TWO-DIMENSIONAL SHALLOW WATER FLOWS IN UNSTRUCTURED GRIDS

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Abstract

A multidimensional upwinding technique is applied to the simulation of 2D shallow water flows. It is adapted from fluctuation splitting methods recently proposed for the solution of the Euler system of equations on unstructured triangular grids. The basis of the numerical method is stated and the particular adaptation to the shallow water system is described. A test case of interest to hydraulic engineers is presented.

1. INTRODUCTION

Classical methods and central difference schemes still dominate the software products for the shallow water equations, with Preissmann's, Abbott's (see [3] for example) and McCormack's [6] schemes the most commonly used. Some years after their adoption for solving problems in gas dynamics, upwind and TVD (Total Variation Diminishing) numerical schemes have been successfully used for the solution of the shallow water equations, with similar advantages [7]. Their use is nevertheless only gradually gaining acceptance in this sector.
In a philosophy different from concentrating on finite volumes and the changes of the variables across the cell sides, Deconinck et al. [4] consider solutions on triangular grids in which the unknowns are associated with the vertices and updates to these nodal values are through the advection of linear wave solutions. This avoids the problems of taking the normal to the cell interfaces as a privileged direction.
In this paper we consider the use of this technique for 2D shallow water flows and the question of whether they may be of practical use. In the next sections, the basis of the numerical method is stated and the adaptation to the shallow water system is described. Finally, some numerical results are presented. Although this work is at an early stage, our results indicate that the advantages may outweigh the disadvantages and that these schemes may have a future for hydraulic engineering applications.

2. BASIC TECHNIQUE FOR SYSTEMS OF EQUATIONS

If the equation to be solved is non-linear, a suitable linearization must be performed before the existing techniques for linear equations are applied. An averaged advection speed which satisfies discrete conservation can be found by assuming linear variation of the conserved quantities w over the cell and therefore constant gradient ∇w. Since the advection schemes used for the shallow water equations are no different for those used for the Euler equations we will not go into further detail about their particular construction and description. We refer the reader to the very good reviews in [9] and [14].
The application of multi-dimensional upwinding to a general non-linear 2D system of conservation laws

$$\frac{\partial \mathbf{w}}{\partial t} + \nabla.(\mathbf{F}(\mathbf{w})) = 0 \quad , \quad \mathbf{F} = (\mathbf{f}, \mathbf{g})$$

requires a discrete form of the linearization

$$\frac{\partial \mathbf{w}}{\partial t} + (A,B)\nabla(\mathbf{w}) = 0.$$

Where, in particular, a consistent approximation for the cell residual is sought

$$\mathbf{R}_T = (\bar{A},\bar{B})_T \cdot \nabla \mathbf{w}_T \tag{1}$$

where $\bar{A}_T, \bar{B}_T$ are discrete equivalents of the cell averaged Jacobian matrices, calculated using the nodal values. The important assumption of linear variation of $\mathbf{w}$ on each cell, enables to write

$$\mathbf{R}_T = \frac{1}{S_T}\int_T (A,B)\cdot\nabla \mathbf{w} dS = \frac{1}{S_T}\nabla \mathbf{w}_T \int_T (A,B) dS$$

where discrete cell gradients and cell Jacobians can be defined

$$\nabla \mathbf{w}_T = \frac{1}{S_T}\sum_{i=1}^{3} \mathbf{w}_i \mathbf{n}_i \ , \ \bar{A} = \frac{1}{S_T}\int_T A dS \ , \ \bar{B} = \frac{1}{S_T}\int_T B dS$$

Unfortunately, the exact evaluation of the above integrals is not practical either for the Euler or for the shallow water equations. Roe [12] suggested the introduction of a parameter set of variables for a simpler treatment of the former system. The strategy we have followed for the shallow water equations makes use of the set of primitive variables and is described in the next section.

