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Space-Time Residual Distribution Schemes on Moving Meshes

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Consider the scalar conservation law

$$\partial_t u + \nabla \cdot \mathbf{f} = 0 \quad \text{or} \quad \partial_t u + \mathbf{a} \cdot \nabla u = 0$$

on a domain Ω .

- $\mathbf{a} = \frac{\partial \mathbf{f}}{\partial u}$ is the advection velocity for the flow.
- $u(\mathbf{x}, 0)$ is specified.
- $u(\mathbf{x}, t)$ is specified on inflow boundaries.

Integrating the conservation law gives

$$\partial_t u + \nabla \cdot \mathbf{f} = 0 \quad \longrightarrow \quad \int_{\Omega} \partial_t u + \nabla \cdot \mathbf{f} \, d\mathbf{x} \, dt = 0$$

- u can be continuous or discontinuous.
- Attempt to integrate the equations exactly.
- Distribute the integrals between the unknowns.
- For conservation, apply **Gauss' divergence theorem**.

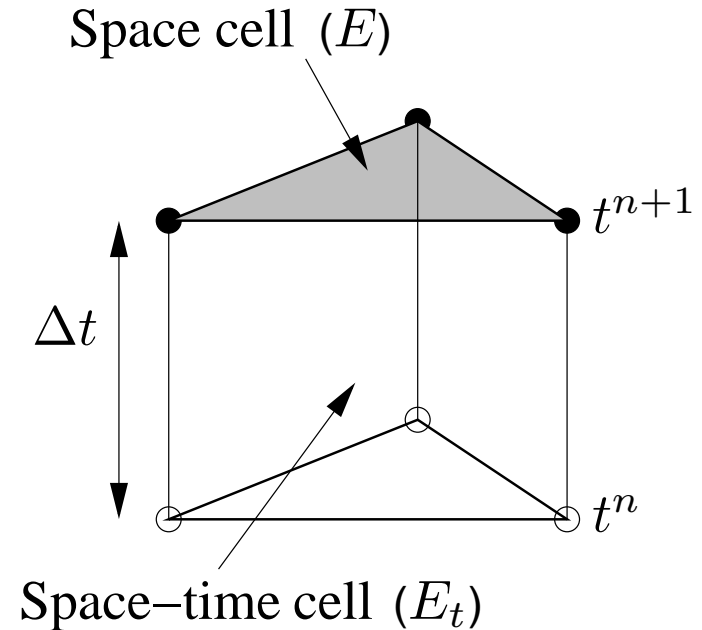
This is related to the finite element approach.

For a space-time mesh element (E_t), consider

$$\phi_{E_t} = \int_{E_t} \partial_t u + \nabla \cdot \mathbf{f} \, d\mathbf{x} \, dt$$

- For simplicity, assume that u is stored at mesh nodes and **varies linearly** in space and in time.
- In simple cases ϕ can be evaluated exactly using an appropriate **conservative linearisation**.
 - This leads to schemes with nice properties.
 - Conservation can be imposed in other cases.

- It is simplest to treat time and space slightly differently.
- Integrating over a space-time prism gives



$$\begin{aligned}\phi_{E_t} &= \int_E u^{n+1} - u^n d\mathbf{x} + \int_{t^n}^{t^{n+1}} \int_E \nabla \cdot \mathbf{f} d\mathbf{x} dt \\ &= \int_E u^{n+1} - u^n d\mathbf{x} - \int_{t^n}^{t^{n+1}} \oint_{\partial E} \mathbf{f} \cdot d\mathbf{n} dt\end{aligned}$$



Residual Distribution

The aim is to solve the equations given by

$$\partial_t u + \nabla \cdot \mathbf{f} = 0 \quad \longrightarrow \quad \sum_{E_t | i \in E_t} \beta_i^{E_t} \phi_{E_t} = 0 \quad \forall \text{ nodes } i$$

This is done iteratively, at each time level, by:

- **distributing** each residual ϕ_{E_t} to adjacent nodes;
- carefully choosing the **distribution coefficients** $\beta_i^{E_t}$;
- applying a simple pseudo-time-stepping algorithm,

$$(u_i)^{(m+1)} = (u_i)^{(m)} - \frac{\Delta \tau}{|S_i|} \sum_{E_t | i \in E_t} \beta_i^{E_t} \phi_{E_t}$$

Ideally, a residual distribution scheme would have the following properties.

- **Conservative**: for discontinuity capturing.
- **Positive**: to prohibit unphysical oscillations.
- **Linearity Preserving**: for accuracy.
- **Continuous**: for convergence of the iteration.
- **Compact**: for efficiency (and parallelism).
- **Upwind**: for physical realism.

For d -dimensional linear advection, assume that

- the spatial mesh is composed of **simplices**,
- the space-time mesh is **prismatic**,
- u **varies linearly** within each simplex and in time,

and write each element residual in the form

$$\phi_{E_t} = \sum_{i \in E_t} k_i u_i \quad \text{where} \quad k_i = \frac{\Delta t}{2d} \tilde{\mathbf{a}} \cdot \mathbf{n}_i \pm \frac{|E|}{d+1}$$

is an upwinding parameter.

