A Hybrid Central/Upwind Approach to the Solution of the One Dimensional Euler Equations[1]

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Abstract

It is a fact of life that you rarely get something for nothing. In CFD there is a balance which needs to be struck between the speed and the accuracy of a method. To get results quickly and cheaply it is inevitable that the solution will be of questionable quality and budgets restrict the accuracy which can be achieved, so a compromise is often necessary.

However, it is usually the case that the faster methods are able to model smooth flows quite adequately and only fail when confronted with more complex flow features such as shocks or vortices, which occur in relatively small regions of the entire flow domain. Therefore, it should be possible to use a simple and fast scheme for the majority of the flow, and more appropriate ones for local phenomena.

This report documents a preliminary study in which such a method is used to model the one dimensional Euler equations. It is demonstrated that a domain decomposition approach can be used to gain significant savings with little or no loss of accuracy.

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1 Introduction

This is a brief report on the work done in one dimension, in preparation for the multidimensional work to be completed for the DRA funded contract, *A Simplified, Genuinely Multidimensional Scheme For The Euler Equations*, which is aimed at examining the possibility of using different schemes to model different features in the same flow.

This preliminary work has been carried out for two main reasons

- to familiarise the author with the theory of Roe's scheme in one dimension before moving on to more dimensions with increased complexity.
- to study the effects of, in particular, dividing the flow domain into separate regions, each of which uses a scheme appropriate to the flow there. This can be done in this simplified environment before attempting to implement such changes in higher dimensions.

Usually the flow is fairly smooth in a large proportion of the region, so a very simple approach can be used in the modelling. In this report two 'simple' schemes are considered: a central differencing scheme, with artificial diffusion added to endow it with a limited amount of stability, and Lax-Wendroff. It should be possible to use other schemes which could improve the modelling of the flow, but these are generally more complicated and expensive to use, and the question arises as to whether it is worth bothering to use the simpler scheme at all if it isn't significantly cheaper than the alternative.

However, both of the above schemes are inadequate for modelling flow features such as shocks, so it is necessary to use a more appropriate method. These features are only found in relatively small regions of the flow, so it may be possible to model them using a much better, but far more computationally expensive method without significantly increasing the overall run time. In one dimension, the method used is Roe's upwinding scheme which is known to give a very good shock definition. The expense of this method comes from the use of an approximate Riemann solver, which decomposes the fluctuation in each cell into waves, which are then distributed individually in an upwind sense. This is not required for either the central differencing or Lax-Wendroff schemes.

This, though, raises a number of other points, such as how to monitor the regions where the more complicated scheme is to be used, and what the effects of the interfaces between the regions using different schemes are. It is these points in particular that this study in one dimension is intended to clarify.

In two and three dimensions there are many other complications, the most significant of which is probably the choice of wave decomposition model to be used. Until recently, Roe's scheme has been extended to higher dimensions, with

a reasonable amount of success, using an operator splitting technique. However, it has long been known that these methods are incapable of properly modelling certain flow features, such as shear and shocks not aligned with the mesh. There is much work being carried out at present to produce a truly multidimensional wave decomposition model [3, 7, 10, 11], but it is by no means clear which will prove to be the best. It may even be that in this domain decomposition method, different models need to be used for different flow features. The work will be done solely on unstructured triangular or tetrahedral meshes.

The result will hopefully be a 'hybrid' scheme incorporating the best of both worlds; having accurate modelling of the flow where it is needed but producing answers quickly and cheaply.

In the next section, a brief description is given of the three schemes used in this study, Roe's scheme, central differencing and Lax-Wendroff, and their application to the one dimensional Euler equations. Section 3 describes the results obtained for a number of test cases, used initially to validate the individual schemes, and then to evaluate the domain decomposition approach, *ie.* using different schemes in different regions. Finally, section 4 gives a summary of the results and conclusions drawn from the study.

2 The One Dimensional Schemes

The work discussed in this report is a brief study of the effects of domain decomposition in one dimension. This is being used to gain an insight into the work, in a simplified environment, before it is extended to more dimensions. In this way, problems related to the use of two different schemes in a single flow domain can be solved, before the introduction of the fluctuation distribution scheme and the wave decomposition model necessary for multidimensional upwinding, neither of which can be considered to be perfected. This section gives a brief description of the three schemes used in this study.

2.1 Upwinding (Roe's Scheme)

The first of the three schemes used here is upwinding and, in particular, Roe's scheme is used for solving the Euler equations. The scheme is most easily described in conjunction with a single, scalar conservation law, *ie.* the one dimensional linear advection equation

$$u_t + a u_x = 0 \tag{1}$$

and the upwinding scheme applied to this gives

$$u_i^{n+1} = u_i^n - \frac{a\Delta t}{\Delta x}(u_i^n - u_{i-1}^n). \tag{2}$$

It is explicit and first order accurate, and for $a > 0$ it is stable for CFL numbers

$$\nu = \frac{a\Delta t}{\Delta x} \leq 1. \tag{3}$$

For $a < 0$ it is unconditionally unstable. In this case, a 'left hand' version of the scheme is required

$$u_i^{n+1} = u_i^n - \frac{(-a)\Delta t}{\Delta x}(u_i^n - u_{i+1}^n) \tag{4}$$

which is again stable for $\nu \leq 1$.

The Euler equations are a nonlinear system of conservation laws

$$\underline{\mathbf{u}}_t + \underline{\mathbf{f}}(\underline{\mathbf{u}})_x = \underline{\mathbf{0}} \tag{5}$$

where in one dimension

$$\underline{\mathbf{u}} = \begin{pmatrix} \rho \\ \rho u \\ e \end{pmatrix}, \quad \underline{\mathbf{f}} = \begin{pmatrix} \rho u \\ p + \rho u^2 \\ u(e+p) \end{pmatrix} \tag{6}$$

in which ρ is density, u is velocity, p is pressure and e is the total energy, related to the other variables by an equation of state which, for a perfect gas, is

$$e = \frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2. \tag{7}$$

The method used here to solve this system is Roe's scheme, which is described in detail from a theoretical point of view in [8, 9], and from a computational point of view in [12]. This method involves the decomposition of the fluctuation, $(\underline{\mathbf{f}}_R - \underline{\mathbf{f}}_L)\Delta t/\Delta x$, in each cell, on to the eigenvectors of the Jacobian matrix, $\tilde{A}$, of $\underline{\mathbf{f}}(\underline{\mathbf{u}})$, where $\underline{\mathbf{f}}_R$ and $\underline{\mathbf{f}}_L$ are the flux vectors at the right and left hand nodes of the cell respectively, Δt is the time step and Δx is the cell width. The fluctuation can then be distributed to increment the nodal values of $\underline{\mathbf{u}}$, being subtracted from the left node if the corresponding eigenvalue is negative, or from the right node if it is positive. Thus, the nodal update formula can be written

$$\underline{\mathbf{u}}_i^{n+1} = \underline{\mathbf{u}}_i^n - \frac{\Delta t}{\Delta x}\left(\left(\sum_{k=1}^{3}\lambda_k^+\alpha_k\underline{\mathbf{r}}_k\right)_{i-1/2}^{n} + \left(\sum_{k=1}^{3}\lambda_k^-\alpha_k\underline{\mathbf{r}}_k\right)_{i+1/2}^{n}\right) \tag{8}$$

where λ_k^+ and λ_k^- are the positive and negative eigenvalues of the local Jacobian matrix, $\tilde{A}$, $\underline{\mathbf{r}}_k$ is the corresponding eigenvector and α_k is the wave strength. In effect, the Euler equations have been locally linearised and decoupled leaving three independent linear advection problems to be solved in each cell.

An interesting point in the solution of this problem is the introduction of a parameter vector

$$\underline{\mathbf{w}} = \rho^{1/2}\begin{pmatrix} 1 \\ u \\ H \end{pmatrix} \tag{9}$$

where H is the enthalpy. This has the useful property that each component of $\underline{\mathbf{u}}$ and $\underline{\mathbf{f}}$ is merely quadratic in the components of $\underline{\mathbf{w}}$, so transforming into these variables simplifies the algebra a great deal. It has also been suggested that these parameter vector variables might have other desirable properties, and that it may be advantageous to solve the Euler equations in these variables rather than the conserved variables.