3. THE 2D SHALLOW WATER SYSTEM

In the conservative formulation of the system of equations, with $\mathbf{U} = (h\ ,\ uh\ ,\ vh)^T$, where h, u and v are the depth and x and y velocities respectively, the fluctuation is defined as

$$\Phi_T = \int_T \mathbf{U}_t dS = -\int_T (\mathbf{E}_x + \mathbf{F}_y) dS.$$

We are seeking a conservative discrete approximation of the Jacobians like in eq.(1) satisfying

$$\Phi_T \approx -S_T(\bar{\mathbf{E}}_x + \bar{\mathbf{F}}_y) = -S_T(\bar{A}\bar{\mathbf{U}}_x + \bar{B}\bar{\mathbf{U}}_y).$$

We can use the transformation matrix M between the conserved variables $\mathbf{U}$ and the primitive variables $\mathbf{V} = (h\ ,\ u\ ,\ v)^T$ to define new matrices R and S,

$$R = \frac{\partial \mathbf{E}}{\partial \mathbf{V}} = \frac{\partial \mathbf{E}}{\partial \mathbf{U}}\frac{\partial \mathbf{U}}{\partial \mathbf{V}} = AM \ , \ S = \frac{\partial \mathbf{F}}{\partial \mathbf{V}} = \frac{\partial \mathbf{F}}{\partial \mathbf{U}}\frac{\partial \mathbf{U}}{\partial \mathbf{V}} = BM$$

so that:

$$\mathbf{E}_x + \mathbf{F}_y = \mathbf{E}_V \mathbf{V}_x + \mathbf{F}_V \mathbf{V}_y = R\mathbf{V}_x + S\mathbf{V}_y.$$

Then, provided that the variables $\mathbf{V}$ are linear over the cells T, the gradients, $\mathbf{V}_x$ and $\mathbf{V}_y$, are constant, and this enables us to write the fluctuation as

$$\Phi_T = -(\int_T (R(\mathbf{V})\mathbf{V}_x + S(\mathbf{V})\mathbf{V}_y) dS = -S_T[\bar{R}\mathbf{V}_x + \bar{S}\mathbf{V}_y]$$

with the definitions:

$$\bar{R} = \frac{1}{S_T}\int_T R(\mathbf{V}) dS \ , \ \bar{S} = \frac{1}{S_T}\int_T S(\mathbf{V}) dS. \tag{2}$$

We now replace $\bar{R}$, $\bar{S}$ by $\bar{R} = R(\bar{\mathbf{V}})$, $\bar{S} = S(\bar{\mathbf{V}})$, where the averaged variables are simply calculated summing over the nodal values at the vertices of the triangle T. Note that with this definition of $\bar{R}$, $\bar{S}$ we are only approximating equation (2), unlike in the Euler equations where an exact representation of the integral is obtained because in this case R is linear in $\mathbf{V}$.
It is easy to identify $\bar{\mathbf{E}}_x = \bar{R}\mathbf{V}_x$ and $\bar{\mathbf{F}}_y = \bar{S}\mathbf{V}_y$, and we can use the change of variables to rewrite the fluctuation in terms of suitable averages of the conserved variables [8].
The next thing to do is to compute the fluctuations and distribute them to the vertices of every cell by means of an advection scheme. For that purpose it is necessary to work out the gradients $\nabla\mathbf{V} = (\mathbf{V}_x, \mathbf{V}_y)$ within each triangle, and decompose the residual into parts that can be explained as due to the passage of a wave. The latter step will require a description of wave models.

4. WAVE MODELS

Considering the linearized system of equations written in primitive variables

$$\frac{\partial \mathbf{V}}{\partial t} + \bar{G}\frac{\partial \mathbf{V}}{\partial x} + \bar{H}\frac{\partial \mathbf{V}}{\partial y} = 0.$$

A simple wave solution can be found, as in Roe [11,12].
It is then possible to express the gradient as the sum

$$\nabla\mathbf{V} = \sum_{k=1}^{n} \alpha^k \mathbf{r}^k \mathbf{n}^k$$

that is,

$$\mathbf{V}_x = \sum_{k=1}^{n} \alpha^k \mathbf{r}^k cos\theta^k \ , \ \mathbf{V}_y = \sum_{k=1}^{n} \alpha^k \mathbf{r}^k sin\theta^k.$$

The vectors $\mathbf{r}^k$ are the right eigenvectors of the matrix $M^* = \bar{G}cos\theta + \bar{H}sin\theta$. The variables α^k represent weighting coefficients of the sum and θ^k are the different angles of each wave. The celerity, c, is the equivalent of the speed of sound in gas-dynamics and is the velocity of small perturbations in still water, given by $c = \sqrt{gh}$.
The connection between the gradient of the primitive variables and that of the averaged conservative variables can be used to develop the latter as

$$\bar{\mathbf{U}}_x = \sum_{k=1}^{n} \alpha^k \mathbf{r}_c^k cos\theta^k \ , \ \bar{\mathbf{U}}_y = \sum_{k=1}^{n} \alpha^k \mathbf{r}_c^k sin\theta^k$$

where now, $\mathbf{r}_c^k$ represent the right eigenvectors of the matrix $M_c^* = \bar{A}cos\theta + \bar{B}sin\theta$, and can be worked out through $\mathbf{r}_c^k = M(\bar{\mathbf{V}})\mathbf{r}^k$. The two matrices M^* and M_c^* share the unique set of eigenvalues, λ^k. The residual then can be split into a sum of waves

$$R_T = \bar{A}\bar{\mathbf{U}}_x + \bar{B}\bar{\mathbf{U}}_y = \sum_{k=1}^{n} \alpha^k \lambda^k \mathbf{r}_c^k.$$

