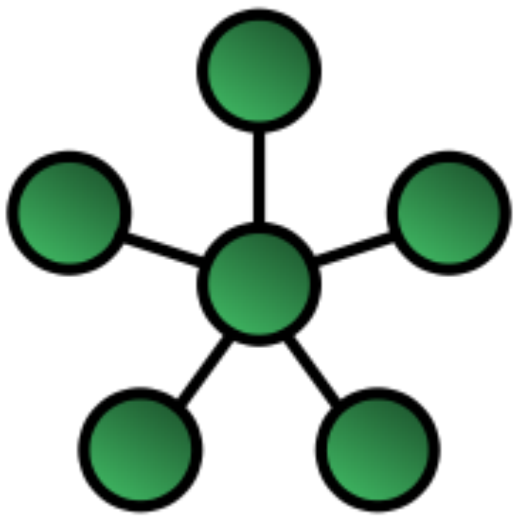
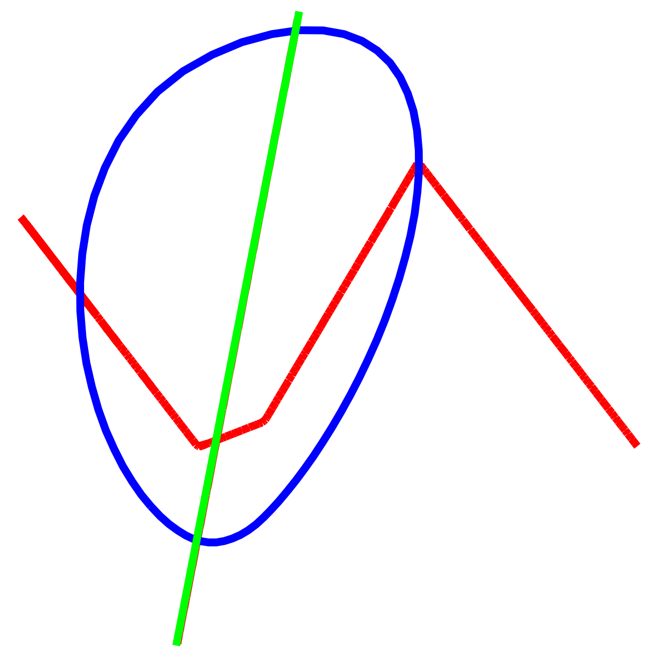
Networks of ~~Non~~smooth Oscillators & Applications in Neuroscience