There are in fact two ways of constructing Roe's linearisation. The first, mentioned above, uses the analogy of solving an approximate Riemann problem within each cell, assuming the variables to be piecewise constant in space, and develops the linearisation from there. A second derivation, developed afterwards and favoured in more recent papers on the subject [1, 2], considers the parameter vector variables to be piecewise linear in space. The latter has the advantage of being readily generalised to higher dimensions.

In one dimension, Roe's scheme is easily extended to give second order accuracy with monotonicity by the use of anti-diffusive terms and flux limiters. A description of the theory of flux limiters can be found in [5], while their implementation is described in [12]. In the course of this work the minmod, van Leer and superbee limiters were all used, but only the results produced using the van Leer harmonic mean were presented because the differences were negligible. The same

two references also give a thorough description of entropy conditions, necessary to ensure the correct treatment of sonic points by removing the possibility of the scheme picking up a spurious, entropy violating solution.

A detailed description of the extension of Roe's upwinding scheme to two dimensions can be found in [13], and [2] gives a brief description of its extension to three dimensions. A more general view of multidimensional upwinding techniques is given in [14]. Information on the wave decomposition model is more difficult to find [3, 10, 11], as this is the least understood aspect of these schemes and there seems to be no general consensus as to the best model.

2.2 Central Differencing

The first 'cheap' scheme chosen for this study is central differencing. It is being used because it is simple and very fast, and since it will only be used in smooth parts of the flow, it doesn't matter that it isn't the best scheme available. It is an explicit scheme with second order accuracy, but on its own is unconditionally unstable. Therefore, a small amount of artificial diffusion is required to gain sensible results. If the scheme is applied to the linear advection equation then it looks like

$$u_i^{n+1} = u_i^n - \frac{a\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n) + \frac{\epsilon\Delta t}{(\Delta x)^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) \tag{10}$$

in which the parameter ϵ regulates the amount of artificial diffusion being used.

The main problem with this scheme is that, due to the necessary inclusion of artificial diffusion, a new tuning parameter, ϵ, has been introduced which is problem dependent and, as such, undesirable. Even at its optimal value, the maximum CFL number achievable which still retains stability is significantly less than one (the CFL stability limit for Roe's scheme), and in the de Laval nozzle test case used here, a value of only 0.4 could be used, thus reducing the savings made by using the simpler scheme. This problem suggests that it might be necessary to use another method as the cheap scheme. However, this alternative must still be significantly faster than Roe's scheme, otherwise it is simpler to use the superior method everywhere.

Central differencing is easily extended to the Euler equations since no decomposition of the fluctuation is necessary. In order that it is compatible with Roe's scheme, it can be thought of in terms of distributing the fluctuation within each cell. This is most easily shown by writing the scheme as

$$\begin{aligned}\underline{\mathbf{u}}_i^{n+1} = \underline{\mathbf{u}}_i^n &- \frac{\Delta t}{2\Delta x}(\underline{\mathbf{f}}_{i+1}^n - \underline{\mathbf{f}}_i^n) + \frac{\epsilon\Delta t}{(\Delta x)^2}(\underline{\mathbf{u}}_{i+1}^n - \underline{\mathbf{u}}_i^n) \\ &- \frac{\Delta t}{2\Delta x}(\underline{\mathbf{f}}_i^n - \underline{\mathbf{f}}_{i-1}^n) - \frac{\epsilon\Delta t}{(\Delta x)^2}(\underline{\mathbf{u}}_i^n - \underline{\mathbf{u}}_{i-1}^n)\end{aligned} \tag{11}$$

which illustrates how central differencing can be thought of as a fluctuation distribution scheme where half the fluctuation is sent to each of the left and right nodes of the cell. It also shows that the artificial diffusion term is dealt with just as simply. The only slight problem is the calculation of the time step, which involves finding the value of the largest eigenvalue, λ_{max}, of the local Jacobian matrices of $\underline{\mathbf{f}}(\underline{\mathbf{u}})$, so

$$\Delta t = \frac{\nu \Delta x}{\lambda_{max}} \tag{12}$$

where ν is the CFL number. However, this only requires the calculation of u, the velocity of the flow, and c, the local speed of sound, since the eigenvalues of the Jacobian are known to be $u - c$, u and $u + c$.

2.3 Lax-Wendroff

The third and final scheme used here is Lax-Wendroff. It is being considered as an alternative cheap scheme for modelling the smooth parts of the flow, despite the seemingly greater computational expense, because it doesn't have the problems of central differencing mentioned above. Applied to the linear advection equation the scheme looks like

$$u_i^{n+1} = u_i^n - \frac{a\Delta t}{2\Delta x}(u_{i+1}^n - u_{i-1}^n) + \frac{1}{2}\left(\frac{a\Delta t}{\Delta x}\right)^2 (u_{i+1}^n - 2u_i^n + u_{i-1}^n). \tag{13}$$

It is again explicit and second order accurate but is stable for CFL numbers

$$-1 \leq \nu \leq 1, \tag{14}$$

a great improvement on central differencing. Lax-Wendroff also has the advantages of not having the extra tuning parameter, ϵ.

The extension of the scheme to the Euler equations is again relatively straightforward. As with the derivation of Equation 13, the 'Lax-Wendroff trick' is used to express the time derivatives in terms of space derivatives

$$\underline{\mathbf{u}}_{tt} = -\underline{\mathbf{f}}_{xt} = -\underline{\mathbf{f}}_{tx} = -(\tilde{A}\underline{\mathbf{u}}_t)_x = (\tilde{A}\underline{\mathbf{f}}_x)_x \tag{15}$$

where $\tilde{A}$ is the Jacobian matrix of $\underline{\mathbf{f}}(\underline{\mathbf{u}})$. This can then be substituted into the Taylor series expansion for $\underline{\mathbf{u}}_i^{n+1}$, giving

$$\underline{\mathbf{u}}_i^{n+1} = \underline{\mathbf{u}}_i^n - \Delta t \underline{\mathbf{f}}_x + \frac{(\Delta t)^2}{2}(\tilde{A}\underline{\mathbf{f}}_x)_x. \tag{16}$$

The appropriate central differences can now be chosen to approximate the space derivatives to give the Lax-Wendroff scheme. Using simple algebraic manipulation the scheme can be written

$$\begin{aligned}\underline{\mathbf{u}}_i^{n+1} = \underline{\mathbf{u}}_i^n &- \frac{\Delta t}{2\Delta x}\left(I - \frac{\Delta t}{\Delta x}\tilde{A}_{i+1/2}\right)(\underline{\mathbf{f}}_{i+1}^n - \underline{\mathbf{f}}_i^n) \\ &- \frac{\Delta t}{2\Delta x}\left(I + \frac{\Delta t}{\Delta x}\tilde{A}_{i-1/2}\right)(\underline{\mathbf{f}}_i^n - \underline{\mathbf{f}}_{i-1}^n).\end{aligned} \tag{17}$$

This shows how the scheme can be formulated in terms of the distribution of a fluctuation within each cell, with each node getting a contribution from the left and the right cells. This makes the scheme compatible with Roe's scheme at interfaces in the domain decomposition environment. As with central differencing, the calculation of the time step again involves determining, λ_{max}, the largest eigenvalue of the local Jacobian matrices of $\underline{\mathbf{f}}(\underline{\mathbf{u}})$, so

$$\Delta t = \frac{\nu \Delta x}{\lambda_{max}} \tag{18}$$

where ν is the CFL number.

There are many variations of the Lax-Wendroff scheme designed to speed it up by avoiding the calculation of the matrix $\tilde{A}$, and multiplying by it. However, none of these have been used here, and this should be taken into account when considering the speed of the method compared with Roe's scheme. It should also be noted that throughout this study, global time stepping has been used, and convergence of all of the schemes to the steady state solution could be accelerated enormously by the use of local time stepping techniques.