Since the k_i sum to zero in an element

$$\phi_{E_t} = \sum_{i \in E_t} k_i (u_i - u_{in}), \quad u_{in} = \left(\sum_{i \in E_t} k_i^+ \right)^{-1} \left(\sum_{i \in E_t} k_i^+ u_i - \phi_{E_t} \right)$$

- The **N scheme** (linear, positive)

$$\phi_i^{E_t} = \beta_i^{E_t} \phi_{E_t} = k_i^+ (u_i - u_{in})$$

- The **LDA scheme** (linear, linearity preserving)

$$\phi_i^{E_t} = \beta_i^{E_t} \phi_{E_t} = \left(\sum_{i \in E_t} k_i^+ \right)^{-1} k_i^+ \phi_{E_t}$$

To achieve positivity *and* linearity preservation:

- The **PSI scheme** limits the distribution coefficients of the N scheme:

$$\beta_i^{E_t} \longleftarrow \frac{(\beta_i^{E_t})^+}{\sum_{k \in E_t} (\beta_k^{E_t})^+} \Rightarrow \beta_i^{E_t} \in [0, 1]$$

- **Blended schemes** use weighted averages:

$$\phi^{\text{Blend}} = \theta \phi^{\text{N}} + (1 - \theta) \phi^{\text{LDA}} \quad \theta \in [0, 1]$$

- This is more robust and flexible but may not be positive for the most common choices of θ .



Forcing continuity at element faces can be restrictive.

- It is difficult to change the representation locally, within mesh elements, since it has a knock-on effect on neighbours. This interferes with:
 - conservation, particularly at boundaries;
 - h - and p -adaptivity;
 - the limiting of high order schemes for positivity;
 - **the stability of time-dependent schemes.**
- Discontinuous flow cannot be represented exactly.

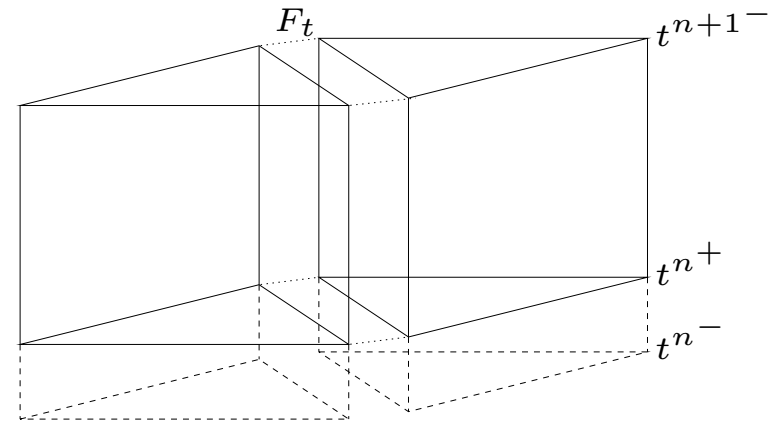
If u is allowed to be discontinuous then

$$\begin{aligned} & \int_{\Omega} \partial_t u + \nabla \cdot \mathbf{f} \, d\mathbf{x} \, dt \\ = & \sum_{E_t} \int_E u^{n+1-} - u^{n+} \, d\mathbf{x} - \sum_{E_t} \int_{t^n}^{t^{n+1}} \oint_{\partial E} \mathbf{f} \cdot d\mathbf{n} \, dt \\ & + \sum_{F_t} \int_{t^n}^{t^{n+1}} \int_{F_t} [\mathbf{f}] \cdot d\mathbf{n} \, dt + \sum_E \int_E u^{n+} - u^{n-} \, d\mathbf{x} \end{aligned}$$

It is possible to consider the face integrals separately.

$$\int_{\Omega} \partial_t u + \nabla \cdot \mathbf{f} \, d\mathbf{x} \, dt = \sum_{E_t} \phi_{E_t} + \sum_{F_t} \psi_{F_t} + \sum_E \psi_E$$

- The ψ are simply integrals over an interface of the **flux difference** across it.
- The ψ_{F_t} will be ignored here.



Integrating across temporal discontinuities gives

$$\psi_E = \int_E [u] d\mathbf{x}$$

- Upwinding always distributes forward in time.
- This removes the necessity for the past shield condition on the space-time distribution.
- Schemes can now be positive for *any* time-step.

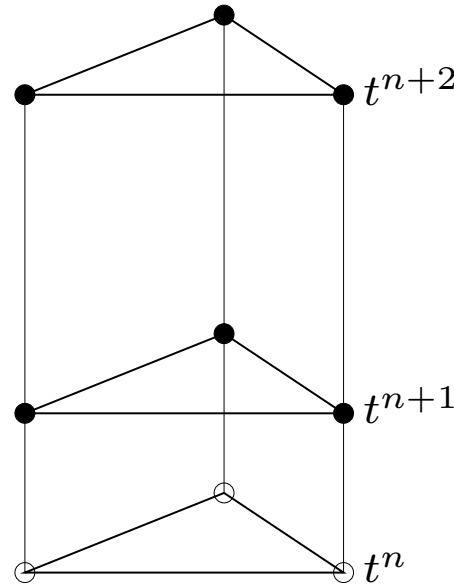
Discontinuities in Time



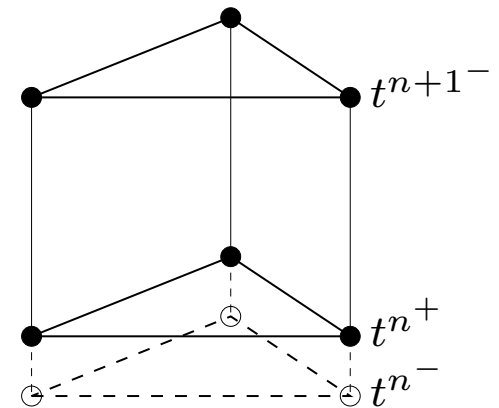
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This is a degenerate two-layer scheme.

- The distribution is much simpler in the discontinuity.



