Very High Order, Non-Oscillatory Fluctuation Distribution Schemes

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1 Introduction

Fluctuation distribution schemes for approximating multidimensional systems of conservation laws have developed to a stage where they can be used reliably to produce accurate simulations of complex steady state fluid flow phenomena using unstructured meshes [DSA00]. These methods are often required to avoid producing unphysical, numerically induced, oscillations in the flow field, and this has so far restricted them to second order accuracy.

More recent research has led to higher order methods (for both steady state and time-dependent flows) which use the fact that a k^{th} order method can be derived by evaluating the fluctuation exactly with respect to a $(k-1)^{\text{th}}$ degree polynomial representation of the dependent variable and then distributing it in a linearity preserving manner [AR03]. So far, two successful approaches have been proposed for constructing this high order interpolant within each mesh cell [AR03, CF02]. A third alternative will be presented here.

As they stand, none of these three approaches can guarantee the absence of spurious oscillations from the flow without the application of an additional smoothing stage. This paper will briefly describe a technique which can be combined with any of the above procedures to provide schemes which are both higher than second order accurate and free of spurious oscillations. Brief results will be shown to demonstrate its effectiveness in approximating the scalar advection equation on two-dimensional, unstructured, triangular meshes.

2 Fluctuation Splitting

Consider the two-dimensional scalar conservation law given by

$$u_t + f_x + g_y \;=\; 0 \qquad \text{or} \qquad u_t + \boldsymbol{\lambda} \cdot \boldsymbol{\nabla} u \;=\; 0 \tag{1}$$

on a domain Ω, with $u(x, y, t) = g(x, y, t)$ imposed on the inflow part of the boundary $\partial\Omega$. $\boldsymbol{\lambda} = \left(\frac{\partial f}{\partial u}, \frac{\partial g}{\partial u}\right)^T$ defines the advection velocity associated with

the conservation law (1). This equation has an associated fluctuation, assumed here to be calculated over a triangular mesh cell $\triangle$ and given by

$$\phi \;=\; -\iint_{\triangle} \boldsymbol{\lambda}\cdot\boldsymbol{\nabla} u \; d\Omega \;=\; \oint_{\partial\triangle} u\,\boldsymbol{\lambda}\cdot d\mathbf{n}\,, \tag{2}$$

in which $\mathbf{n}$ represents the inward pointing normal to the cell boundary. It will be assumed throughout this paper that u has a continuous piecewise polynomial representation and the integration in (2) is carried out exactly (though this isn't always necessary [AB02]). Given this, a simple forward Euler discretisation of the time derivative leads to an iterative update of the nodal solution values which is generally written [DSBR94] as

$$u_i^{n+1} \;=\; u_i^n + \frac{\Delta t}{S_i} \sum_{j\in\cup\triangle_i} \alpha_i^j\,\phi_j\,, \tag{3}$$

where Δt is the time-step, S_i is the area of the median dual cell corresponding to node i, α_i^j is the distribution coefficient which indicates the appropriate proportion of the fluctuation ϕ_j to be sent from cell j to node i, and $\cup\triangle_i$ represents the set of cells with vertices at node i. Conservation is assured as long as $\sum_{i\in\triangle_j}\alpha_i^j = 1\,,\ \forall j$, where $\triangle_j$ represents the set of nodes at the vertices of cell j.

2.1 A Second Order Positive Scheme

The basis of most positive fluctuation distribution schemes is the N scheme [DSBR94]. Its derivation relies on the fluctuation being evaluated exactly, as

$$\phi \;=\; -\frac{1}{2}\sum_{i\in\triangle} u_i\,\hat{\boldsymbol{\lambda}}\cdot\mathbf{n}_i \;=\; -\sum_{i\in\triangle} u_i\,k_i\,, \tag{4}$$

where the symbol $\hat{}$ indicates an appropriately linearised quantity and $\mathbf{n}_i$ is the inward unit normal to the i^{th} edge (opposite the i^{th} vertex) multiplied by the length of that edge. The dependent variable u is assumed to vary in a continuous piecewise linear manner, with the unknowns stored at the mesh nodes. Now, for each triangle it is always possible to locate a pair of vertices i_2 and i_3 for which $k_{i_2}k_{i_3}\geq 0$. The fluctuation can then be written

$$\phi^{LO} \;=\; k_{i_2}(u_{i_1}-u_{i_2}) + k_{i_3}(u_{i_1}-u_{i_3}) \;=\; \phi_{i_2}^{LO} + \phi_{i_3}^{LO}\,. \tag{5}$$