2.4 Flow With Area Variation

The de Laval nozzle test case presented in this report provides one dimensional nozzle flow with a variation in cross-sectional area. This requires a modification to the Euler equations in one dimension which takes the form of a source term

$$\underline{\mathbf{U}}_t + \underline{\mathbf{F}}_x = \underline{\mathbf{g}} \tag{19}$$

where

$$\underline{\mathbf{g}} = \begin{pmatrix} 0 \\ \frac{PS_x}{S} \\ 0 \end{pmatrix} \tag{20}$$

and S is the cross-sectional area of the nozzle, $\underline{\mathbf{U}} = S\underline{\mathbf{u}}$, $P = Sp$ and $\underline{\mathbf{F}} = S\underline{\mathbf{f}}$. Thus we are now solving for the conserved variables multiplied by the cross-sectional area of the nozzle.

A comprehensive description of how the source term is incorporated into Roe's scheme can be found in [4]. Essentially the extra term is decomposed, along with the fluctuation, on to the eigenvectors of the matrix, $\tilde{A}$, and each component is distributed as before. When central differencing is used, then a cell averaged value of the source term is distributed simply by sending half each to the left and right nodes of the cell.

The extension of the Lax-Wendroff scheme is slightly more complicated in that the introduction of the source term affects the 'Lax-Wendroff trick', giving

a different expression for the first and second derivative terms in the Taylor expansion, so

$$\underline{\mathbf{u}}_i^{n+1} = \underline{\mathbf{u}}_i^n - \Delta t(\underline{\mathbf{F}}_x - \underline{\mathbf{g}}) + \frac{(\Delta t)^2}{2}(\tilde{A}(\underline{\mathbf{F}}_x - \underline{\mathbf{g}}))_x + \underline{\mathbf{g}}_t \tag{21}$$

and each of the derivative terms here is then approximated using central differences as with the homogeneous equations.

For all three of the above schemes, the boundaries are treated using a combination of Riemann invariant boundary conditions and extrapolation boundary conditions, depending on the speed and direction of the flow.

2.5 A Domain Decomposition Approach

The approach to using two schemes in a single flow domain is treated very simply here. At every time step, each cell is designated as using either Roe's scheme or central differencing/Lax-Wendroff. There is no special treatment given to cells at the interface between the two schemes - something which may have to change in the future by introducing some form of parameter smoothing between the two regions.

Two methods are used to specify the cells which will use Roe's scheme. The first simply designates a number of cells which will use Roe's scheme throughout the calculation. The second method uses a monitor which flags cells with a high density gradient at each time step. This is intended to pick out shocks as they develop and use the appropriate scheme for them. The velocity magnitude gradient has also been suggested as a monitor [14] but hasn't been used here.

3 Results

Initially, Roe's scheme was used with three standard test cases

- Sod's shock tube problem [8]. This is a straightforward problem used simply to test the program.
- Woodward and Colella's blast tube problem [15]. This highlighted the need to extend Roe's scheme to second order in order to accurately model strong interacting shocks. This is not so important when converging to a steady state solution.
- Converging cylindrical shock [4]. This involved the inclusion of a source term in the Euler equations in a similar way to that required for flow with area variation.

Brief results are shown in Figures 1, 2 and 3, which agree satisfactorily with those presented by other authors.

However, all of these test cases are time dependent problems and are of limited use for considering cases where time stepping is used as a means of converging to a steady state solution. Therefore, a fourth test case was used for studying domain decomposition. This is a one dimensional nozzle flow on the interval $-1 \leq x \leq 1$ with the cross sectional area taken [6] to be

$$S(x) = \pi \left(1 - \frac{1}{10}(1 + \cos \pi x)\right)^2. \tag{22}$$

This is known as a de Laval nozzle. In all the results presented here the interval is spanned by 200 cells.

Initially, a completely subsonic case was studied where the freestream Mach number $M_\infty = 0.4$, which gives a smooth flow, symmetric about $x = 0$. This immediately highlighted the problem of how much artificial diffusion is necessary to obtain a converged solution with central differencing. If the value of ϵ is too small or too large then the scheme is unstable except for very small CFL numbers. After some experimenting, a value of $\epsilon = 0.006$ was chosen, allowing a CFL number of 0.4 to be used. Figure 4 shows how the artificial diffusion is needed to smooth out oscillations produced at the boundaries. The pictures show the solution after 200 time steps (not converged) using different amounts of artificial diffusion. In the first two cases, where $\epsilon = 0.0$ and $\epsilon = 0.001$, the solution blows up after a few more time steps, but in the third case, $\epsilon = 0.006$, a converged solution was obtained, Figure 6.

This suggested that, since the boundary seemed to be the source of the oscillations, then it may be necessary to use Roe's scheme here instead of central

differencing. Also, in this study the boundaries are treated using a simple Riemann invariant boundary condition which is far better suited to the upwinding scheme. The results of using Roe's scheme at the boundary are shown in Figure 5. Here, the same subsonic test case was run again for 200 time steps, but using Roe's scheme in the ten cells at either end of the domain, while the rest of the region used central differencing. There are now two interfaces, where adjacent cells use different schemes. These are the sources of oscillations which are, if anything, worse that those produced by the boundaries. In the case where no artificial diffusion was used, the solution blew up after only 130 iterations, but as before, increasing the value of ϵ slowly removes the oscillations, until with $\epsilon = 0.006$, a converged solution could be obtained, Figure 7. It should be noted here that every figure after 7 shows solutions which have converged to machine accuracy, *ie.* the residuals have fallen to below 10^{-15}, and from now on the use of central differencing also implies that this particular value of artificial diffusion is being used, unless otherwise stated.

Figures 6 and 7 show the converged solutions, using central differencing everywhere, and using Roe's scheme at the boundaries only, compared with the results of using Roe's scheme everywhere. It is slightly worrying that the solutions do not coincide, even in this subsonic case. This appears to be because the introduction of artificial diffusion alters the solution of the problem at convergence to steady state, and suggests that an alternative scheme might have to be used, even if it is a little more expensive. Because of this, the same case was run using Lax-Wendroff, and the results of this are shown in Figure 8 compared with those of Roe's scheme. There is no longer any visible difference between the two sets of results and the actual difference in Mach number is of the order of 10^{-5} throughout the domain, a great improvement on central differencing.

The subsonic case has no other features which suggest the need to use Roe's scheme rather than central differencing/Lax-Wendroff, so a second case was used which involved increasing the freestream Mach number to $M_\infty = 0.5$ to produce transonic flow with a strong shock downstream of the nozzle throat.

A comparison of the results obtained by using Roe's scheme only and central differencing only is shown in Figure 9. Again the results are slightly different, as expected, with the shock produced using central differencing being weaker, further upstream and a little smeared. Obviously a better treatment of the shock is needed than central differencing can provide, so a monitor was used to detect the shock as it appeared in the calculation by flagging cells with a high density gradient. Figure 10 shows the converged results when Roe's scheme was used in these cells and central differencing everywhere else. In fact, since the shock is captured across two cells, Roe's scheme is only used in these, so the interfaces between the two schemes are extremely close to the shock, and they have introduced wiggles

in the solution both upstream and downstream of the shock. However, the use of Roe's scheme has sharpened the shock, even though it is still too far upstream.

In an attempt to remove the wiggles a second run was carried out which used Roe's scheme in the flagged cells and in two more cells on either side (six cells in total). Figure 11 shows that this reduced the wiggles significantly, although they can still be detected at the interfaces, and even though the shock is again very sharp, it is still in the wrong place. A third run, shown in Figure 12, which used Roe's scheme in five additional cells on either side of the shock hardly reduced the wiggles any more. It simply moved them further from the shock (with the interfaces), and the problem with the shock position remained.

The positioning of the sharpened shock too far upstream seems to be due to the fact that central differencing, used for the majority of the flow, gives a slightly different steady state solution to Roe's scheme. Thus, rather than moving the shock to the correct position, using Roe's scheme only in a small region merely sharpens the shock in its previous position. This is also indicated by repeating the second 'hybrid' run above with a larger amount of artificial diffusion, $\epsilon = 0.01$. This has the effect of increasing the size of the wiggles at the interfaces and of moving the sharpened shock even further upstream, Figure 13.