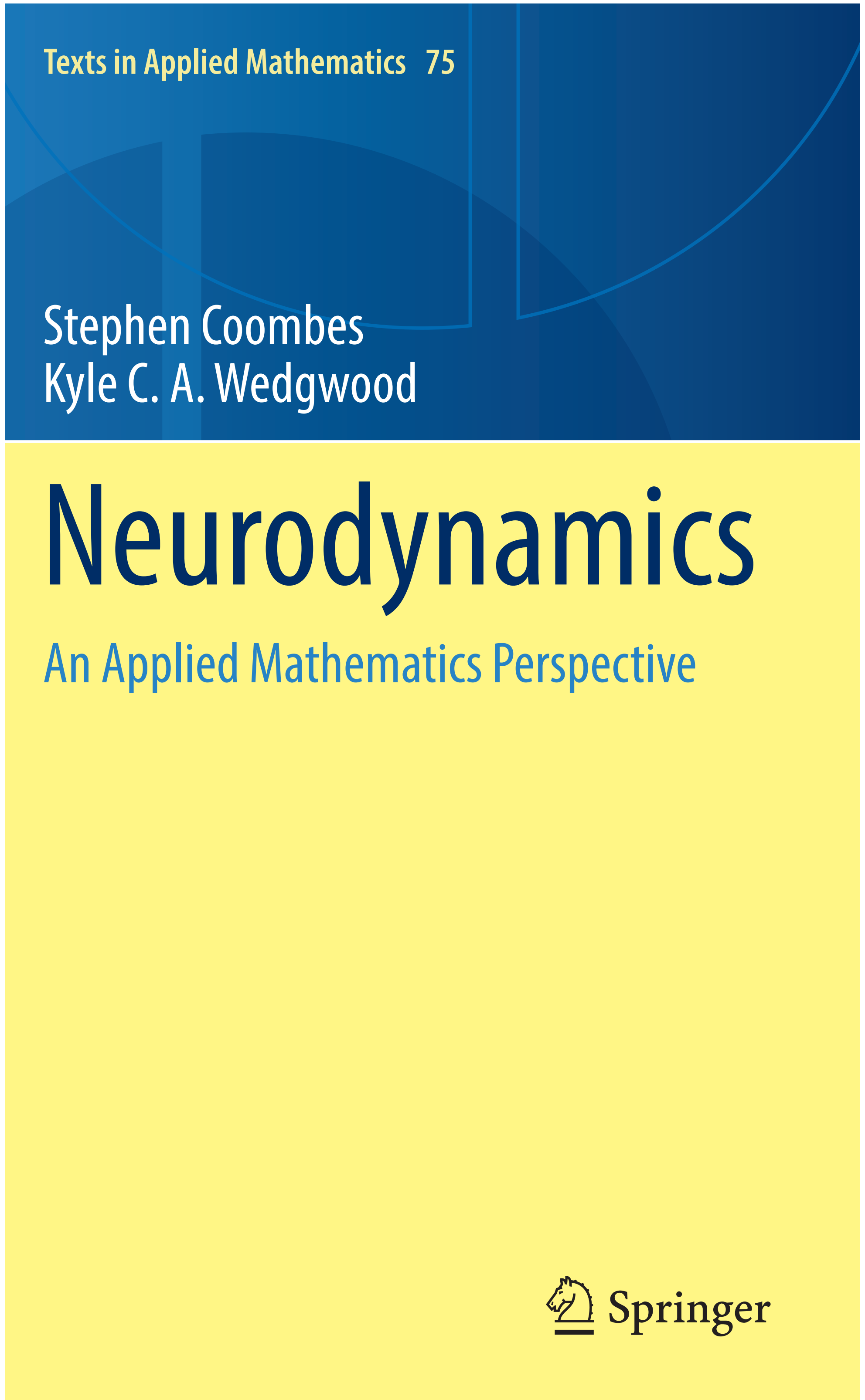


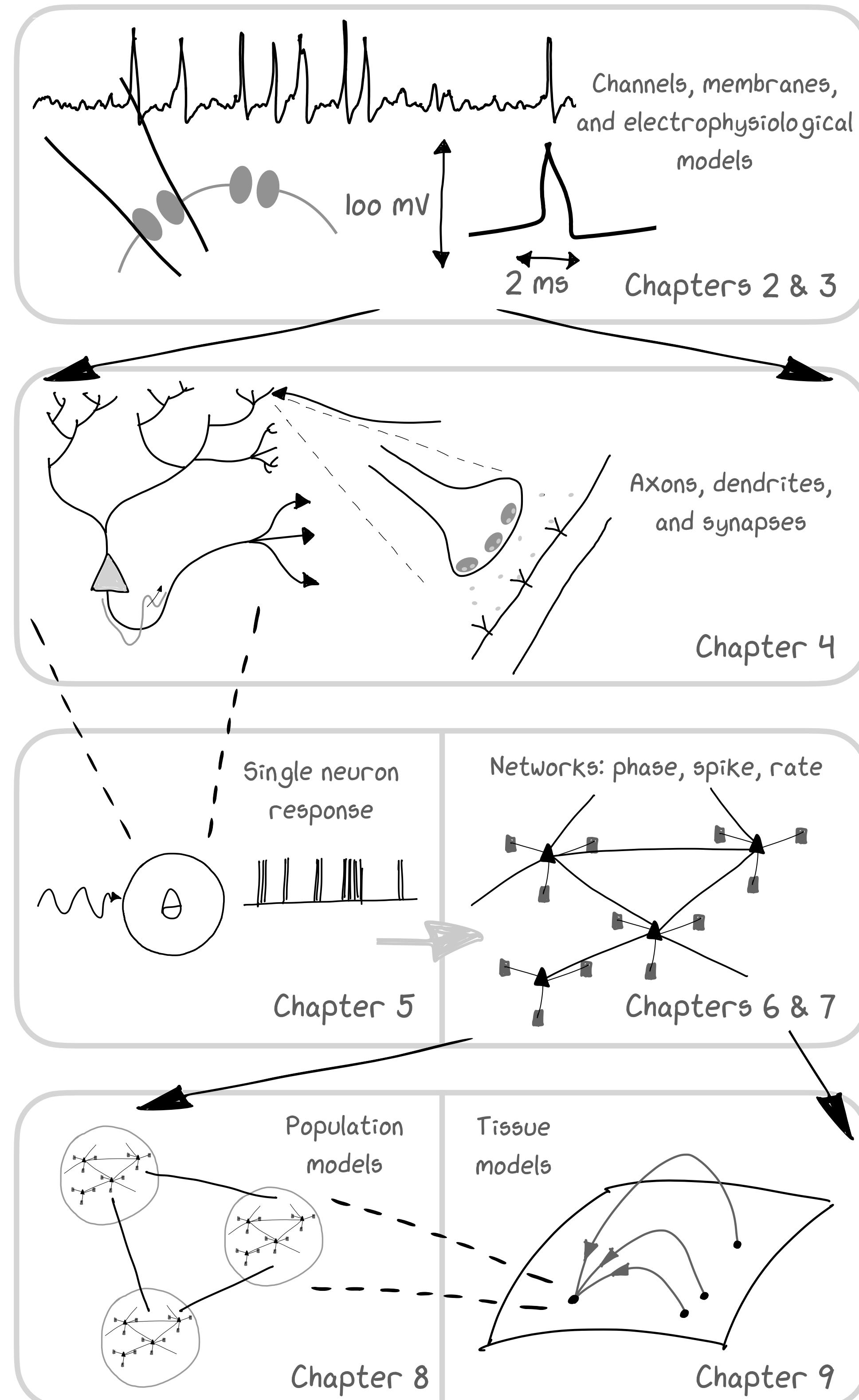


Centre for Mathematical Medicine and Biology (CMMB)

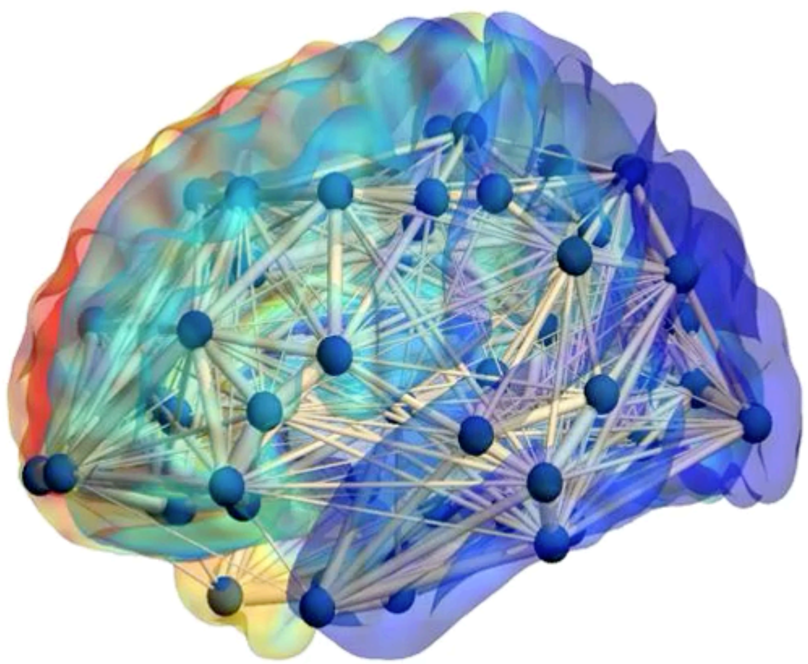
Steve Coombes

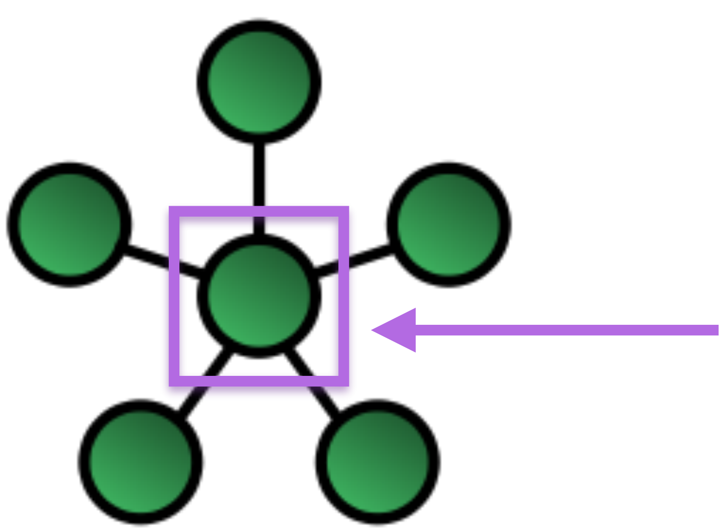
The University of Nottingham


Texts in Applied Mathematics 75
Stephen Coombes
Kyle C. A. Wedgwood
Neurodynamics
An Applied Mathematics Perspective
Springer


Channels, membranes, and electrophysiological models
100 mV
2 ms
Chapters 2 & 3
Axons, dendrites, and synapses
Chapter 4
Single neuron response
Chapter 5
Networks: phase, spike, rate
Chapters 6 & 7
Population models
Chapter 8
Tissue models
Chapter 9