Continuous Double Layer



Discontinuous
in time

A positive, linearity preserving, distribution is

$$\psi_{i,n-}^E = 0 \quad \psi_{i,n+}^E = \beta_{i,n+}^E \psi_E = \frac{|E|}{d+1} (u_i^{n+} - u_i^{n-})$$

The aim is to solve the equations given by

$$\partial_t u + \nabla \cdot \mathbf{f} = 0 \quad \longrightarrow \quad \sum_{E_t | i \in E_t} \beta_i^{E_t} \phi_{E_t} + \sum_{E | i \in E} \beta_{i,n+}^E \psi_E = 0$$

This is done iteratively, at each time level, by:

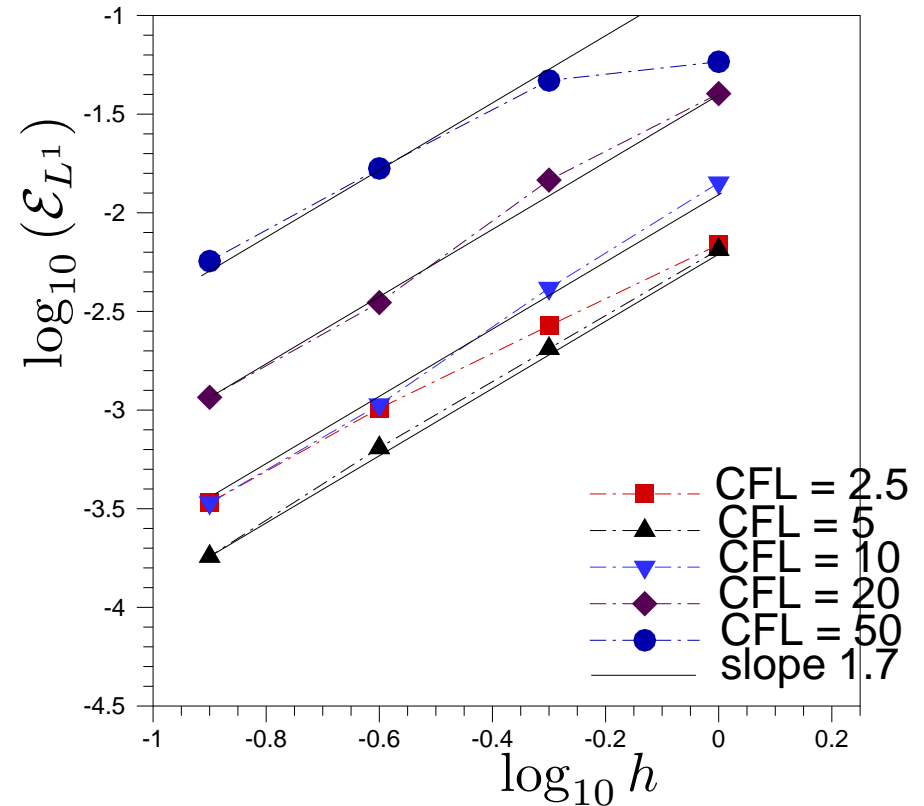
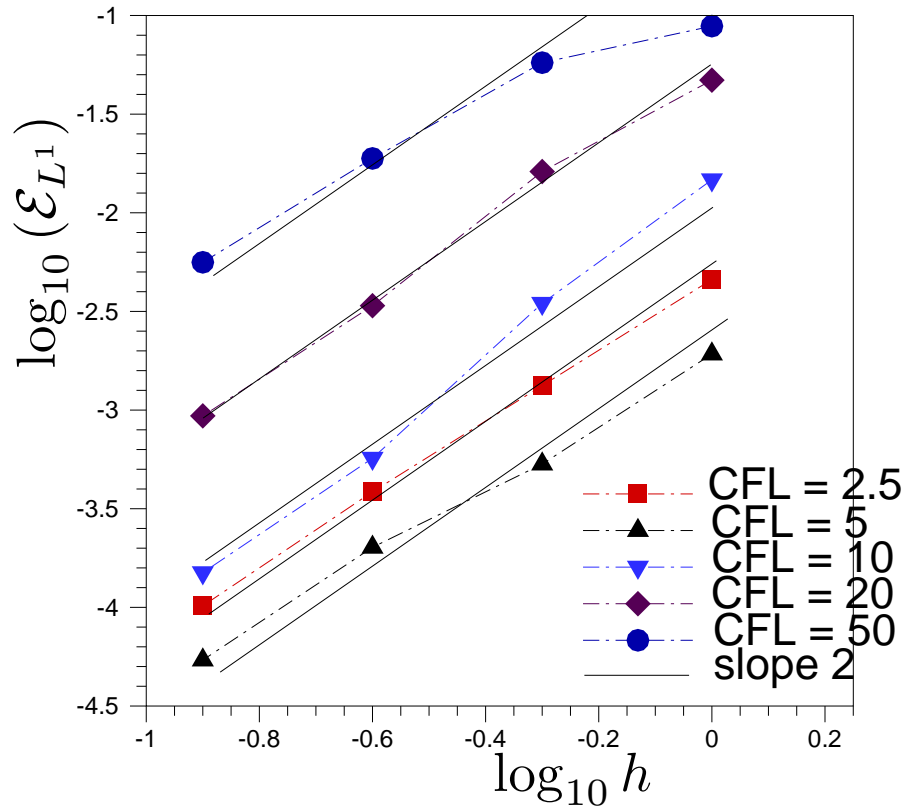
- **distributing** the ϕ_{E_t} and ψ_E to adjacent vertices;
- choosing the **distribution coefficients**, $\beta_i^{E_t}$ and $\beta_{i,n+}^E$;
- applying a simple pseudo-time-stepping algorithm,

$$(u_i)^{(m+1)} = (u_i)^{(m)} + \frac{\Delta \tau}{|S_i|} \left(\sum_{E_t | i \in E_t} \beta_i^{E_t} \phi_{E_t} + \sum_{E | i \in E} \beta_{i,n+}^E \psi_E \right)$$

Results: Linear Advection (2D)



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Mesh convergence for constant advection of a smooth profile: L^1 error for the ST LDA (left) and ST LDA-N (right) schemes.