The same test case was then run using Lax-Wendroff instead of central differencing, in the hope of producing improved results. Figure 14 shows the comparison of the results for Roe's scheme only and Lax-Wendroff only. This already looks more promising since the shock is now in the same place for both schemes, despite the oscillations characteristic of the Lax-Wendroff scheme. As with central differencing, a monitor was then introduced to detect cells with a high density gradient, and Figure 15 shows the result of using Roe's scheme only in the cells spanning the shock. This now compares very favourably with the results obtained from using Roe's scheme in the whole domain, although there is a small discrepancy just upstream of the shock. This difference has been removed simply by using Roe's scheme in four additional cells next to the shock, two upstream and two downstream. Figure 16 shows that the wiggles at the shock have completely disappeared and, as in the subsonic case, there is no longer any visible difference between the two sets of results.

In the results described above, central differencing is only about three times as fast, per time step, as Roe's scheme, so because of the difference in their CFL limits (but otherwise similar convergence rates), a saving of only about 25% is gained by using the simpler scheme. In the 'hybrid' runs, Roe's scheme was used in such a small proportion of the domain that the saving was still about 20% over using Roe's scheme everywhere. Lax-Wendroff, however, can use a similar CFL number to Roe's scheme and is more than twice as fast per time step, an advantage which is maintained in the 'hybrid' runs and would be increased by us-

ing a two-step Lax-Wendroff scheme. This, in fact, makes Lax-Wendroff slightly faster than central differencing overall. It should be noted that all these times have been obtained using a very general code in which computational overheads become increasingly significant as the schemes become simpler. Also, in two and three dimensions the difference in speed will become far more marked, as the multidimensional upwinding scheme is much more complex than its one dimensional counterpart.

An alternative to the domain decomposition method used above would be to use central differencing or Lax-Wendroff to obtain a reasonably converged solution, and then use this as the starting point for a second calculation using Roe's scheme everywhere. This removes the problem of the two schemes giving different answers, but unfortunately saves little time and is not a viable alternative to the domain decomposition method. Although it decreases the time taken for the second calculation to reach convergence by about 10%, most of this is used in gaining the partially converged solution. The same problem will be found in two dimensions so it is not worth pursuing any further.

4 Conclusions

The work presented in this report shows that it is possible to obtain savings in one dimension by using a cheap and simple scheme in most of the flow domain and only using a more accurate, but expensive, scheme (Roe's upwinding scheme) in the small regions where it is necessary.

In this study two 'cheap' schemes were considered, central differencing and Lax-Wendroff. Unfortunately, the savings were not as large as might be expected for central differencing, due to a restrictive CFL limit, and both schemes suffered from using a very small number of nodes in the test case considered, which caused the iteration times to be swamped by overheads. These problems will be of less significance in higher dimensions where many more nodes are used and Roe's scheme is relatively far more expensive.

The domain decomposition method has been successfully used to produce sharp shocks, while using Roe's scheme in only a very small part of the domain. Shocks were found to be the only one dimensional flow feature which required special attention.

However, for central differencing, the interfaces between the two schemes produced wiggles which could not be completely removed by moving the interfaces away from the shock. Also, the sharpened shock was too far upstream. This seems to be due to the fact that central differencing with artificial diffusion and Roe's scheme do not converge to precisely the same steady state solution and, most importantly, predict different shock positions. If Lax-Wendroff is used instead, these problems disappear and an extremely good comparison can be made between the 'hybrid' scheme and Roe's scheme. This strongly suggests that a Lax-Wendroff type scheme and not central differencing should be used as the simple scheme when this work is extended to two dimensions. The Lax-Wendroff 'hybrid' scheme in one dimension takes only half the time to reach a converged solution that Roe's scheme does, a speed advantage which will be increased greatly in two dimensions due to the greater complexity of Roe's multidimensional scheme and the reduced significance of the computational overheads.

This work has also given a valuable insight into the domain decomposition approach, highlighting problems to be solved before the extension to two dimensions is attempted. This should make it simpler to tackle the essentially multidimensional problems related to the fluctuation distribution scheme and the wave decomposition model.

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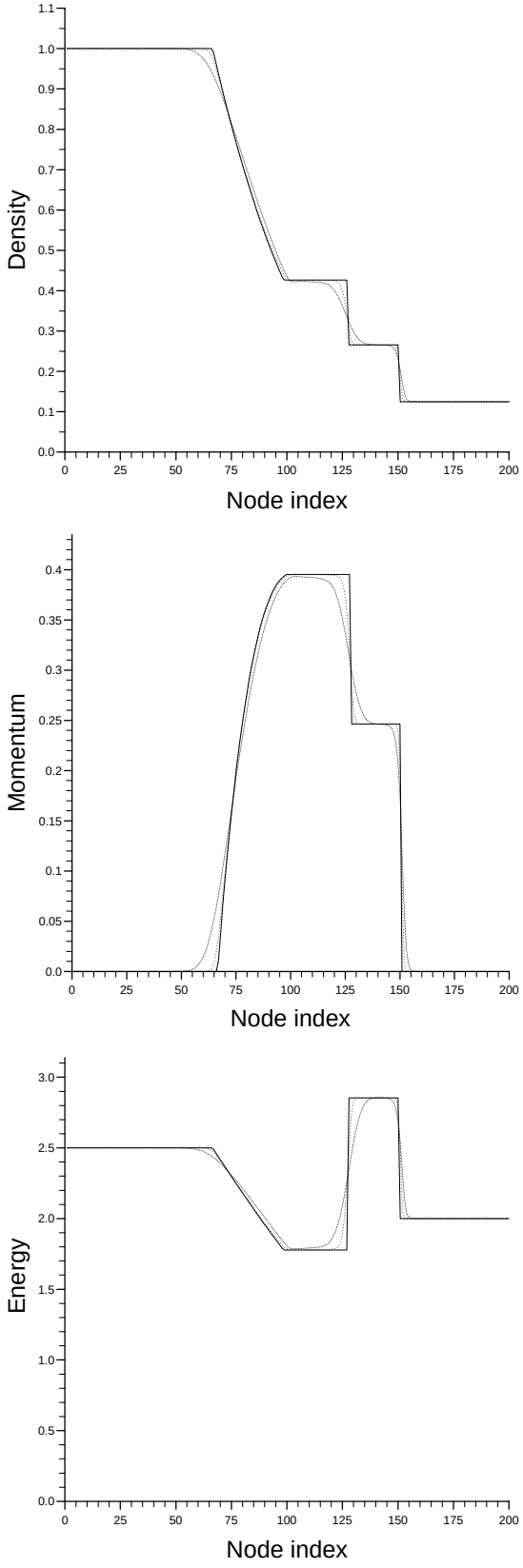

Figure 1: Sod's problem: comparisons of density, momentum and energy between the exact solution (solid line), first order Roe's scheme (dotted line) and second order Roe's scheme using the van Leer flux limiter (broken line).