When the choice of i_2 and i_3 is not unique, *i.e.* $k_i = 0$ for some vertex i, this node can be paired with either of the other vertices. This leads to an alternative formulation of the N scheme, given by

$$\begin{aligned}
S_{i_1}u_{i_1} &\;\rightarrow\; S_{i_1}u_{i_1} + \Delta t\,k_{i_2}^-(u_{i_1}-u_{i_2}) + \Delta t\,k_{i_3}^-(u_{i_1}-u_{i_3})\\
S_{i_2}u_{i_2} &\;\rightarrow\; S_{i_2}u_{i_2} + \Delta t\,k_{i_2}^+(u_{i_1}-u_{i_2})\\
S_{i_3}u_{i_3} &\;\rightarrow\; S_{i_3}u_{i_3} + \Delta t\,k_{i_3}^+(u_{i_1}-u_{i_3})\,,
\end{aligned} \tag{6}$$

in which $\cdot^{\pm}$ denotes the positive/negative part of the quantity. This scheme is locally (and hence globally) positive, so the iteration given by (3) is conditionally stable, the appropriate restriction on the time-step being

$$\Delta t \;\leq\; \frac{S_i}{\sum_{j\in\cup\triangle_i} \max\left(0, \frac{1}{2}\hat{\boldsymbol{\lambda}}\cdot\mathbf{n}_i^j\right)}\,. \tag{7}$$

The contribution made by cell j to node i by the N scheme can be written as $(\phi_i^j)^N = (\alpha_i^j)^N\phi_j$, where ϕ_j is the fluctuation in cell j (see (6)). The contributions due to the PSI scheme, the most commonly used of the second order non-oscillatory fluctuation distribution schemes, can then be defined by

$$(\phi_i^j)^{PSI} \;=\; \frac{[(\alpha_i^j)^N]^+}{\sum_{k\in\triangle_j}[(\alpha_k^j)^N]^+}\phi_j \;=\; (\alpha_i^j)^{PSI}\phi_j\,. \tag{8}$$

This scheme can easily be shown to be conservative, positive for the time-step given by (7), and linearity preserving, *i.e.* $(\alpha_i^j)^{PSI}$ is bounded so the order of accuracy of the steady state scheme is one higher than the degree of the polynomial used to represent u (in this case second order) [Abg01, AM03].

3 Higher Order Positive Schemes

The first stage in the creation of higher order fluctuation distribution schemes is the construction of a higher order representation of the dependent variable u in (2). In this work a continuous piecewise quadratic interpolant of the nodal data is sought, though higher degree polynomials are also possible. In previous work, quadratics have been produced in each mesh cell by either (1) reconstructing $\boldsymbol{\nabla}u$ at the mesh vertices and combining them with the nodal values of u to satisfy the necessary degrees of freedom [CF02], or (2) storing and updating values of u at additional nodes at the midpoints of each cell edge [AR03]. A third approach is proposed here, which uses the values of u at additional mesh nodes beyond the given cell to construct the local polynomial: for a quadratic, three additional nodes are required and these are chosen to be the vertices of the three neighbouring triangles opposite the given cell's edge (except for boundary cells, which are treated as special cases, and some extreme mesh topologies, which do not occur here). This does not immediately produce a continuous representation, which is imposed here by averaging the two interpolants at each edge.

Unfortunately, it is easy to show that the fluctuations which result from any of these three interpolants cannot lead to a locally positive scheme if they are distributed locally within the mesh cell (or subcell in the case method (2)) [Hub06]. It is, for example, possible to have a non-zero fluctuation in a (sub)cell for which $u_{i_1} = u_{i_2} = u_{i_3}$. The possibility of distributing the fluctuation farther afield is currently under investigation but has yet to yield a scheme of practical value.

An alternative approach is considered here which overcomes this obstacle by modifying the high order interpolant. Let $\bar{u}(\mathbf{x})$ be the linear interpolant of the values of u at the vertices of a given triangular (sub)cell and $u(\mathbf{x}) = \bar{u}(\mathbf{x}) + \delta u(\mathbf{x})$ be a higher order representation of the data within that triangle. If the high order correction $\delta u(\mathbf{x})$ on each (sub)cell edge $i_1 i_2$ is limited to give $\delta u'(\mathbf{x})$ along that edge which satisfies

$$|\delta u'_{i_1 i_2}(\mathbf{x})| \;\le\; C|u_{i_1} - u_{i_2}| \qquad \forall\, \mathbf{x} = \mu\, \mathbf{x}_{i_1} + (1-\mu)\, \mathbf{x}_{i_2}\,,\;\; 0 \le \mu \le 1\,, \quad (9)$$

for some finite constant $C \ge 0$ then, subject to an appropriate restriction on the time-step in (3), it is possible to distribute the fluctuation (2) due to the modified interpolant $u'(\mathbf{x}) = \bar{u}(\mathbf{x}) + \delta u'(\mathbf{x})$ to the vertices of the specified (sub)cell in a locally positive manner [Hub06].

