, illustrating the 'sandwiching' of a sinusoidally perturbed mesh for the PME with $m = 4$.*

**9. Conclusions.** A moving mesh method is described using a form of Lagrangian fluid invariance by replacing the density with a monitor function. The resulting monitor velocity gives a moving frame on which to solve the problem. For scalar problems the solution can be reconstructed from the invariance equation itself.

A moving finite element method is described based on partial invariances and a weighted form of the equations. The method, which involves no background mesh or iteration, needs only the solution of three symmetric matrix equations and the integration of one ODE system.

The method is demonstrated on the 2-D porous medium equation (7.1) with the choice of monitor informed by scale invariance.

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