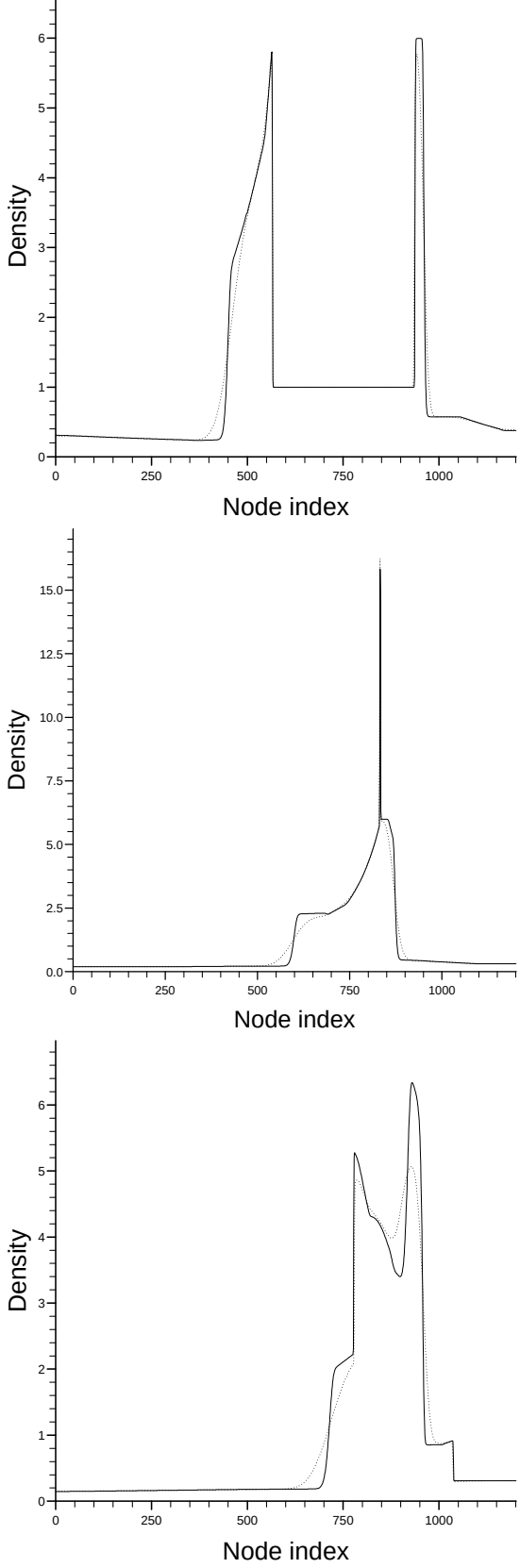

Figure 2: Woodward and Colella's problem: comparison of density between first order Roe's scheme (dotted line) and second order Roe's scheme using the van Leer flux limiter (solid line), at times $t = 0.016, 0.028, 0.038$.

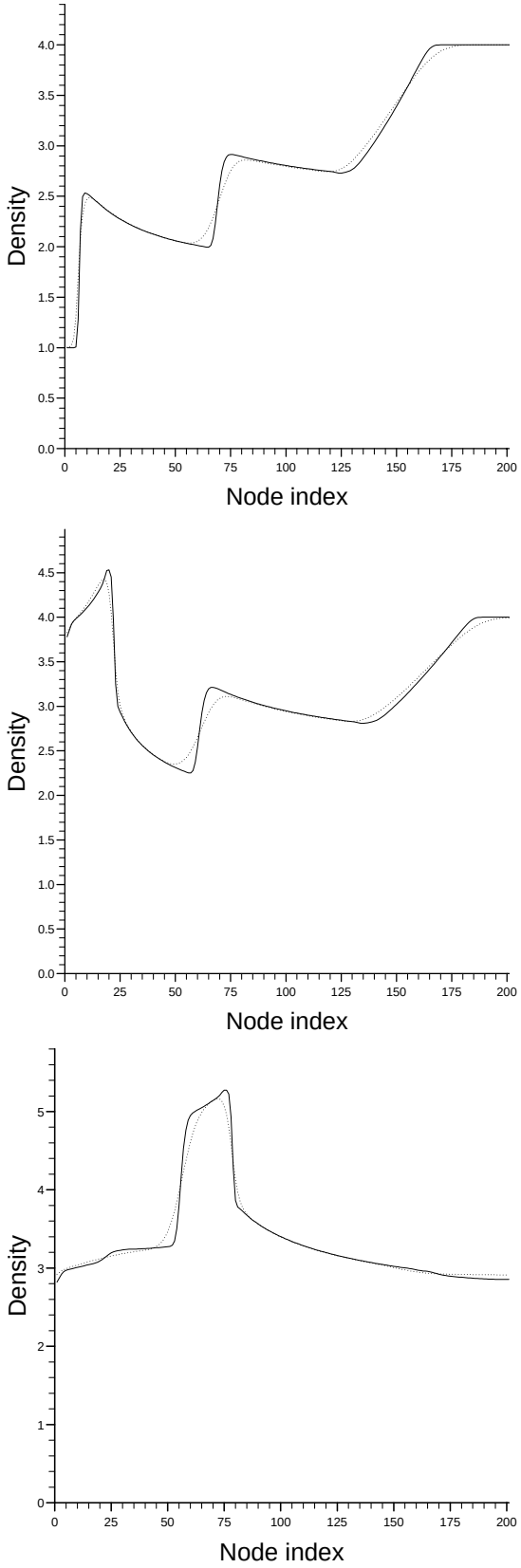

Figure 3: Converging cylindrical shock: comparison of density between first order Roe's scheme (dotted line) and second order Roe's scheme using the van Leer flux limiter (solid line), at times $t = 54, 70, 110$.

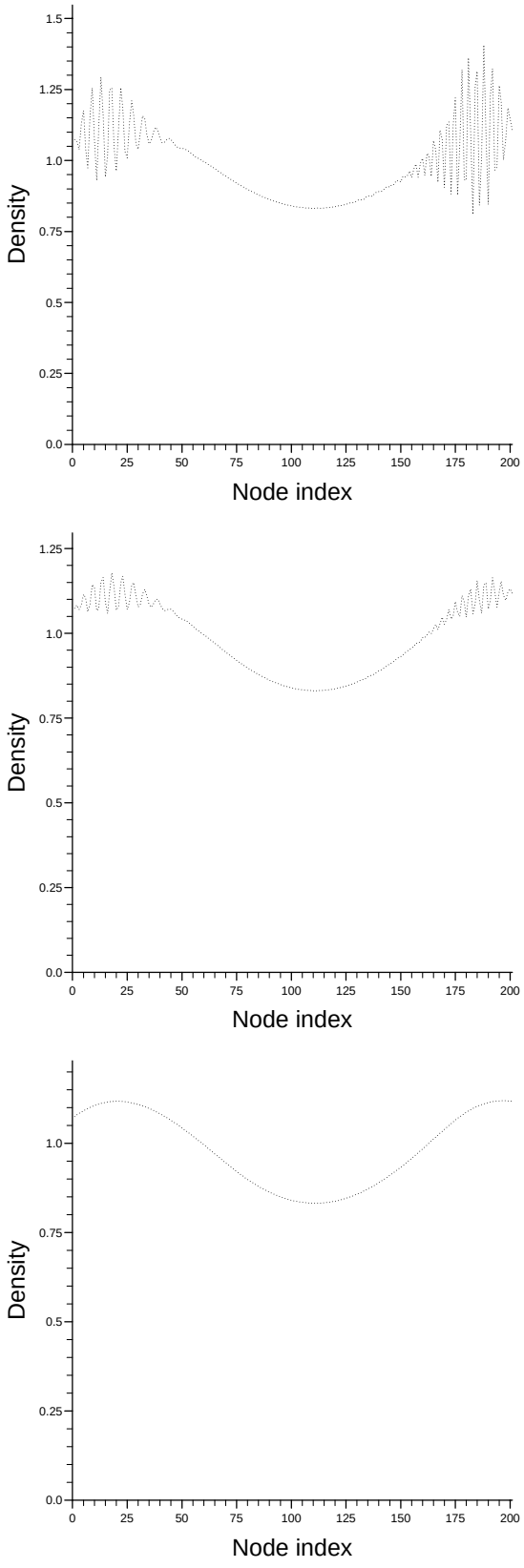

Figure 4: Central differencing with $\epsilon = 0.0$ (top), $\epsilon = 0.0004$ (middle) and $\epsilon = 0.006$ (bottom).

