Multidimensional Upwind Fluctuation Distribution Schemes for Time Dependent Problems

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1 Introduction

Over the last ten years a family of cell vertex finite volume methods for the solution of the two-dimensional scalar advection equation has evolved known as multidimensional upwind fluctuation distribution schemes [1]. For the approximation of steady state flows on unstructured triangular grids these have reached a degree of maturity whereby the multidimensional schemes reproduce most of the advantages of upwind schemes in one dimension: smooth, second order accurate solutions and rapid convergence to the steady state without the necessity for additional artificial viscosity.

Unfortunately, all of the upwind distribution schemes developed for steady state problems are only first order accurate for time-dependent flows. In this paper an approach to creating monotonic high resolution fluctuation distribution schemes will be described which combines the well known PSI and Lax-Wendroff schemes in a manner which generalises the flux-corrected transport approach [1]. The new method includes a fluctuation *re*distribution step in which the distribution coefficients are altered in such a way as to avoid the creation of new extrema by the nodal updates whilst retaining conservation and as much of the accuracy of the original scheme as possible.

2 Steady state schemes

Consider the two-dimensional scalar advection equation,

$$u_t + f_x + g_y \;=\; 0 \quad \text{or} \quad u_t + \vec{\lambda}\cdot\vec{\nabla}u \;=\; 0\,, \tag{2.1}$$

where $\vec{\lambda} = \left(\frac{\partial f}{\partial u}, \frac{\partial g}{\partial u}\right)^{\mathrm{T}}$ defines the advection velocity. The fluctuation associated with this equation is the cell-based quantity given by

$$\phi \;=\; -\int\!\!\int_{\triangle} \vec{\lambda}\cdot\vec{\nabla}u \,\mathrm{d}x\,\mathrm{d}y \;=\; \oint_{\partial\triangle} u\,\vec{\lambda}\cdot\mathrm{d}\vec{n}\,, \tag{2.2}$$

where $\vec{n}$ represents the inward pointing normal to the boundary of the cell.

The quantity ϕ is evaluated within each cell under an appropriate linearisation [1] and distributed it to the nodes of the grid. A simple forward Euler discretisation of the time derivative leads to an iterative update of the nodal solution values which is generally written [1] as

$$u_i^{n+1} \;=\; u_i^n + \frac{\Delta t}{S_i}\sum_{\cup\triangle_i}\alpha_i^j\phi_j\,, \tag{2.3}$$

where S_i is the area of the median dual cell for node i, α_i^j is the distribution coefficient which indicates the appropriate proportion of the fluctuation ϕ_j to be sent from cell j to node i, and $\cup\triangle_i$ represents the set of cells with vertices at node i. Conservation is assured as long as

$$\sum_i \alpha_i^j \;=\; 1 \quad \forall j \;, \tag{2.4}$$

i.e. the whole of each fluctuation is sent to the nodes. The distribution coefficients for the PSI and Lax-Wendroff schemes may be found in [1].

3 Limiting by fluctuation redistribution

Fluctuation redistribution is a generalisation of the flux-corrected transport (FCT) technique for cell vertex fluctuation distribution schemes and can be described along the lines of [3] in the following six steps:

1. Compute the Low order Element Contributions (LEC) from the PSI scheme.
2. Compute the High order Element Contributions (HEC) from the Lax-Wendroff scheme.
3. Calculate the Antidiffusive Element Contributions (AEC) given by
$$\mathrm{AEC} \;=\; \mathrm{HEC} - \mathrm{LEC} \;. \tag{3.1}$$
4. Compute the updated low order solution,
$$u_i^{\mathrm{L}} \;=\; u_i^n + \sum_{\cup\triangle_i} \mathrm{LEC} \quad \forall\, i \;. \tag{3.2}$$
5. Correct the AEC to each vertex in a manner such that conservation is retained and the new solution (as defined in step 6) has no extrema not also found in either u_i^{L} or u_i^n, so
$$\mathrm{AEC}^k \;\rightarrow\; C_T^k \times \mathrm{AEC}^k \quad \text{where} \quad 0 \le C_T^k \le 1 \;. \tag{3.3}$$
6. Calculate the final solution update,
$$u_i^{n+1} \;=\; u_i^L + \sum_{\cup\triangle_i} \mathrm{AEC}^k \;. \tag{3.4}$$