For a quadratic representation of u and linearly varying $\boldsymbol{\lambda}$ only the edge midpoints are required by the quadrature for exact evaluation of the fluctuation, so the limiting on edge $i_1 i_2$ is carried out by using

$$u'_{i_1 i_2} \;=\; \frac{u_{i_1} + u_{i_2}}{2} + \alpha_{i_1 i_2}(u_{i_1} - u_{i_2}) \quad (10)$$

as the limited solution value at the edge midpoint, where

$$\alpha_{i_1 i_2} \;=\; \max\left(-C, \min\left[C, \frac{u_{i_1 i_2} - (u_{i_1} + u_{i_2})/2}{u_{i_1} - u_{i_2}}\right]\right). \quad (11)$$

A value of $C = 0.25$ is chosen here. This is the largest value that guarantees that the limited interpolant along each edge is monotonic (if it is chosen to be of the form $\delta u'(\mathbf{x}) = C'\, \delta u(\mathbf{x})$ where $C' \in [0,1]$). Also, larger values of C tend to reduce the rate of convergence of the iteration (3) to the steady state. The general case requires limiting at additional quadrature points.

As a result of the above procedure, the limited high order fluctuation can be written

$$\phi^{LIM} \;=\; \oint_{\partial\triangle} u'\, \boldsymbol{\lambda} \cdot \mathbf{n}\; d\Gamma \;=\; K_{i_2}(u_{i_1} - u_{i_2}) + K_{i_3}(u_{i_1} - u_{i_3}) \quad (12)$$

where i_1, i_2 and i_3 are chosen to be precisely those vertices designated by the N scheme (5). Explicit, *bounded* expressions can easily be found for K_{i_2} and K_{i_3} [Hub06]. The formulation of the N scheme given in (6) can then be applied directly to this higher order fluctuation, *i.e.*

$$\begin{aligned}
S_{i_1} u_{i_1} &\to S_{i_1} u_{i_1} + \Delta t\, K^-_{i_2}(u_{i_1} - u_{i_2}) + \Delta t\, K^-_{i_3}(u_{i_1} - u_{i_3}) \\
S_{i_2} u_{i_2} &\to S_{i_2} u_{i_2} + \Delta t\, K^+_{i_2}(u_{i_1} - u_{i_2}) \\
S_{i_3} u_{i_3} &\to S_{i_3} u_{i_3} + \Delta t\, K^+_{i_3}(u_{i_1} - u_{i_3})\,.
\end{aligned} \quad (13)$$

This scheme is clearly locally positive for a small enough time-step, the limit on which is approximately inversely proportional to C. The distribution coefficients of the resulting N-like (N^*) scheme take the form $(\phi^j_i)^{N^*} =$

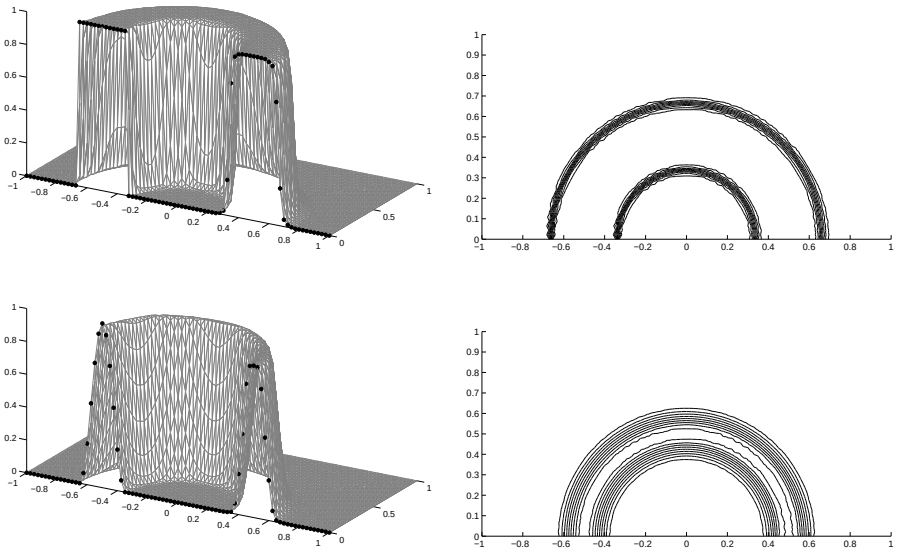

Fig. 1. The new scheme applied to circular advection of a square wave (top) and a cosine-squared profile (bottom).