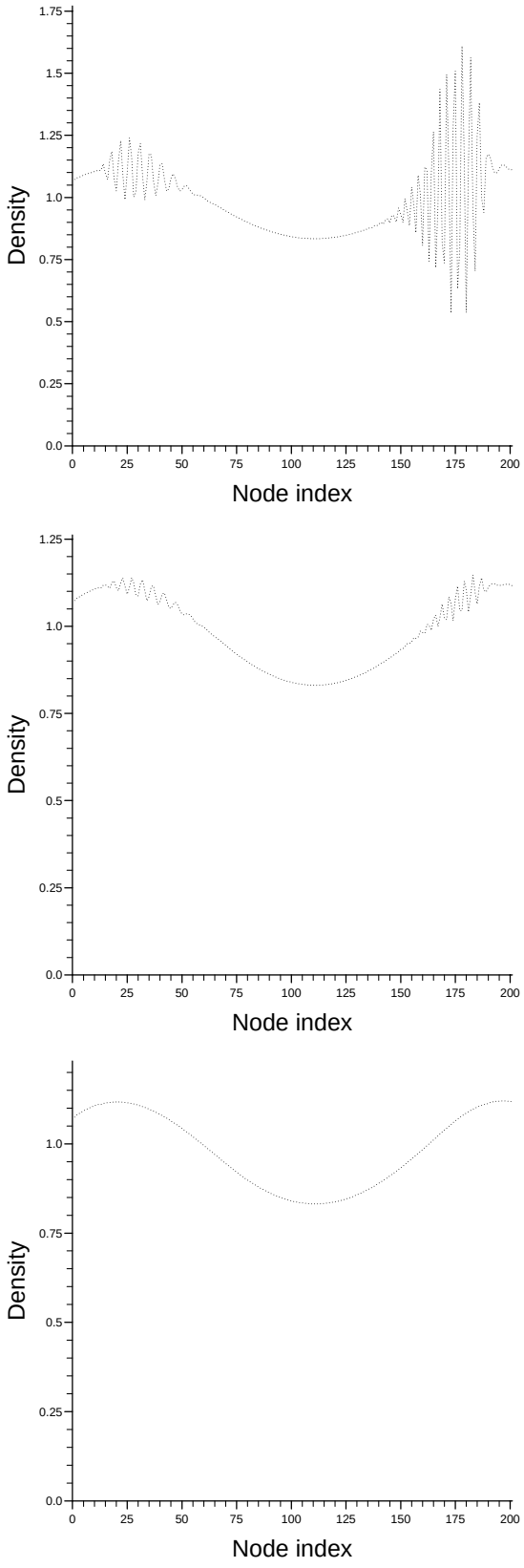

Figure 5: Roe's scheme used in 10 cells at either boundary, central differencing with $\epsilon = 0.0$ (top), $\epsilon = 0.0004$ (middle) and $\epsilon = 0.006$ (bottom) elsewhere.

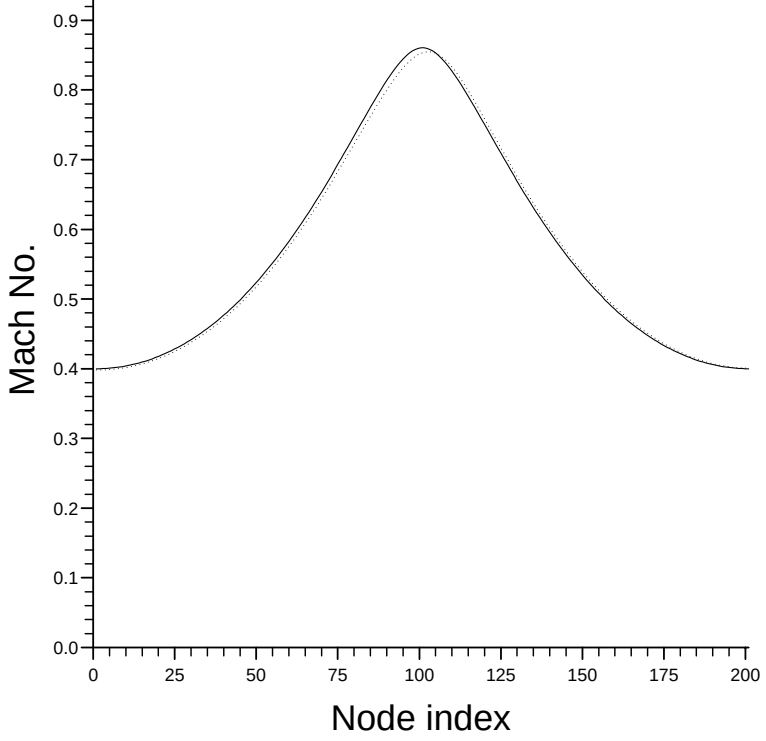

Figure 6: Comparison of Roe's scheme (solid line) and central differencing with $\epsilon = 0.006$ (dotted line) for the de Laval nozzle, $M_\infty = 0.4$.

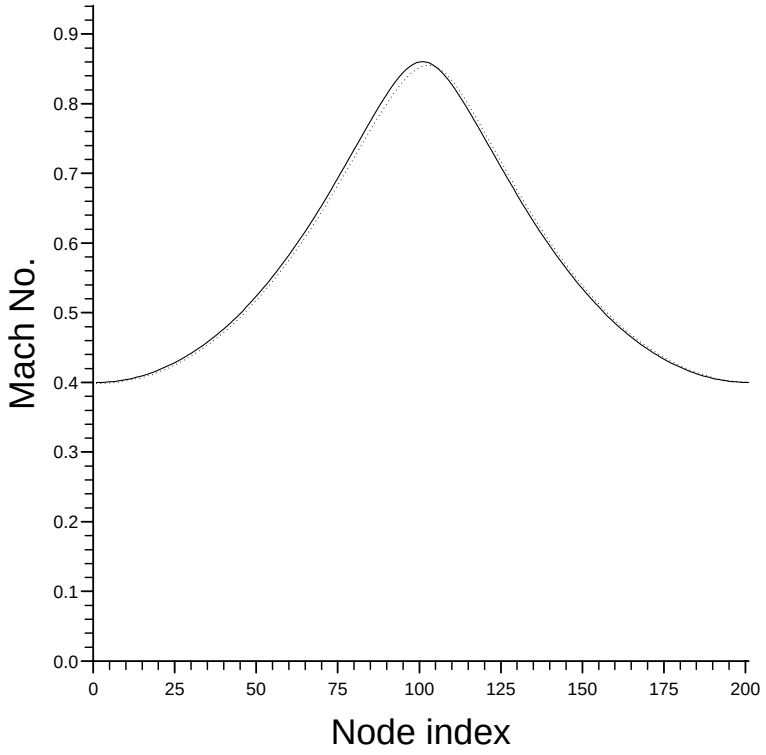

Figure 7: Comparison of Roe's scheme (solid line) and the hybrid scheme with the ten cells at either boundary using Roe's scheme (dotted line) for the de Laval nozzle, $M_\infty = 0.4$.

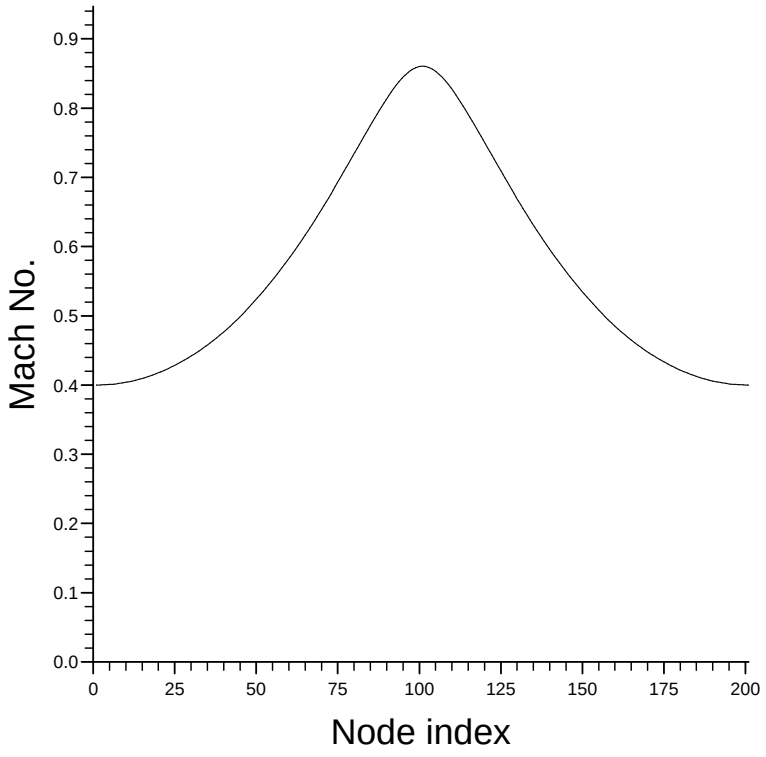

Figure 8: Comparison of Roe's scheme (solid line) and Lax-Wendroff (dotted line) for the de Laval nozzle, $M_\infty = 0.4$.