The limiting procedure of step 5 is designed to make AEC^k as large as possible without introducing new extrema or knowing in advance the nodal updates due to the high order scheme in adjacent cells. It involves the following calculations:

- Evaluate in order the quantities
$$\begin{aligned} u_i^* &= \begin{cases}\max\\ \min\end{cases}(u_i^{\mathrm{L}}, u_i^n) \\ u_T^* &= \begin{cases}\max\\ \min\end{cases}(u_1^*, u_2^*, u_3^*) \\ u_i^{\substack{\max\\ \min}} &= \begin{cases}\max\\ \min\end{cases} u_T^* \qquad \forall\, T \in \cup\triangle_i \;, \end{aligned} \tag{3.5}$$
the last of which give the extreme values of the solution at each node i beyond which the updated solution is not allowed to go.

- Define

$$P_i^{\pm} = \sum_{\cup \triangle_i} \begin{matrix}\max\\ \min\end{matrix} (0, \mathrm{AEC})$$
$$Q_i^{\pm} = u_i^{\begin{matrix}\max\\ \min\end{matrix}} - u_i^{\mathrm{L}} \tag{3.6}$$

  and subsequently

$$W_i^{\pm} = \begin{cases} \min(1, Q_i^{\pm}/P_i^{\pm}) & \text{if } P_i^{+} > 0,\ P_i^{-} < 0 \\ 0 & \text{if } P_i^{\pm} = 0\,, \end{cases} \tag{3.7}$$

  a nodal limiting factor for the antidiffusive contribution which ensures that the new solution value at node i does not violate the prescribed bounds.

- Finally calculate bounds on each element/vertex limiting factor from the nodal limiting factors at those vertices

$$C_T^k \le \begin{cases} W_i^{+} & \text{if } \mathrm{AEC}^k > 0 \\ W_i^{-} & \text{if } \mathrm{AEC}^k < 0\,. \end{cases} \tag{3.8}$$

The above limiting is applied to the difference between the *element* contributions of the two underlying schemes which are easily constructed from the explicit fluctuation distribution scheme [2]. It remains to choose the values of the C_T^k.

3.1 The distribution point

Consider a single grid cell in isolation: the distribution point is defined to be the point whose local area coordinates are the distribution coefficients of the scheme for that triangle. The movement of the distribution point is equivalent to the *re*distribution of the fluctuation within the triangle.

3.2 The equivalent equation

The diffusion vector represents the displacement of the distribution point from the centroid of the triangle (the distribution point of a symmetric central scheme). A scheme with diffusion vector $\vec{d}$ has the second order equivalent equation

$$u_t + \vec{\lambda}\cdot\vec{\nabla}u = \vec{d}\cdot\vec{\nabla}(\vec{\lambda}\cdot\vec{\nabla}u)\,, \tag{3.9}$$

in which the right hand side represents the numerical diffusion of the distribution scheme and this equation can be used to analyse the accuracy of the method.

The diffusion vector of the Lax-Wendroff scheme is $\vec{d} = \frac{1}{2}\Delta t\,\vec{\lambda}$ and can be introduced into the equivalent equation by rewriting (3.9) as

$$u_t + \vec{\lambda}\cdot\vec{\nabla}u = \frac{\vec{\lambda}\Delta t}{2}\cdot\vec{\nabla}(\vec{\lambda}\cdot\vec{\nabla}u) + \left(\vec{d} - \frac{\vec{\lambda}\Delta t}{2}\right)\cdot\vec{\nabla}(\vec{\lambda}\cdot\vec{\nabla}u)\,. \tag{3.10}$$

Hence, any choice of $\vec{d}$ such that

$$\vec{d} - \frac{\vec{\lambda}\Delta t}{2} \perp \vec{\nabla}(\vec{\lambda}\cdot\vec{\nabla}u) = \vec{\nabla}u_t \tag{3.11}$$

will not alter the second order error term in the approximation, so the corresponding distribution scheme should be second order accurate for the given local data. Therefore, moving the distribution point perpendicular to the local value of $\vec{\nabla}(\vec{\lambda} \cdot \vec{\nabla} u)$ should not change the order of accuracy of the local discretisation.

It is important to note here that $\vec{\nabla} u_t$ in (3.11) can be approximated locally using the unlimited high order update (which has already been calculated as part of this FCT-type limiting procedure). This allows the overall algorithm to remain compact.