$(\alpha_i^j)^{N^*}\phi_j^{LIM}$, and these can be limited in precisely the manner which created the PSI scheme (8) by imposing linearity preservation on the N scheme, *i.e.*

$$(\phi_i^j)^{PSI^*} \;=\; \frac{[(\alpha_i^j)^{N^*}]^+}{\sum_{k\in\triangle_j}[(\alpha_k^j)^{N^*}]^+}\,\phi_j^{LIM} \;=\; (\alpha_i^j)^{PSI^*}\,\phi_j^{LIM}\,. \qquad (14)$$

As with the PSI scheme, the limiting procedure will never increase the magnitude of the distribution coefficients, so the positivity condition for the scheme (13) is actually stronger than necessary.

4 Results

The two-dimensional scalar advection equation (1) is approximated, over the domain $[-1,1]\times[0,1]$ with $\boldsymbol{\lambda} = (y,-x)^T$ and $u(x,y,0) = g(x)$ for $x \in [-1,0]$ and $y = 0$ (where $g(x)$ is chosen to take a variety of forms to demonstrate the properties of the schemes) [Hub06].

Figure 1 shows the results obtained from the positive, high order scheme derived from extending the stencil. Replacing $g(x)$ with much smoother function, and carrying out a series of experiments on successively refined meshes suggests an order of accuracy of 2.36 in the L_1 norm and 2.23 in the L_∞ norm. Very similar results are obtained by applying the same limiting procedure to

the submesh reconstruction and gradient recovery schemes [Hub06]. In all cases the oscillations are removed completely and the results are significantly better than those of the PSI scheme. All of the experiments have converged to their steady states to machine accuracy.

5 Conclusions

A new fluctuation distribution scheme has been presented which is demonstrated to give higher than second order accuracy at the steady state for the scalar advection equation without introducing any spurious oscillations. The procedure used to impose positivity on the high order scheme is generally applicable, but is here combined with an approach which reconstructs a quadratic interpolant within each mesh cell by extending the stencil to include the neighbouring cells' nodes, averaging across cell edges to give continuity (and hence conservation). This approach has also been successfully applied to the two-dimensional inviscid Burgers' equation, but it remains to extend it to three-dimensional problems, higher than third order accuracy (both conceptually straightforward) and nonlinear systems of equations (not so). Ongoing research has shown that this limiting procedure can also be used to construct a fully consistent, positive, high order fluctuation distribution scheme for time-dependent situations.

References

[Abg01] Abgrall, R.: Toward the ultimate conservative scheme: Following the quest. J. Comput. Phys., **167**, 277–315 (2001)

[AB02] Abgrall, R., Barth, T.J.: Residual distribution schemes for conservation laws via adaptive quadrature. SIAM J. Sci. Comput., **24**, 732–769 (2002)

[AM03] Abgrall, R., Mezine, M.: Construction of second order accurate monotone and stable residual distributive schemes: the unsteady case. J. Comput. Phys., **188**, 16–55 (2003)

[AR03] Abgrall, R., Roe, P.L.: High order fluctuation schemes on triangular meshes. J. Sci. Comput., **19**, 3–36 (2003)

[CF02] Caraeni, D., Fuchs, L.: Compact third-order multidimensional upwind scheme for Navier-Stokes simulations. Theor. Comp. Fluid Dyn., **15**, 373–401 (2002)

[DSA00] Deconinck, H., Sermeus, K., Abgrall, R.: Status of multidimensional upwind residual distribution schemes and applications in aeronautics. In: AIAA paper 2000–2328 (2000)

[DSBR94] Deconinck, H., Struijs, R., Bourgois, G., Roe, P.L.: High resolution shock capturing cell vertex advection schemes for unstructured grids. In: Computational Fluid Dynamics, VKI-LS 1994-05. Von Karman Institute, Brussels (1994)

[Hub06] Hubbard, M.E.: Non-oscillatory third order fluctuation splitting schemes for steady scalar conservation laws. Submitted to J. Comput. Phys.