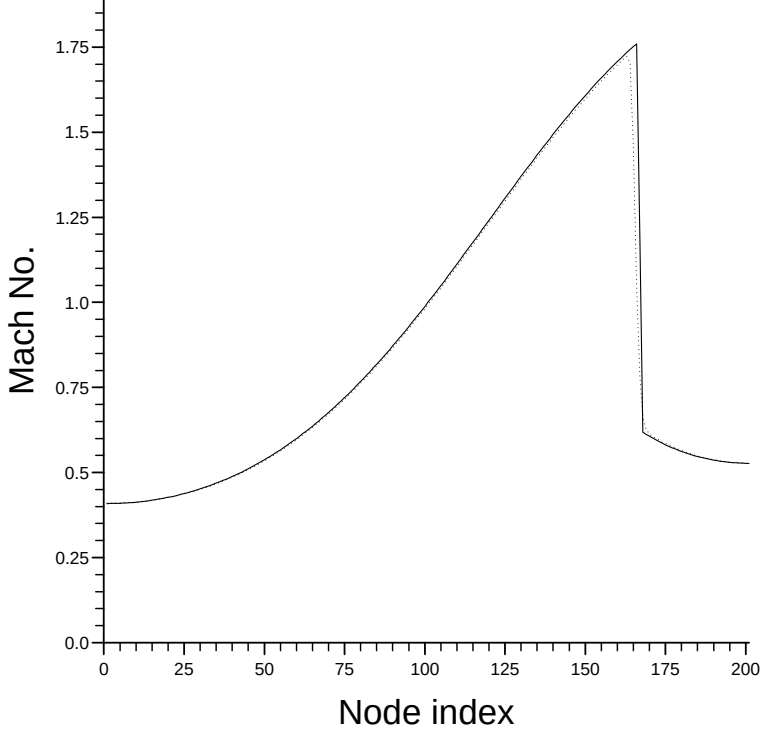

Figure 9: Comparison of Roe's scheme (solid line) and central differencing with $\epsilon = 0.006$ (dotted line) for the de Laval nozzle, $M_\infty = 0.5$.

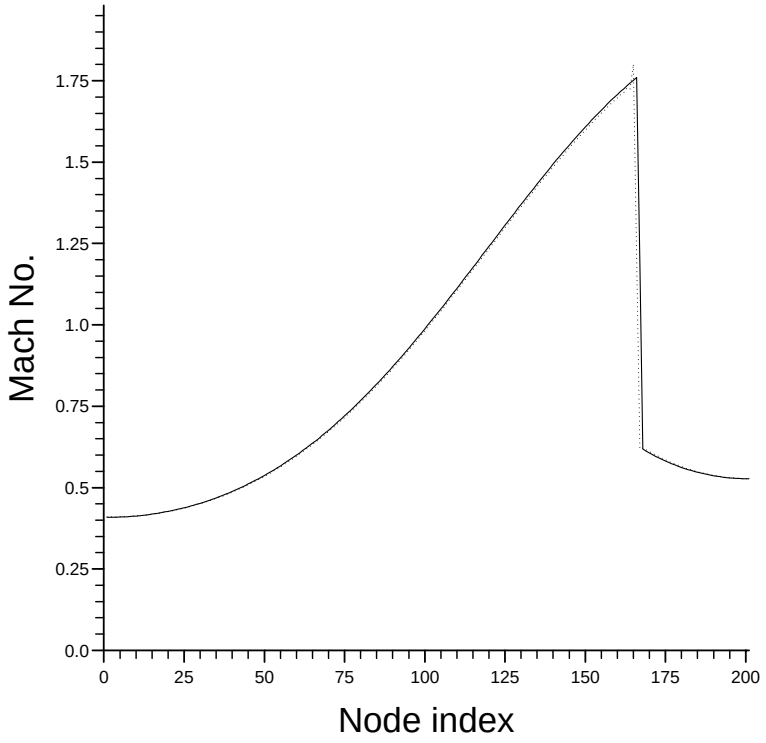

Figure 10: Comparison of Roe's scheme (solid line) and the central differencing hybrid scheme with only the two cells at the shock using Roe's scheme (dotted line) for the de Laval nozzle, $M_\infty = 0.5$.

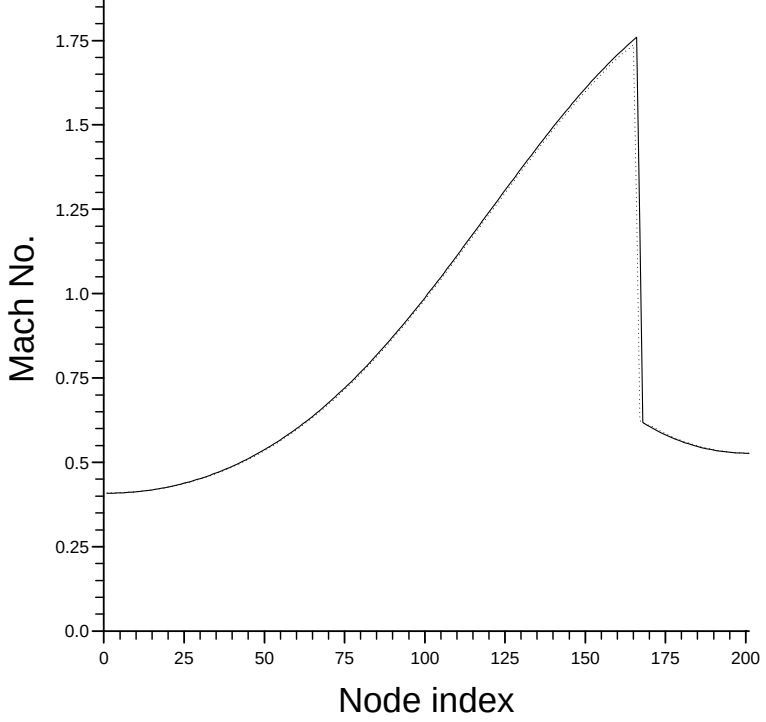

Figure 11: Comparison of Roe's scheme (solid line) and the central differencing hybrid scheme with a total of six cells at the shock using Roe's scheme (dotted line) for the de Laval nozzle, $M_\infty = 0.5$.

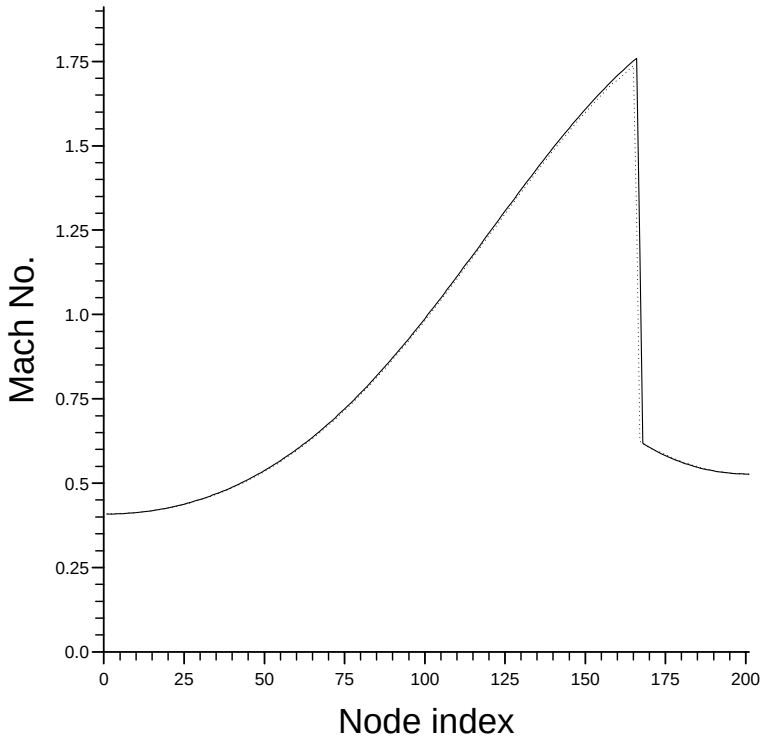

Figure 12: Comparison of Roe's scheme (solid line) and the central differencing hybrid scheme with a total of twelve cells at the shock using Roe's scheme (dotted line) for the de Laval nozzle, $M_\infty = 0.5$.