Models of large scale brain dynamics

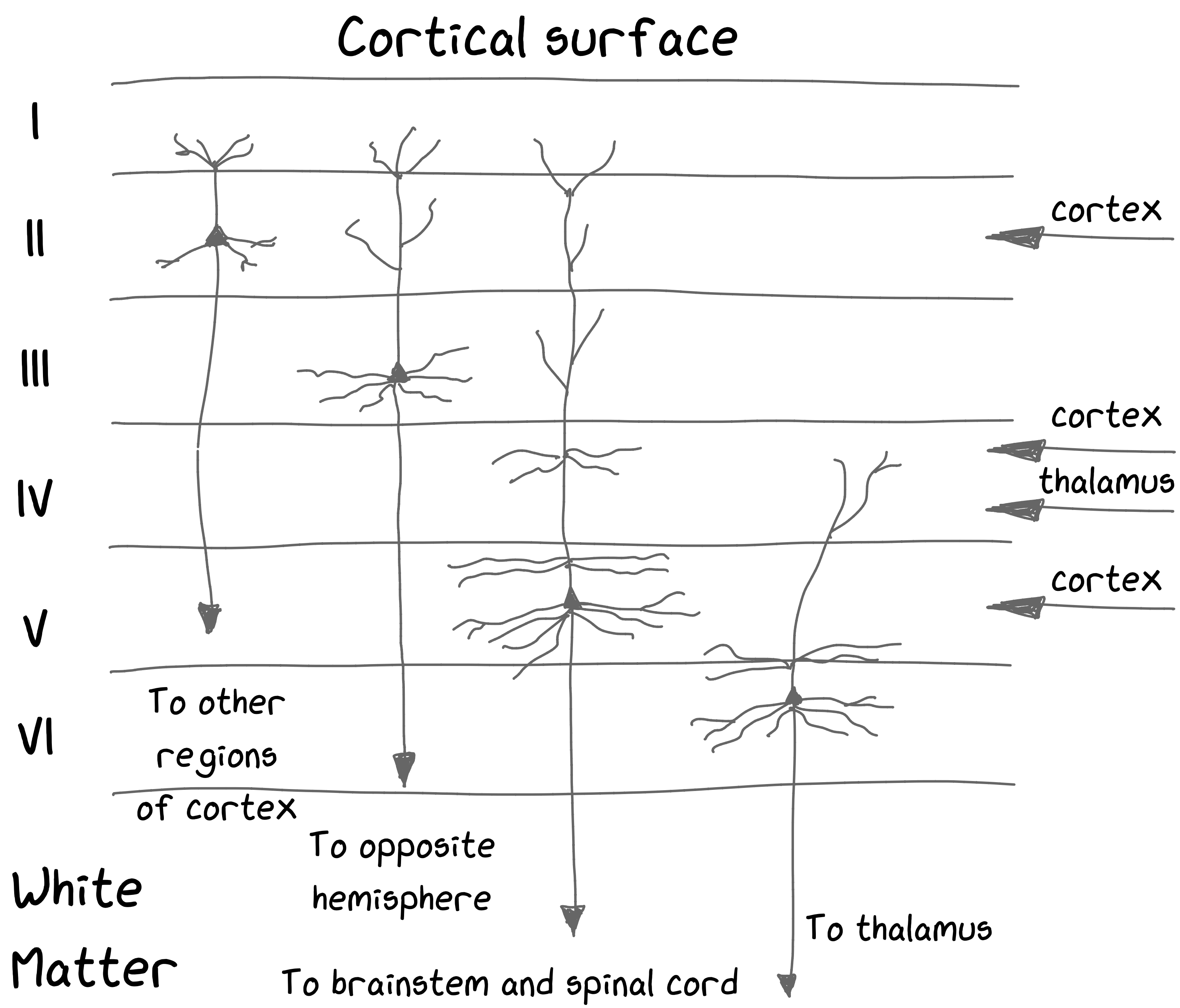


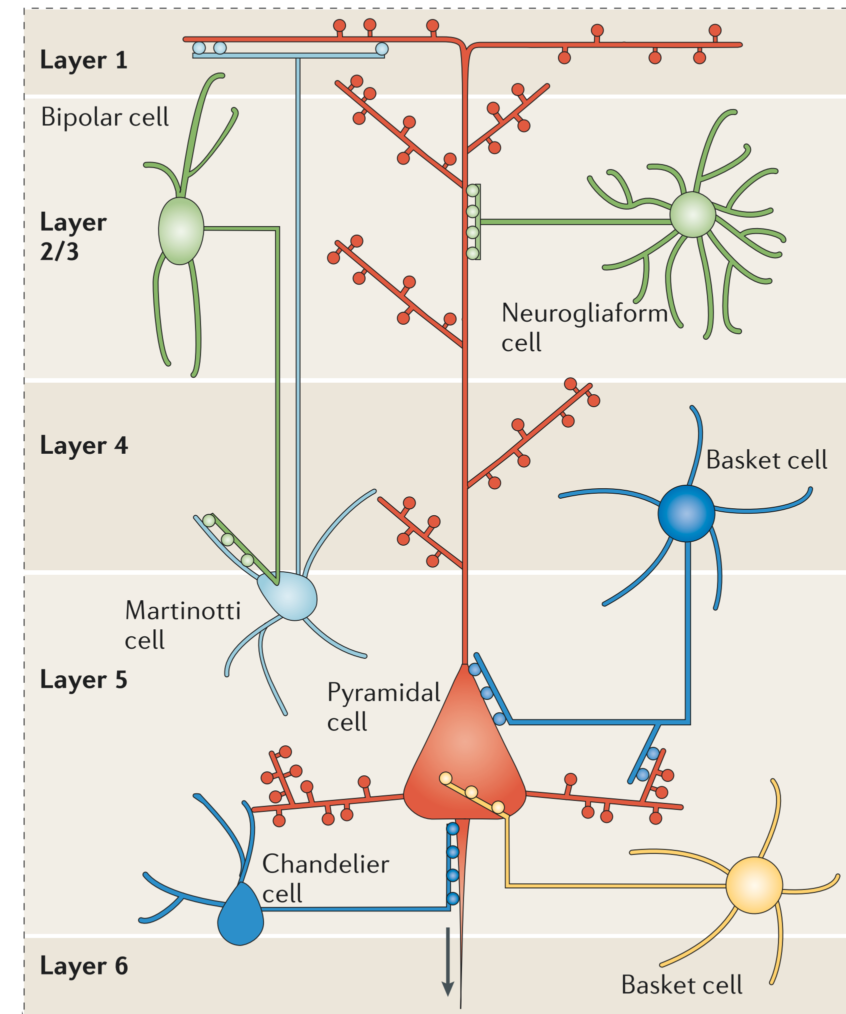


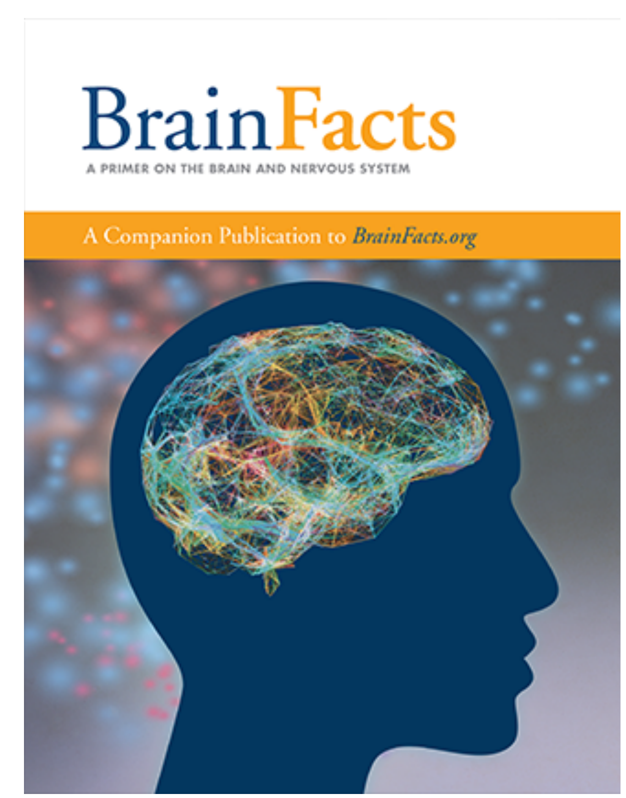
$$\dot{x} = F(x)$$

$$x \in \mathbb{R}^m$$

www.humanconnectome.org



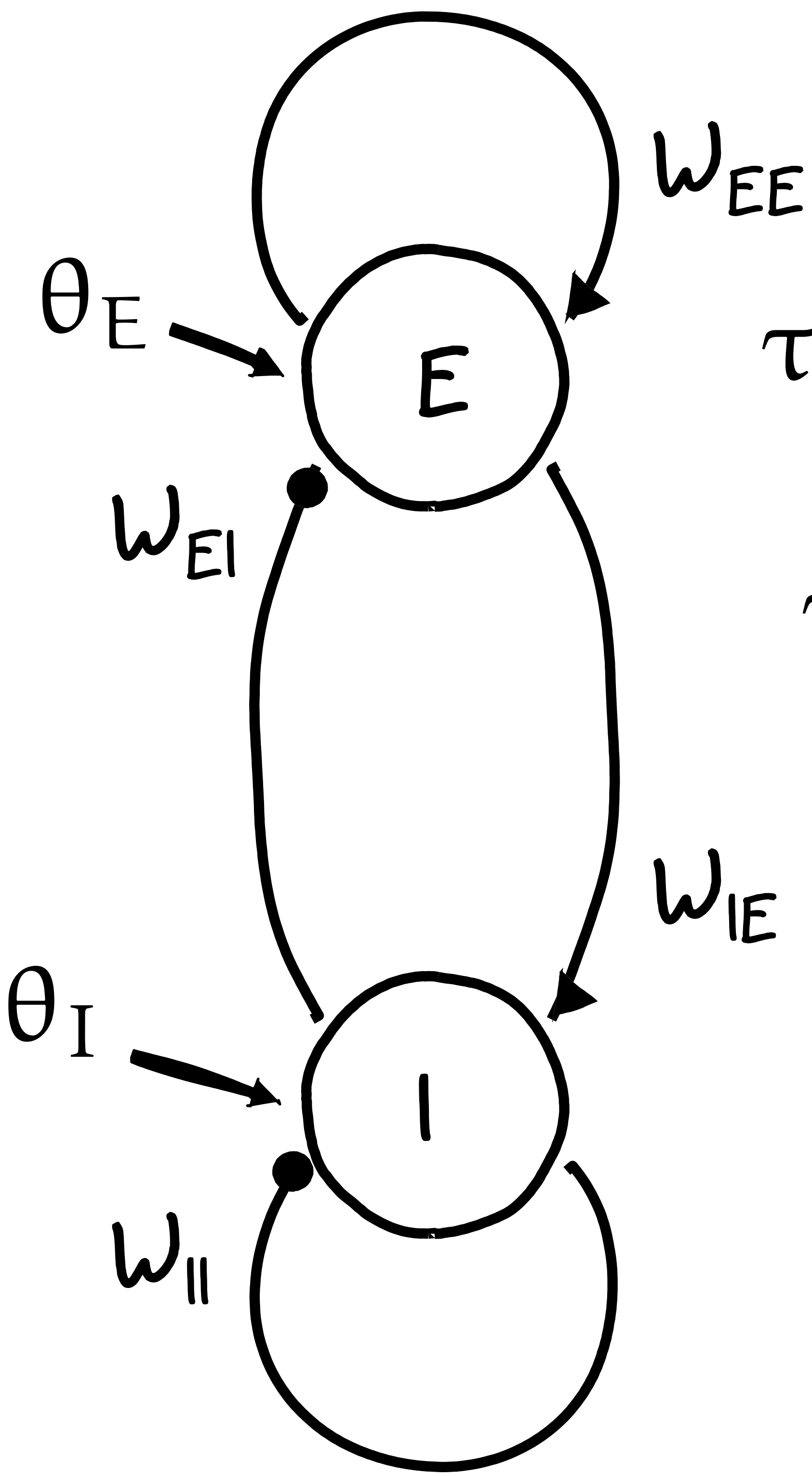




www.brainfacts.org

Marín, O. Interneuron dysfunction in psychiatric disorders. Nat Rev Neurosci 13, 107–120 (2012)