3.3 The monotonicity region

In the limiting procedure described earlier the bounds on the element/vertex contributions in (3.8) define a region, as illustrated in Fig. 1.

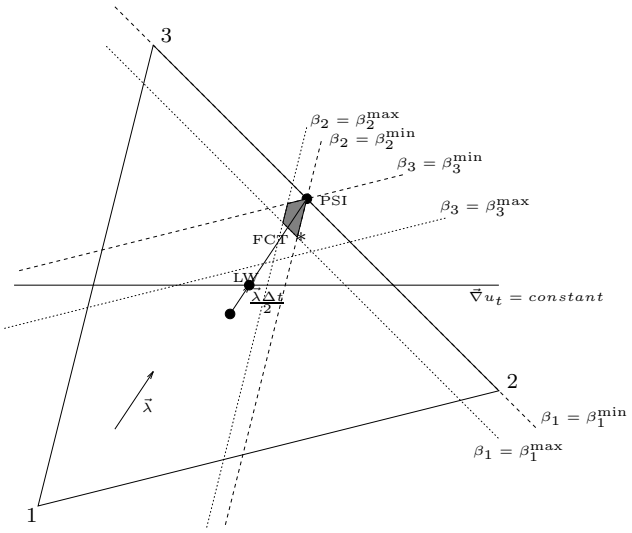

Figure 1: A monotonicity region (shaded dark grey) for the distribution point based on the PSI and Lax-Wendroff schemes.

By considering a general FCT-type algorithm, in which the monotonic scheme is written in terms of low order (LO) and high order (HO) updates, the distribution coefficients can be expressed as

$$\begin{aligned} \alpha_1 &= \alpha_1^{\mathrm{LO}} + \beta_1 \left(\alpha_1^{\mathrm{HO}} - \alpha_1^{\mathrm{LO}} \right) , \\ \alpha_2 &= \alpha_2^{\mathrm{LO}} + \beta_2 \left(\alpha_2^{\mathrm{HO}} - \alpha_2^{\mathrm{LO}} \right) , \\ \alpha_3 &= \alpha_3^{\mathrm{LO}} + \beta_3 \left(\alpha_3^{\mathrm{HO}} - \alpha_3^{\mathrm{LO}} \right) , \end{aligned} \tag{3.12}$$

in which the β_k are limiting coefficients.

The bounds constructed in (3.8) can easily be translated into restrictions on the limiting coefficients since $\beta_k^{\max} = C_T^k$. In general

$$\beta_k^{\min} \;\leq\; \beta_k \;\leq\; \beta_k^{\max} , \quad k = 1, 2, 3, \tag{3.13}$$

which describes three pairs of 'tramlines' parallel to the edges of the triangle, the dotted lines in Fig. 1. (Here, $\beta_k^{\min} = 0$.) The region for which the bounds in (3.13) are satisfied is shaded dark grey in the figure. Placing the distribution point anywhere within this shaded area, the *monotonicity region*, ensures that the subsequent nodal updates will not create any new local extrema at the next time level and as a result imposes stability on the scheme.

Note that FCT, for which

$$\beta_1 \;=\; \beta_2 \;=\; \beta_3 \;=\; \min_{k=1,2,3} \beta_k^{\max}\,, \tag{3.14}$$

will position the distribution point at the intersection of the straight line joining the Lax-Wendroff and PSI distribution points with the boundary of the monotonicity region, as shown in Fig. 1.

3.4 Fluctuation redistribution

The calculation of the limited distribution coefficients takes the following form:

- Find the line passing through the high order distribution point perpendicular to the locally constructed value of $\vec{\nabla} u_t$ (*i.e.* a contour line of u_t).
- Calculate the position of the point in the monotonicity region closest to the line defined above and take this to be the distribution point of the limited scheme. If the line intersects the region then take the point of intersection closest to the high order distribution point.

The new distribution point is indicated by an asterisk in Fig. 1.