Consider the system of conservation laws

$$\partial_t U + \nabla \cdot \mathbf{F} = 0 \quad \text{or} \quad \partial_t U + \mathbf{A} \cdot \nabla U = 0$$

on a domain Ω , where \mathbf{A} gives the **flux Jacobians**.

$$\Phi_{E_t} = \int_E U^{n+1^-} - U^{n^+} d\mathbf{x} - \int_{t^n}^{t^{n+1}} \oint_{\partial E} \mathbf{F} \cdot d\mathbf{n} dt$$

$$\Psi_{F_t} = \int_{t^n}^{t^{n+1}} \int_{F_t} [\mathbf{F}] \cdot d\mathbf{n} dt \quad \Psi_E = \int_E U^{n^+} - U^{n^-} d\mathbf{x}$$

can all (with care) be evaluated exactly, decomposed and distributed to the element vertices.

The element-based residuals take the form

$$\Phi_{E_t} = \sum_{i \in E_t} \mathbf{K}_i U_i \quad \text{where} \quad \mathbf{K}_i = \frac{\Delta t}{2d} \tilde{\mathbf{A}} \cdot \mathbf{n}_i \pm \frac{|E|}{d+1} \mathbf{I}$$

The face-based residuals can be written

$$\Psi_{F_t} = \sum_{i \in F_t} \mathbf{K}_i (U_i^R - U_i^L) \quad \text{where} \quad \mathbf{K}_i = \frac{\Delta t}{2d} \hat{\mathbf{A}}_i \cdot \mathbf{n}$$

$$\Psi_E = \frac{|E|}{d+1} \sum_{i \in E} (U_i^{n^+} - U_i^{n^-})$$

The \mathbf{K}_i can be diagonalised to get the \mathbf{K}_i^+ .

Include the source term in the element residual:

$$\Phi_{E_t} = \int_{E_t} \partial_t U + \nabla \cdot \mathbf{F} - S \, d\mathbf{x} \, dt$$

- With shallow water flows, care is needed to ensure that, when $b + d$ is constant,

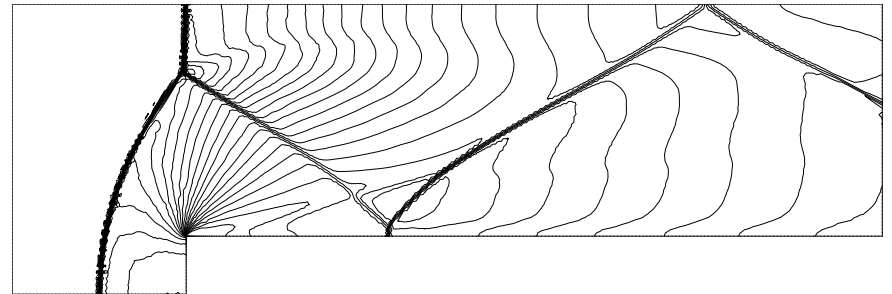
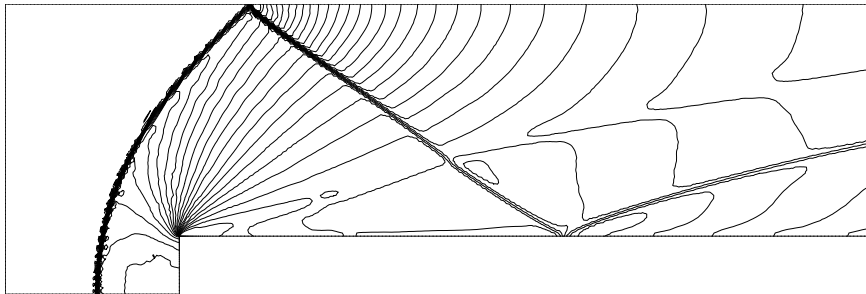
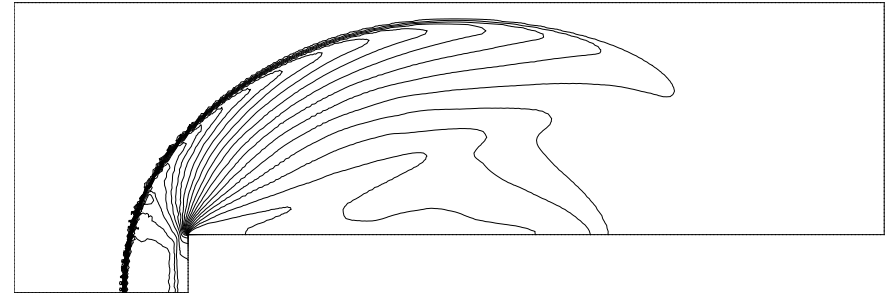
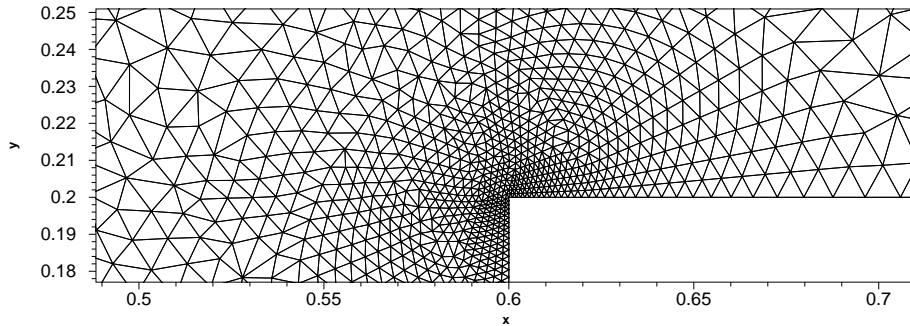
$$\int_E \nabla \left(\frac{gd^2}{2} \right) d\mathbf{x} = - \int_E gd \nabla b \, d\mathbf{x}$$

- The conservative schemes apply as before.
- Discontinuities in space are more challenging.