The wave decomposition of the gradients represents a system of 6 equations in the shallow water case, where we have two spatial derivatives for each of the three variables. Therefore, it allows for 6 unknowns. These must correspond to either the coefficients or angles of a propagation of suitable choices of waves whose advection will represent the total fluctuation. Following Roe's suggestions for the treatment of the Euler equations [11], the splitting can be made into four orthogonal acoustic waves, labelled by their strengths (coefficients) and one angle θ which determines the four directions as well as one shear wave of strength β at an angle ϕ.

Rudgyard's wave models are mainly based on the idea of obtaining the six waves by choosing two propagation angles, θ_1 and θ_2, and performing a decomposition of the gradient,

$$\nabla \mathbf{V} = \sum_{k=1}^{3} \alpha_{\theta 1}^{k} \mathbf{r}_{\theta 1}^{k} \mathbf{n}_{\theta 1} + \sum_{k=1}^{3} \alpha_{\theta 2}^{k} \mathbf{r}_{\theta 2}^{k} \mathbf{n}_{\theta 2}$$

which contains six free parameters, the six α coefficients. The vectors $\mathbf{n}_\theta = (cos\theta, sin\theta)$ are again the unit vectors in the directions θ, and $\mathbf{r}_\theta^k$ are the right eigenvectors of the matrix M^* for each value of θ.

5. NUMERICAL RESULTS

The treatment of the solution at the points on the boundaries of the domain has been kept as close as possible to the theory of characteristics in 2D. For the interior points we used the non-linear PSI advection algorithm [10] but obtained very similar results with the other advection schemes. As for the wave model, the calculations correspond to Rudgyard's decomposition having been found more robust, in general, than the one corresponding to Roe's model D.
The selected example is an excellent test case for a multidimensional shock capturing algorithm. It consists of a series of discontinuities produced when a supercritical flow in a rectangular channel meets a sharp constriction in the cross section. A first shock wave is formed in front of the contraction. It bends towards the downstream direction and is reflected several times on the walls of the narrower part of the channel A 2016 element triangular mesh was used to reproduce the flow in a 3m long and 1m wide rectangular channel with a 20By the time reached in the results shown (approx.3.7s), the flow is steady. As can be seen in the upper part of Fig. 1, the scheme is able to capture all the oblique hydraulic jumps and a discontinuous water surface devoid of oscillations is obtained. Similar accuracy is achieved on a cuadrilateral grid with the TVD in finite volume method reported in [1]. Better resolution is obtained as the grid is refined. This can be seen in the lower part of Fig.1, where a 8064 elements grid was used. A strategy of cell movement proposed by Baines [5] can be exploited for the unstructured grid in order to improve the results. The possibility of using an algorithm capable of making the cells migrate towards the regions of steeper gradients allows the reduction of the total number of cells. A preliminary, but encouraging, result can be found in [9].

6. CONCLUSIONS

Two dimensional wave decomposition and multi-dimensional upwinding seem a promising method of solution for the 2D shallow water equations. Two wave models have been adapted to render the technique suited to hydraulic problems with shocks. As with the 1D TVD schemes, our experience with using the multi-dimensional upwind approach for the shallow water equations has closely followed that of the researchers solving the Euler equations (with both the advection schemes and wave models) showing the same properties as for that system of equations.
Although the procedure is more complicated and costly than present day generalizations of 1D upwinding techniques it is based on a triangular discretization and, by taking advantage of the triangles, the disadvantages can be overcome making the schemes very competitive, and the future for them then looks much more promising. They can clearly be applied to arbitrary geometries, a great advantage for hydraulic engineers working on practical problems, and there is a wide variety of possibilities concerning grid movement and adaption.

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Figure 1: Map of isolines for the steady state solution. Coarse grid.

Figure 2: Map of isolines for the steady state solution. Finer grid