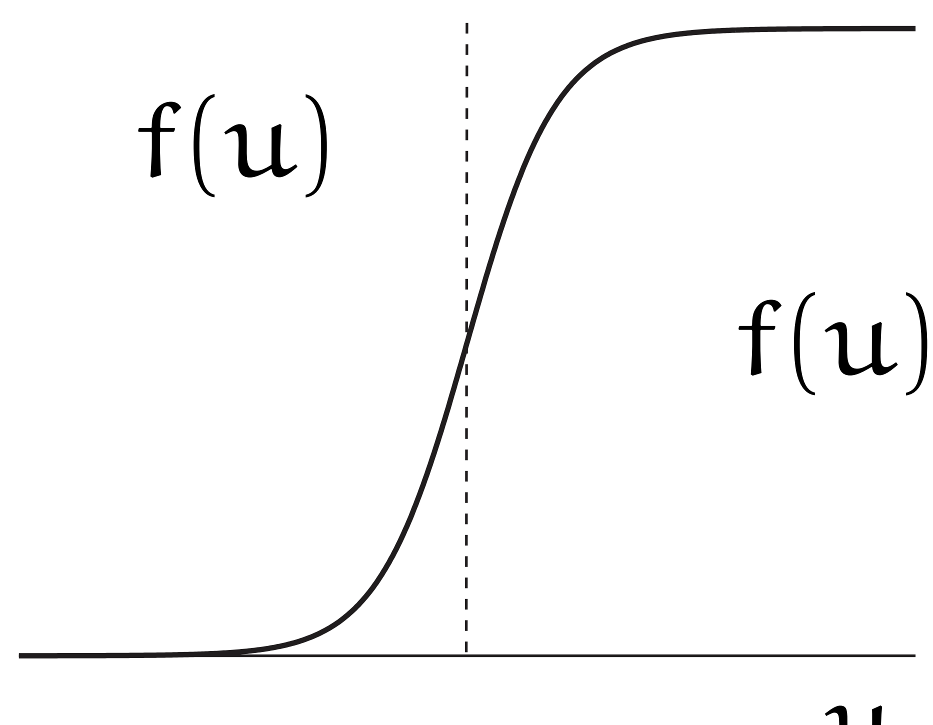
Wilson-Cowan model of cortical activity



excitation-inhibition

$$\tau_E \frac{d}{dt} E = -E + f\left(W_{EE}E - W_{EI}I + \theta_E\right)$$

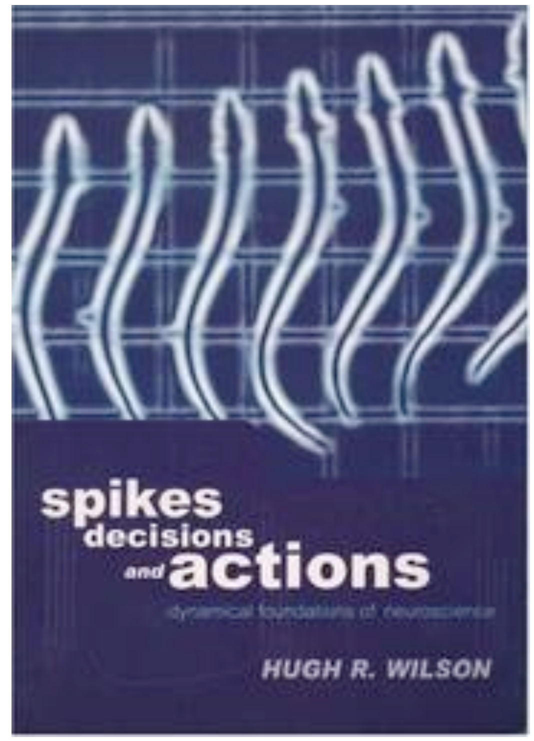
$$\tau_I \frac{d}{dt} I = -I + f\left(W_{IE}E - W_{II}I + \theta_I\right)$$