Results: Euler Equations



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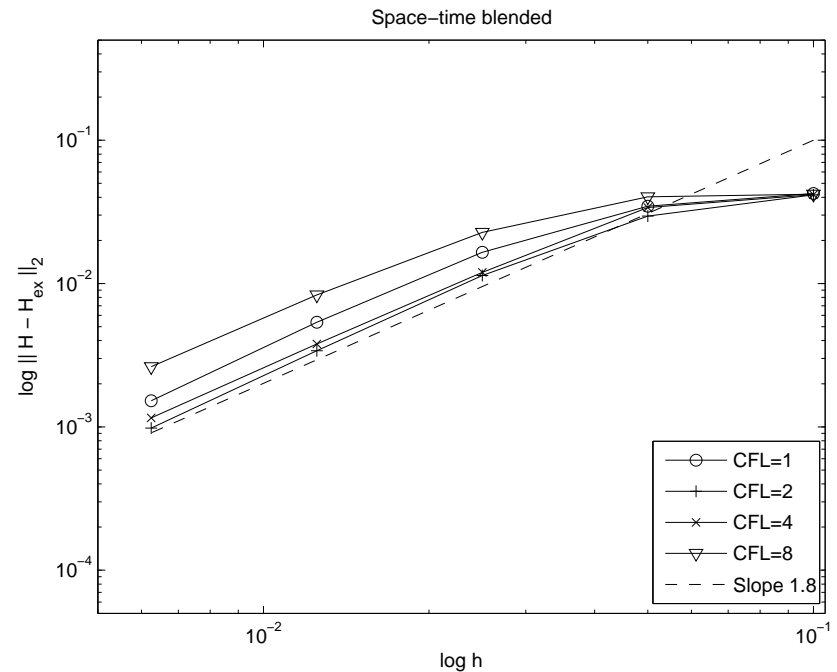
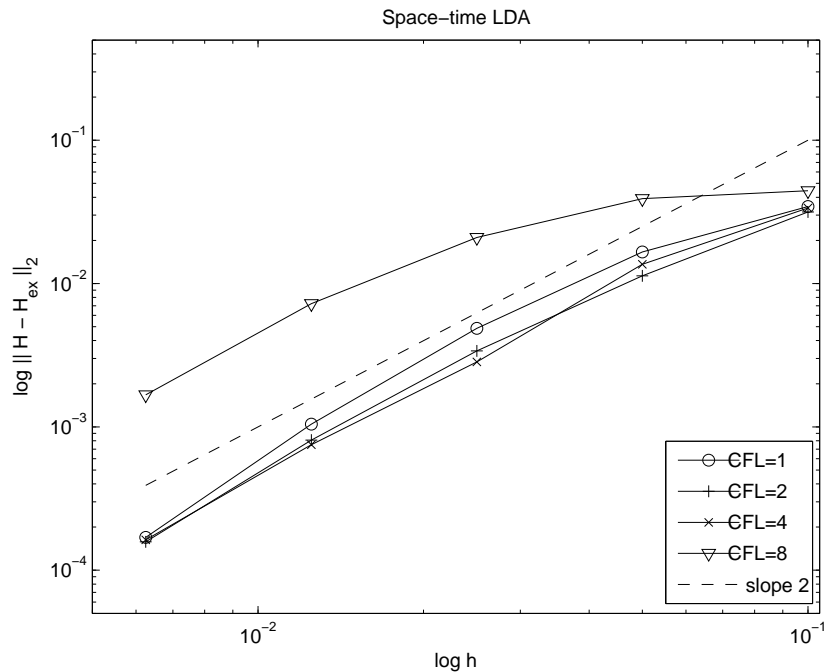


Supercritical backward facing step: ST LDA-N scheme;
density contours with $M_\infty = 3.0$ and $CFL_{\max} = 12.5$.

Results: Shallow Water Flow



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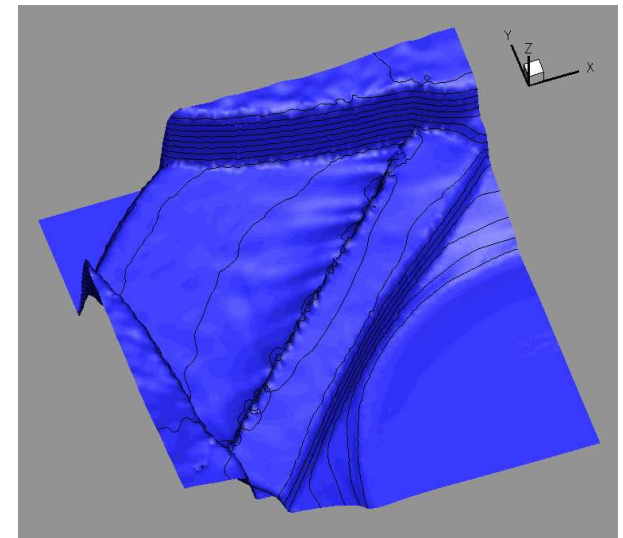
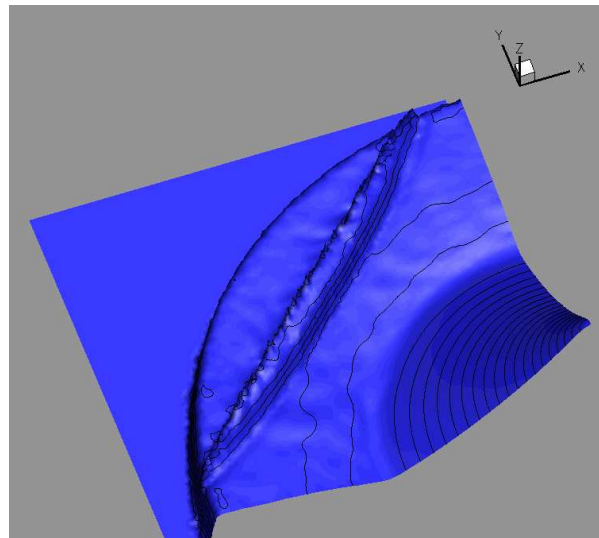
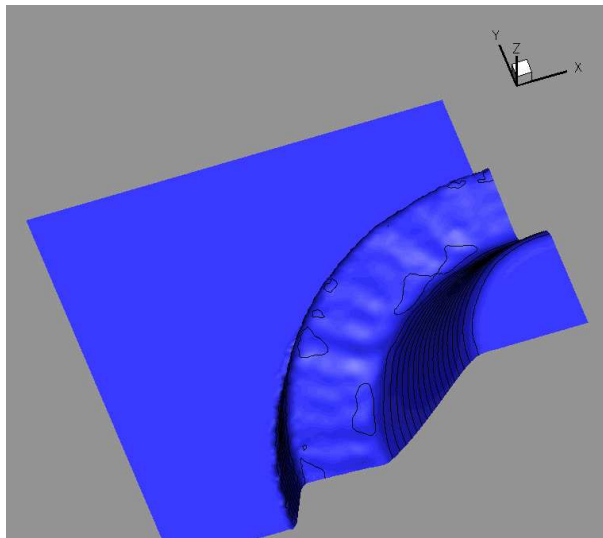