4 Results

The first test case involves the circular advection of the 'cone' given by the initial conditions

$$u \;=\; \begin{cases} \cos^2(2\pi r) & \text{for } \; r \leq 0.25 \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

where $r^2 = (x+0.5)^2+y^2$, with velocity $\vec{\lambda} = (-2\pi y, 2\pi x)^T$ around the domain $[-1,1]\times[-1,1]$, the solution being continually set to zero at each of the inflow boundaries. The initial profile should be advected in a circle without change of shape until it returns to its original position when $t = 1.0$. Both this and the next problem has been solved on regular quadrilateral grids divided into triangles by alternating diagonals.

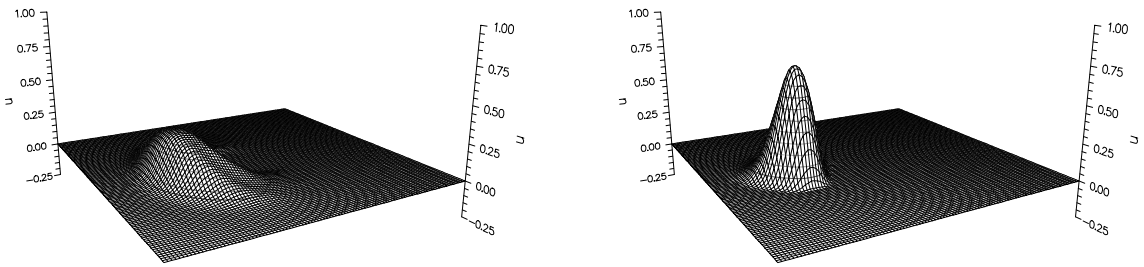

Figure 2: Solution for the rotating cone test case with the PSI scheme (left) and limited second order scheme (right).

Solutions obtained on a 64×64 grid are shown in Fig. 2 for the PSI and limited high resolution schemes. After one revolution the PSI scheme has reduced the height of the peak

from 1.0 to 0.32 and is extremely diffusive, particularly in the streamwise direction. For the new scheme the peak value remains as high as 0.76.

The practical order of accuracy of the scheme has been investigated using the advection of an initial profile given by the double sine wave function

$$u \;=\; \sin(2\pi x)\sin(2\pi y)\;, \tag{4.2}$$

with velocity $\vec{\lambda} = (1,2)^T$ over the domain $[0,1] \times [0,1]$. Periodic boundary conditions are applied.

In Fig. 3 the fluctuation redistribution scheme can be seen to remain close to the unlimited Lax-Wendroff scheme on each of the grids. The numerical order of accuracy in the L_∞ norm on the finest grid is 2.0 for the Lax-Wendroff scheme with or without fluctuation redistribution but only 0.72 for the PSI scheme and 0.68 for a high resolution cell centre upwind scheme.

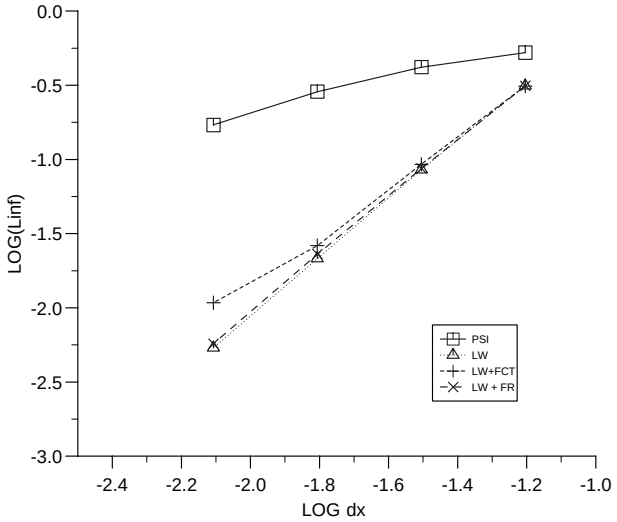

Figure 3: L_∞ errors for the double sine wave test case.

5 Conclusions

In this report the problem of achieving high order accurate numerical solutions to the two-dimensional scalar advection equation using upwind fluctuation distribution schemes on triangular grids has been addressed. A fluctuation redistribution technique has been described which combines the PSI and Lax-Wendroff schemes to attain monotonic, high resolution solutions. Flux-corrected transport is a special case.

The result is a fast, accurate and robust fluctuation distribution scheme based on multidimensional upwind techniques for the solution of the scalar advection equation. Furthermore, it should be straightforward to extend these schemes to nonlinear systems of equations in the same manner as in the steady state case [4] and to three dimensions.

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