$$f(u) = \frac{1}{1 + e^{-\beta u}}$$

The scientists and the science

Hugh R Wilson



Jack D Cowan

- H R Wilson and J D Cowan. Excitatory and inhibitory interactions in localized populations of model neurons. Biophysical Journal, 12:1–24, **1972**. ~4k citations
- H R Wilson and J D Cowan. A mathematical theory of the functional dynamics of cortical and thalamic nervous tissue. Kybernetik, 13:55–80, **1973**. ~2k citations
- Wilson HR, Cowan JD. Evolution of the Wilson-Cowan equations. Biological Cybernetics, **2021** 12; 115(6):643-653.

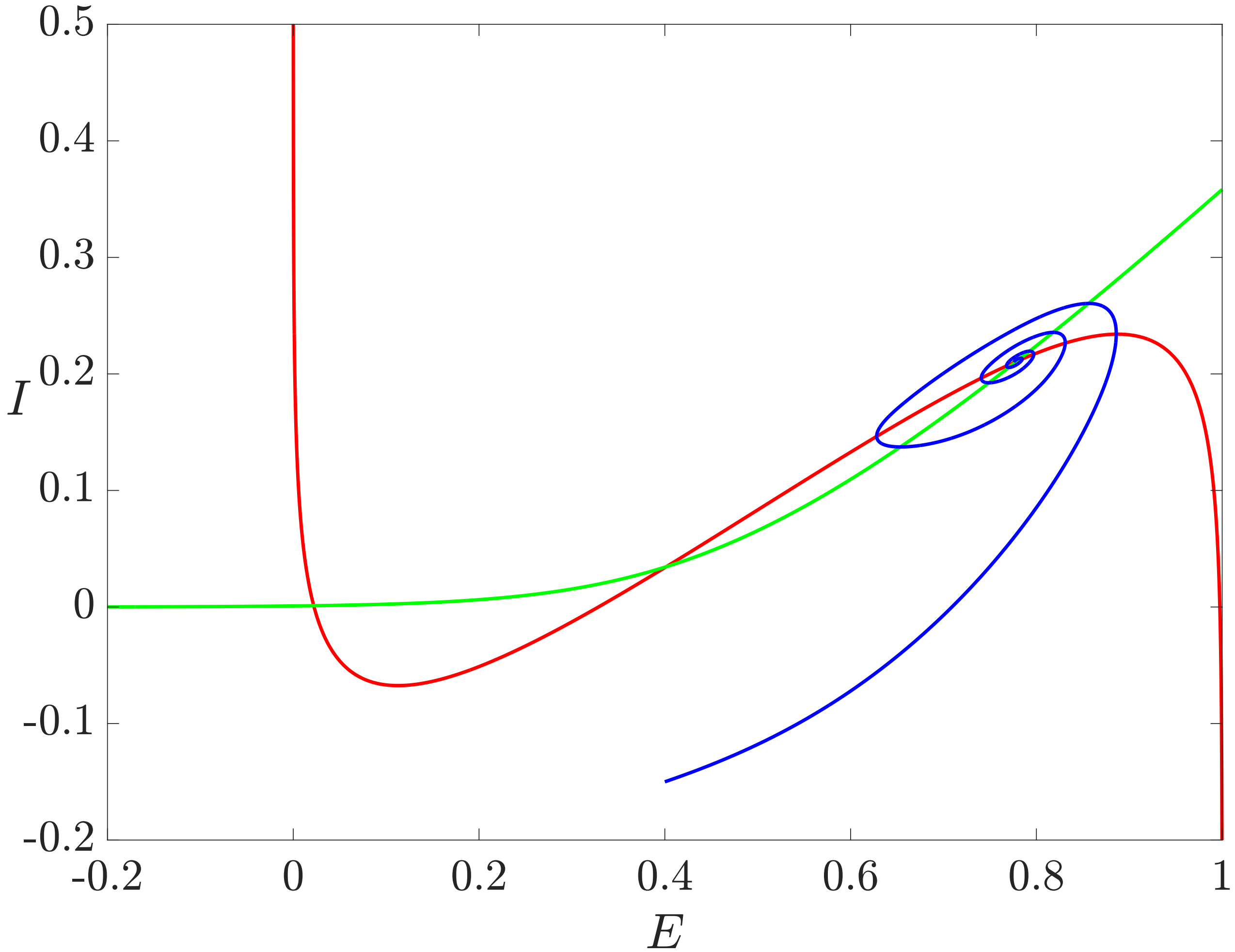
Applications (according to ChatGPT 3.5)

- **Neural Oscillations**
- Visual Perception
- Cortical Maps
- Working Memory
- Attention Mechanisms
- **Epilepsy**
- Neurological Disorders
- Learning and Plasticity
- Brain Network Dynamics
- Computational Neuroscience

Drug induced visual hallucinations (missed by ChatGPT)

Analysis of $Q\,g = f(Wg + \Theta)$ $\quad g = \begin{bmatrix} E \\ I \end{bmatrix}$

$$Q = \begin{bmatrix} \tau_E \frac{d}{dt} + 1 & 0 \\ 0 & \tau_I \frac{d}{dt} + 1 \end{bmatrix} \qquad W = \begin{bmatrix} W_{EE} & -W_{EI} \\ W_{IE} & -W_{II} \end{bmatrix} \qquad \Theta = \begin{bmatrix} \theta_E \\ \theta_I \end{bmatrix}$$