Travelling vortex (exact solution), mesh convergence:
ST LDA (left) and ST LDA-N (right) schemes.

Results: Shallow Water Flow



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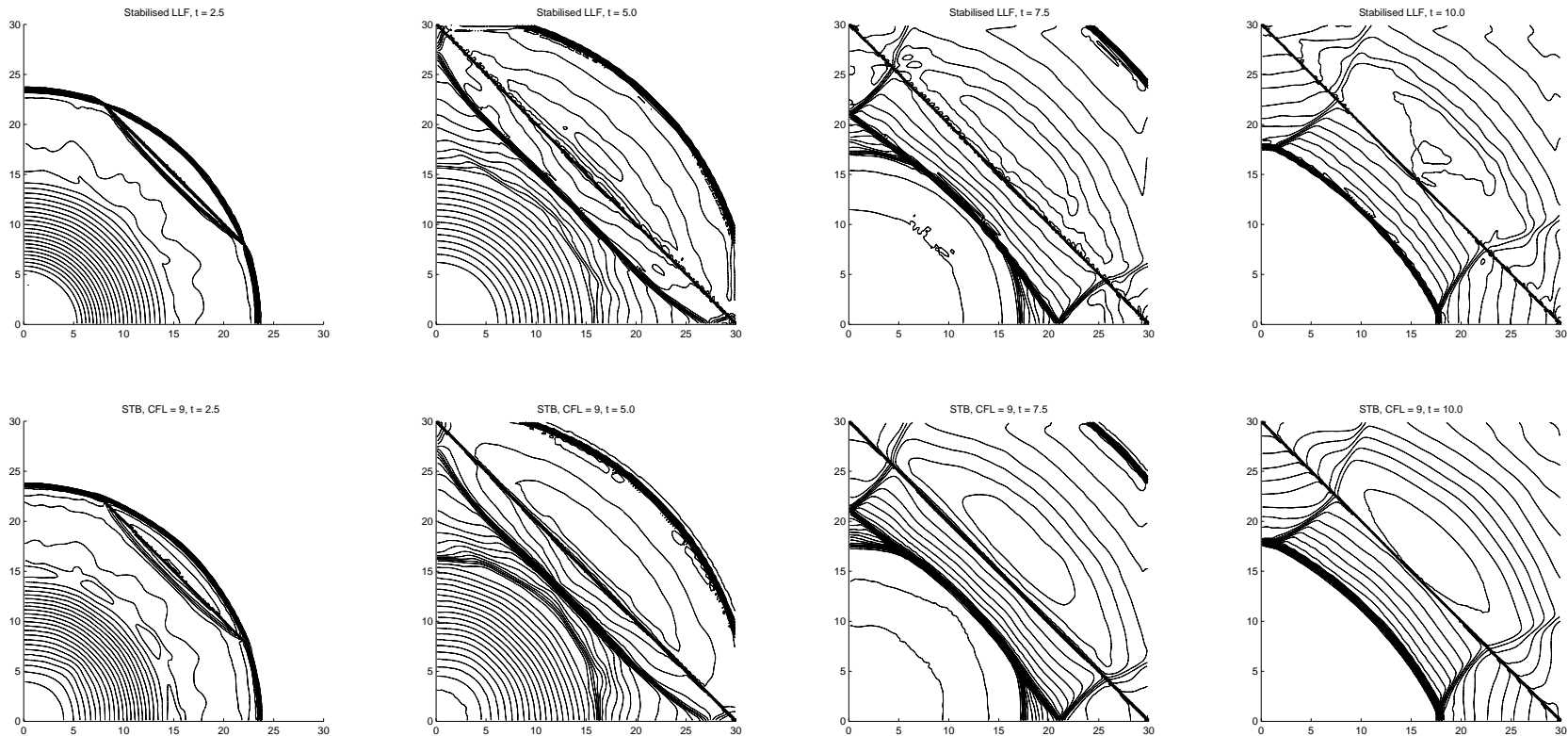


Circular dam break, discontinuous bed, unstructured mesh,
free surface: stabilised LLF scheme.