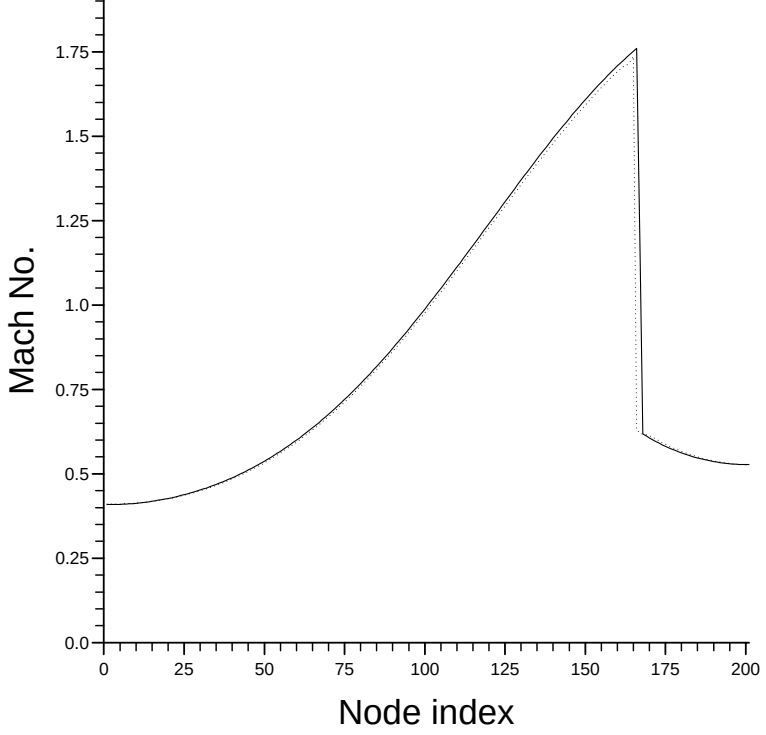

Figure 13: Comparison of Roe's scheme (solid line) and the central differencing hybrid scheme ($\epsilon = 0.01$) with a total of twelve cells at the shock using Roe's scheme (dotted line) for the de Laval nozzle, $M_\infty = 0.5$.

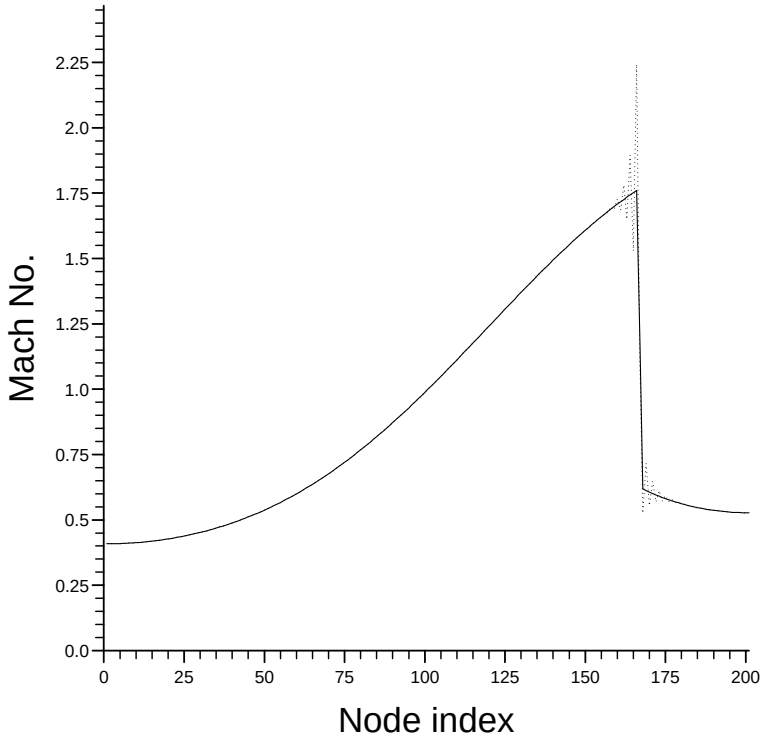

Figure 14: Comparison of Roe's scheme (solid line) and Lax-Wendroff (dotted line) for the de Laval nozzle, $M_\infty = 0.5$.

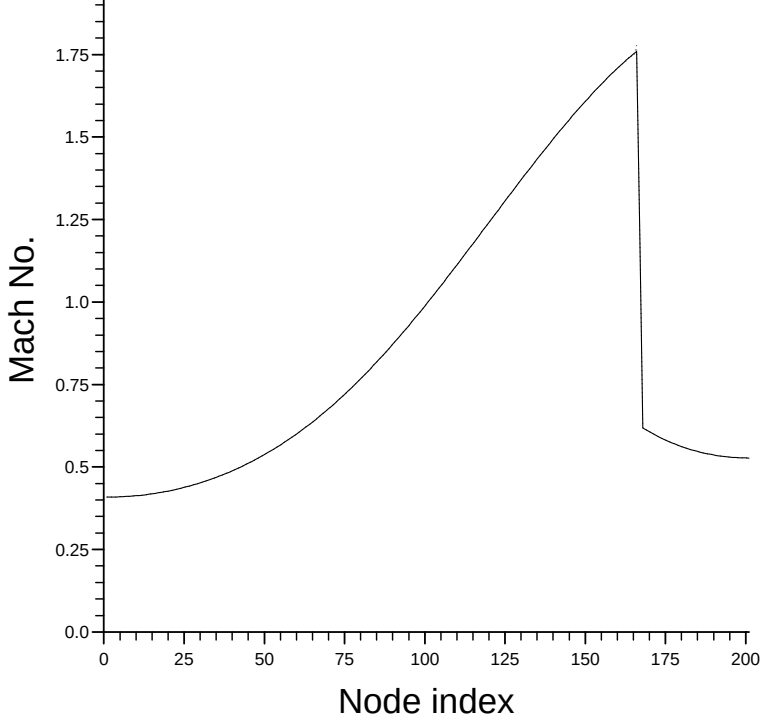

Figure 15: Comparison of Roe's scheme (solid line) and the Lax-Wendroff hybrid scheme with only the two cells at the shock using Roe's scheme (dotted line) for the de Laval nozzle, $M_\infty = 0.5$.

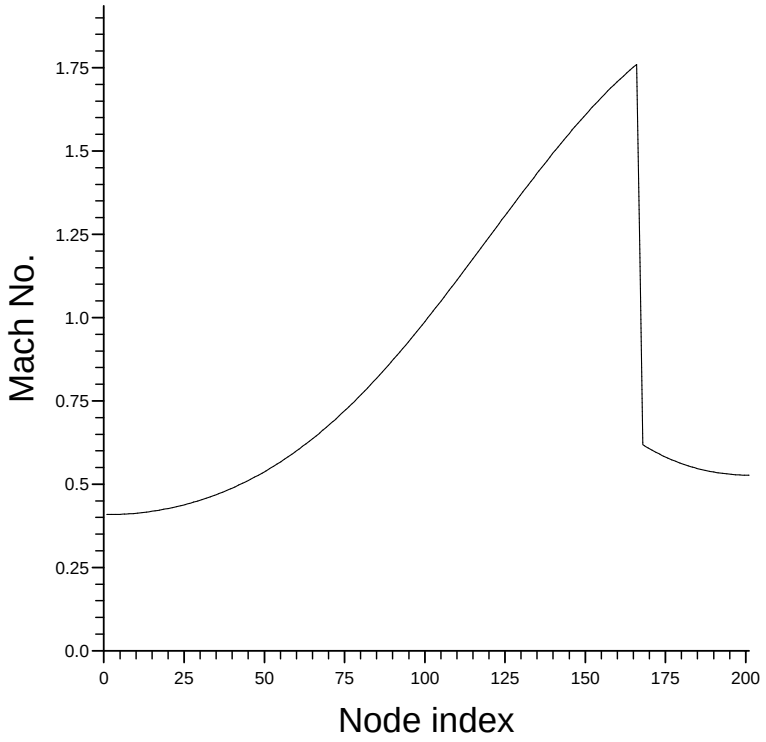

Figure 16: Comparison of Roe's scheme (solid line) and the Lax-Wendroff hybrid scheme with a total of six cells at the shock using Roe's scheme (dotted line) for the de Laval nozzle, $M_\infty = 0.5$.