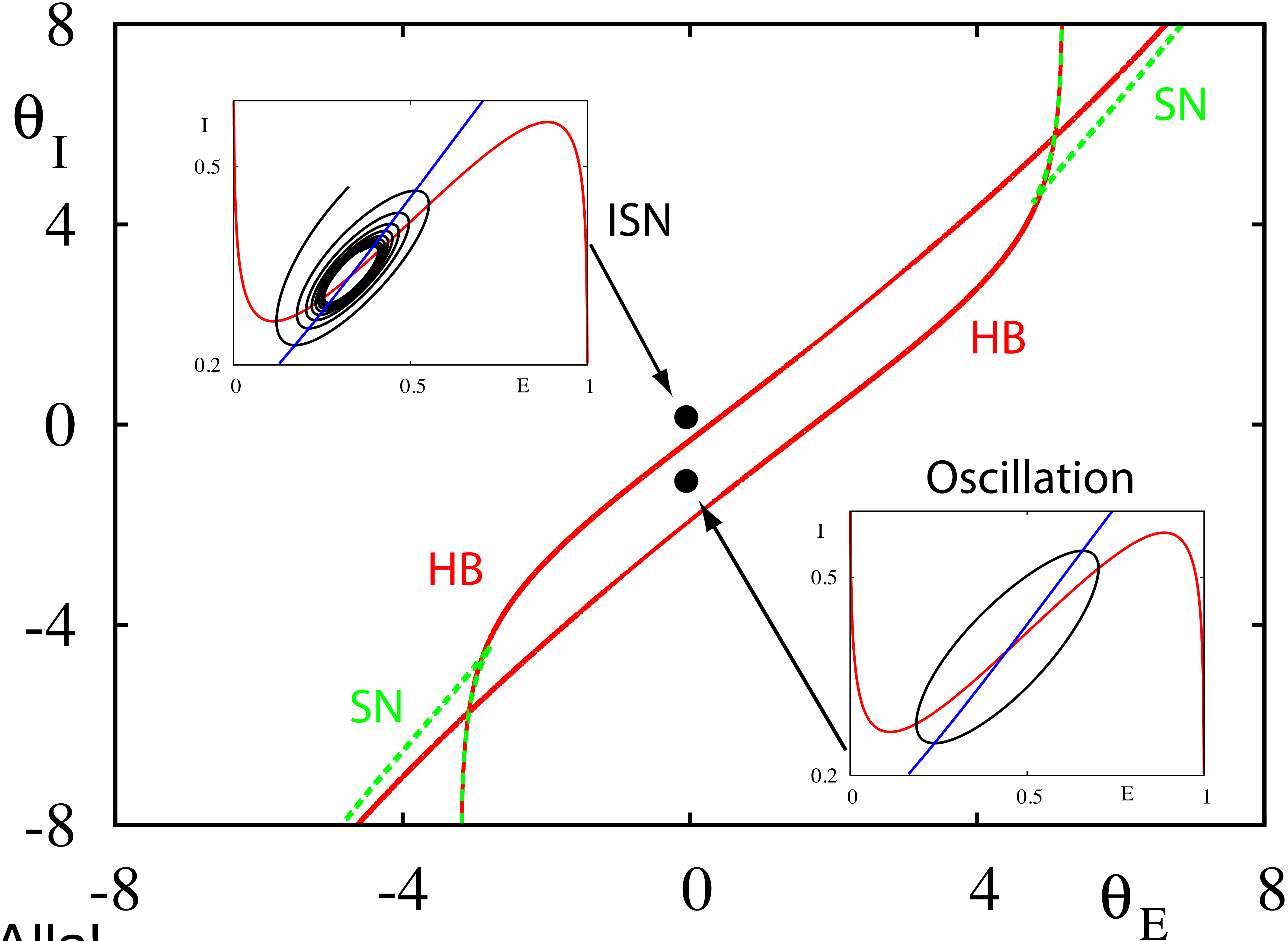



```
%draw the nullcines
[E,I]=meshgrid(linspace(-1,1,500),linspace(-1,1
,500));
contour(E,I,F(E,I,P),[0 0], 'r-',
'LineWidth',2)
hold on
contour(E,I,G(E,I,P),[0 0], 'g-',
'LineWidth',2)

%evolve the model in time
X0 = [0.4; -0.15];
tstart = 0;
dt = 0.01;
tend = 100;
[t, X] = ode45(@RHS, [tstart : dt : tend], X0,
[], P);
```

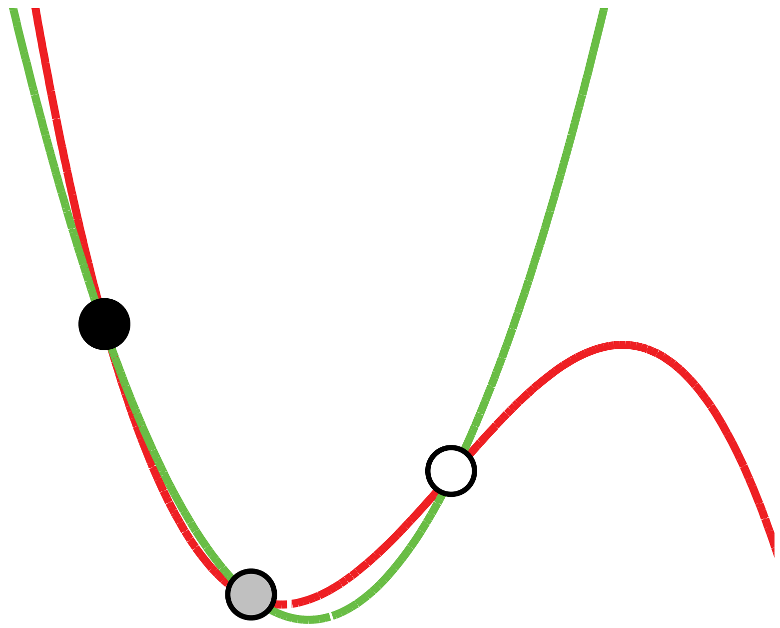
Phase-plane, linear stability, numerical bifurcation analysis, direct simulation, ...

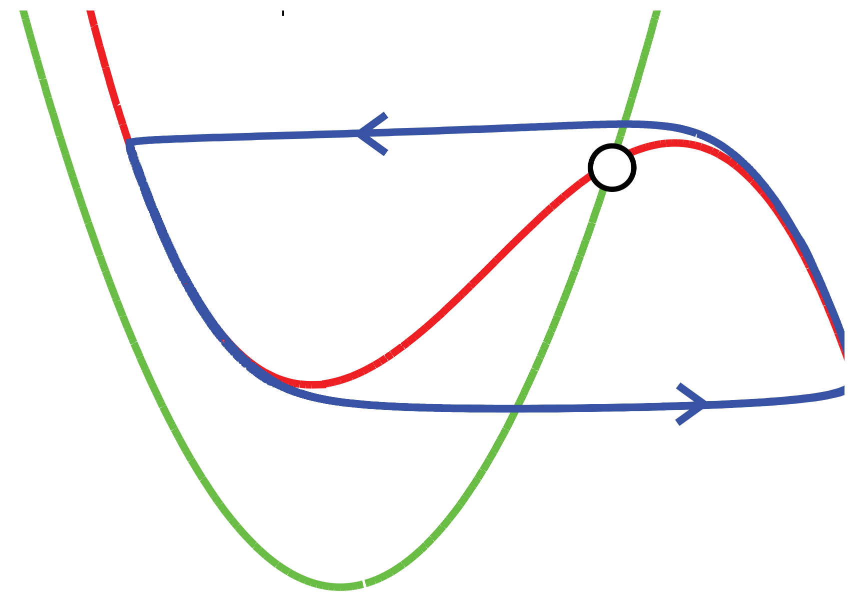
Two parameter bifurcation diagram



Ask Alla!