Results: Shallow Water Flow

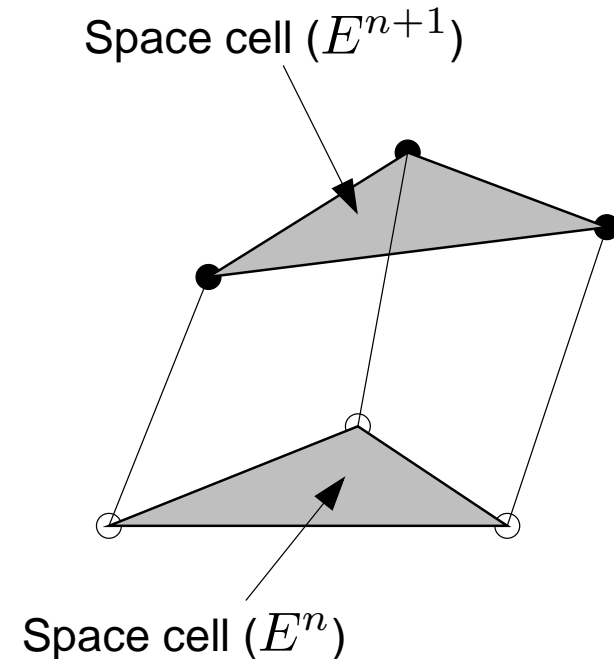


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Circular dam break, discontinuous bed, locally refined mesh,
free surface: stabilised LLF scheme, $CFL_{\max} = 0.8$ (top);
ST LDA-N scheme, $CFL_{\max} = 9.0$ (bottom).

- The mesh at the new time may differ from that at the old time.
- Integrating over a distorted space-time prism gives



$$\begin{aligned}
 \phi_{E_t} &= \int_{E^{n+1}} u^{n+1} d\mathbf{x} - \int_{E^n} u^n d\mathbf{x} - \int_{t^n}^{t^{n+1}} \oint_{\partial E} \mathbf{f}_t \cdot d\mathbf{n}_t dt \\
 &= \int_{E^{n+1}} u^{n+1} d\mathbf{x} - \int_{E^n} u^n d\mathbf{x} - \int_{t^n}^{t^{n+1}} \oint_{\partial E} (\mathbf{f} - u\mathbf{v}) \cdot d\mathbf{n} dt
 \end{aligned}$$

For d -dimensional linear advection, the element residual can still be written in the form

$$\phi_{E_t} \approx \sum_{i \in E_t} k_i u_i \quad \text{where} \quad k_i^* = \frac{\Delta t}{2d} (\tilde{\mathbf{a}}^* - \mathbf{v}_i^*) \cdot \mathbf{n}_i^* \pm \frac{|E^*|}{d+1}$$

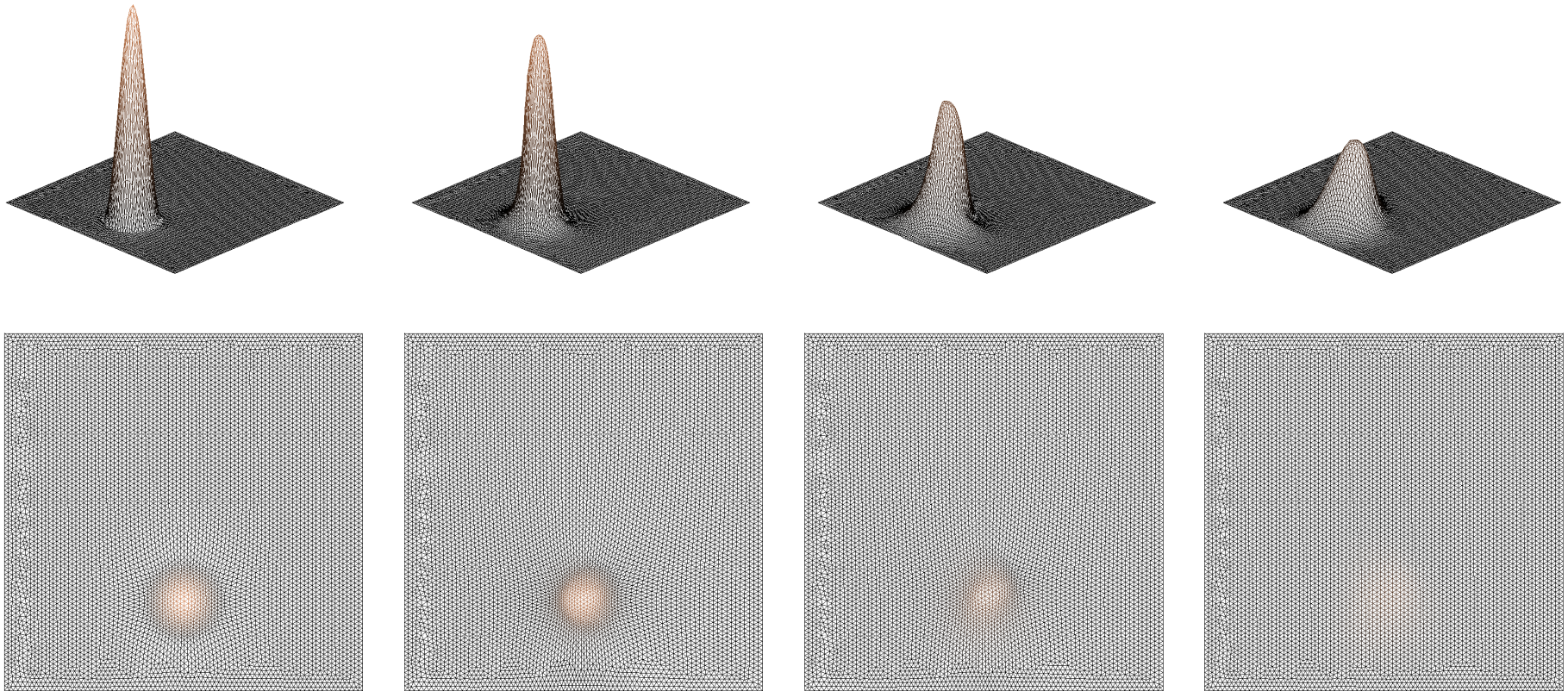
- The superscript \cdot^* indicates the time level.
- The mesh velocity should not be averaged because it may not satisfy $\nabla \cdot \mathbf{v} = 0$.
- The distribution schemes can be applied as before.