Pen & Paper calculations





Steady state

$$0 = F(\bar{x})$$

Linear stability: $x(t) = \bar{x} + u(t)$

$$\frac{d}{dt}u = DF(\bar{x})\,u$$

Saddle-node bifurcation

$$\det DF(\bar{x}) = 0$$

Hopf bifurcation

$$\det DF(\bar{x}) > 0$$

$$\mathrm{Tr}\, DF(\bar{x}) = 0$$

Ricatti equation: $f'(u) = \beta f(u)(1 - f(u))$

Steady state

$$\theta_E = f^{-1}(\bar{E}) - W_{EE}\bar{E} + W_{EI}\bar{I}$$

$$\theta_I = f^{-1}(\bar{I}) - W_{IE}\bar{E} + W_{II}\bar{I}$$

$$DF(\bar{E}, \bar{I}) =$$

$$\begin{bmatrix} (-1 + \beta W_{EE}\bar{E}(1 - \bar{E}))/\tau_E & -\beta W_{EI}\bar{E}(1 - \bar{E})/\tau_E \\ \beta W_{IE}\bar{I}(1 - \bar{I})/\tau_I & (-1 - \beta W_{II}\bar{I}(1 - \bar{I}))/\tau_I \end{bmatrix}$$

e.g., for Hopf, setting Tr=0 gives a quadratic in I that can be solved as

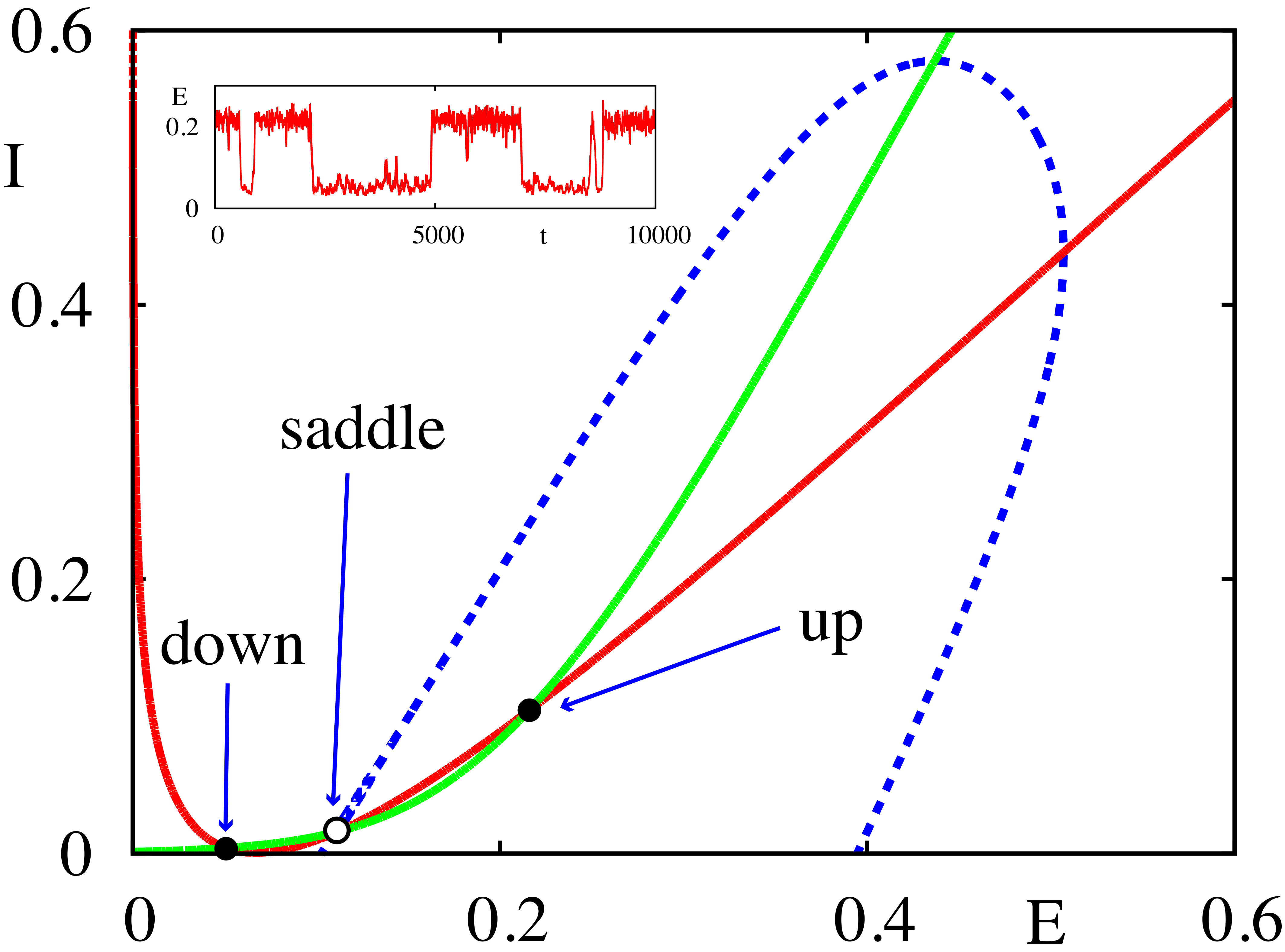
$$\bar{I} = \bar{I}(\bar{E})$$

Hence, a parametric eqn for the locus of Hopf bifurcations:

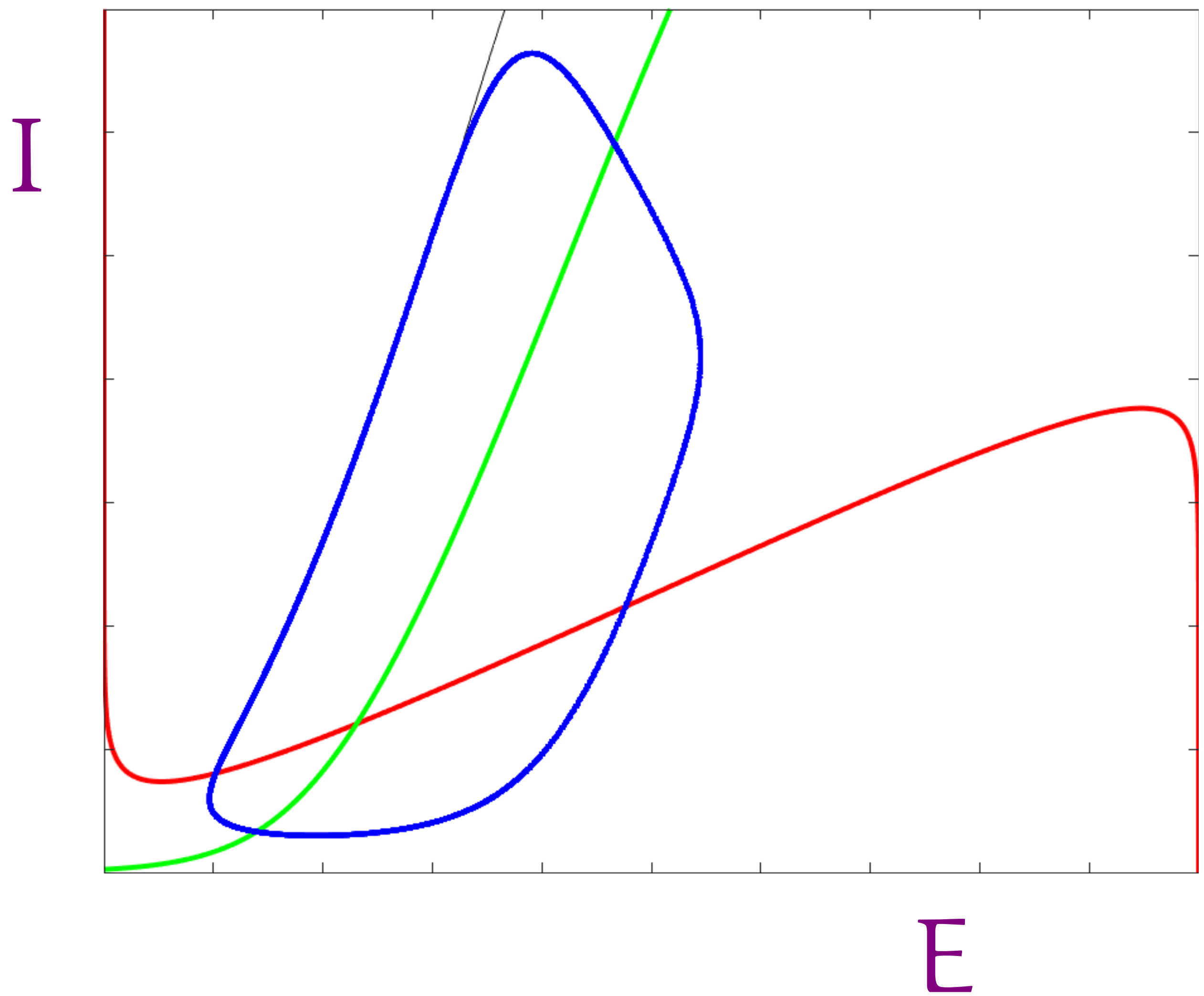
$$(\theta_E, \theta_I) = (\theta_E(\bar{I}), \theta_I(\bar{I}))$$