- The mesh nodes are moved during pseudo-time-stepping, according to

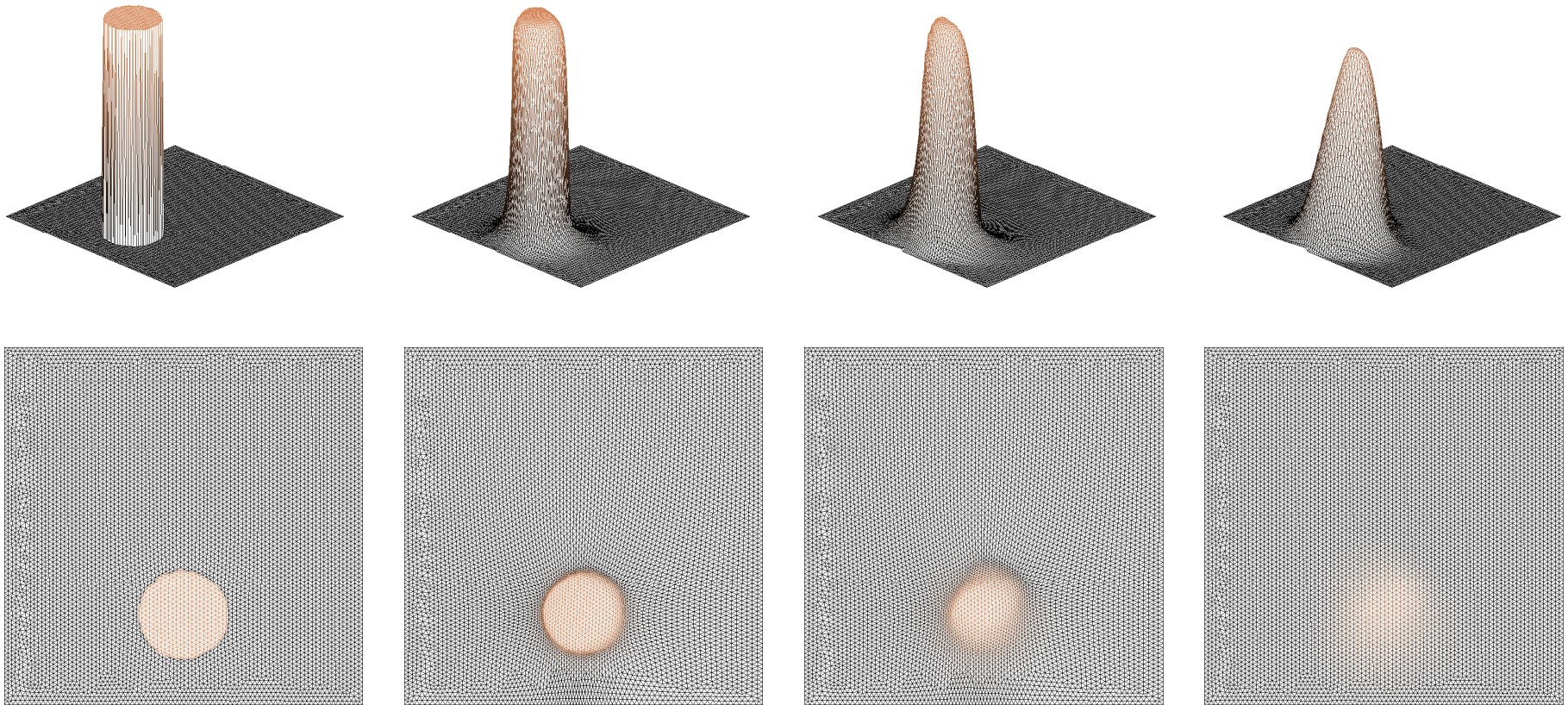
$$\mathbf{x}_i^{(m+1)} = \mathbf{x}_i^{(m)} + \frac{\sum_{E|i \in E} M_E \mathbf{x}_E^{(m)}}{\sum_{E|i \in E} M_E}$$

- A surface area monitor, $M_E = |E| (1 + \alpha |\nabla u|_E^2)^{\frac{1}{2}}$ is interleaved with a Laplacian smoother.
- The pseudo-time-stepping is continued after the movement is stopped, with mesh velocities

$$\mathbf{v}_i = \frac{\mathbf{x}_i^{n+1} - \mathbf{x}_i^n}{\Delta t}$$



Rotating [cosine-squared profile](#), space-time PSI scheme:
moving mesh (0, 1, 5 revolutions), fixed mesh (5 revolutions).



Rotating [cylinder profile](#), space-time PSI scheme:
moving mesh (0, 1, 5 revolutions), fixed mesh (5 revolutions).

For linear advection on fixed meshes the scheme is:

- positive for any time-step;
- conservative, linearity preserving, compact, upwind and continuous;
- second order accurate for smooth profiles.

It also gives good approximations to the Euler and shallow water equations, although it is not yet:

- easy to converge the inner iteration;
- as robust as the best flux-based schemes.

On moving meshes, the scheme is designed to retain all of the fixed mesh properties, but:

- these are only the first results;
- it's not yet clear whether imposing positivity constrains the time-step;
- second order accuracy relies on using appropriate quadrature to evaluate the residual.

At the moment, the moving mesh scheme for nonlinear systems only exists on paper.