… and similarly for saddle-node.

Noise? … ask Benjamin!



Delay induced oscillations



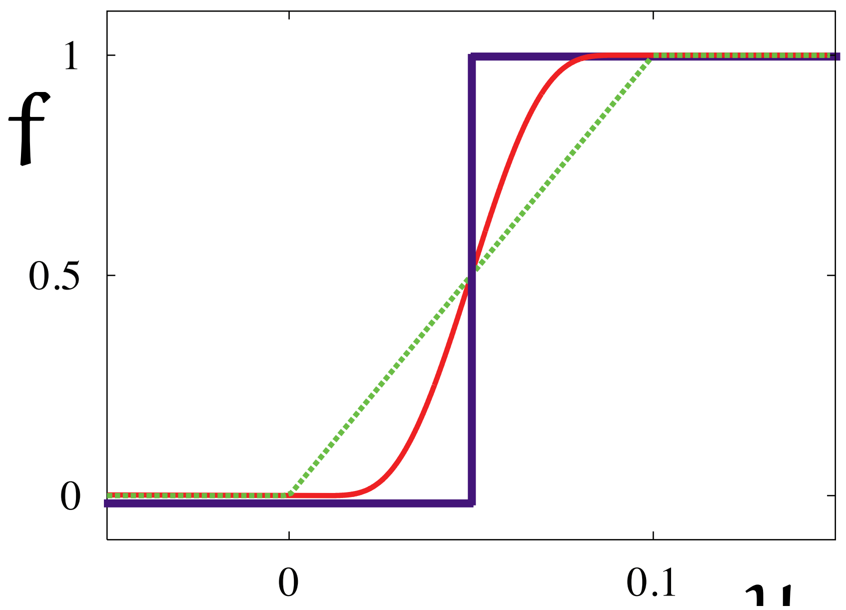
S Coombes and C R Laing 2009 Delays in activity based neural networks, Philosophical Transactions of the Royal Society A, Vol 367, 1117-1129

$$\tau_E \frac{d}{dt}E(t) = -E(t) + f\left(W_{EE}E(t-\tau_{EE}) - W_{EI}I(t-\tau_{EI}) + \theta_E\right)$$

$$\tau_I \frac{d}{dt}I(t) = -I(t) + f\left(W_{IE}E(t-\tau_{IE}) - W_{II}I(t-\tau_{II}) + \theta_I\right)$$

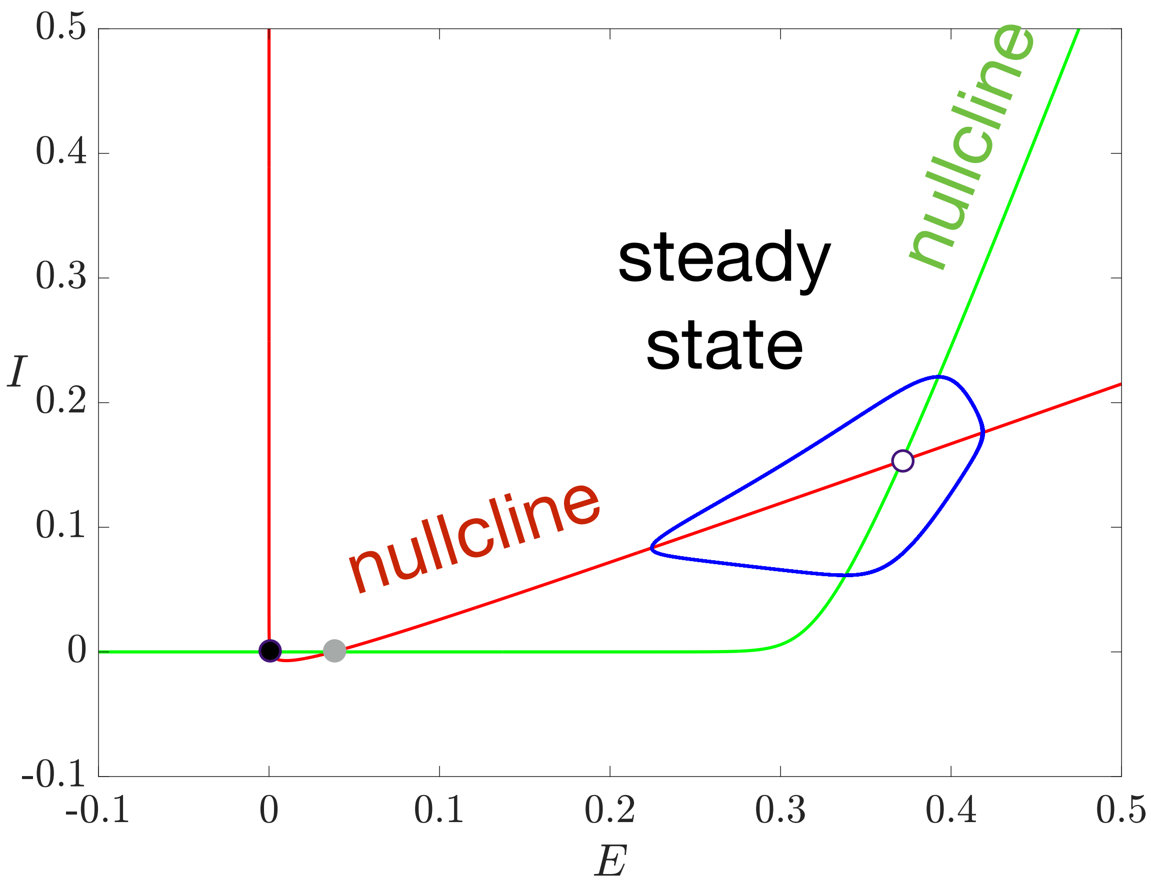
… can still do linear stability analysis of a steady state + numerical bifurcation ~ DDE-BIFTOOL

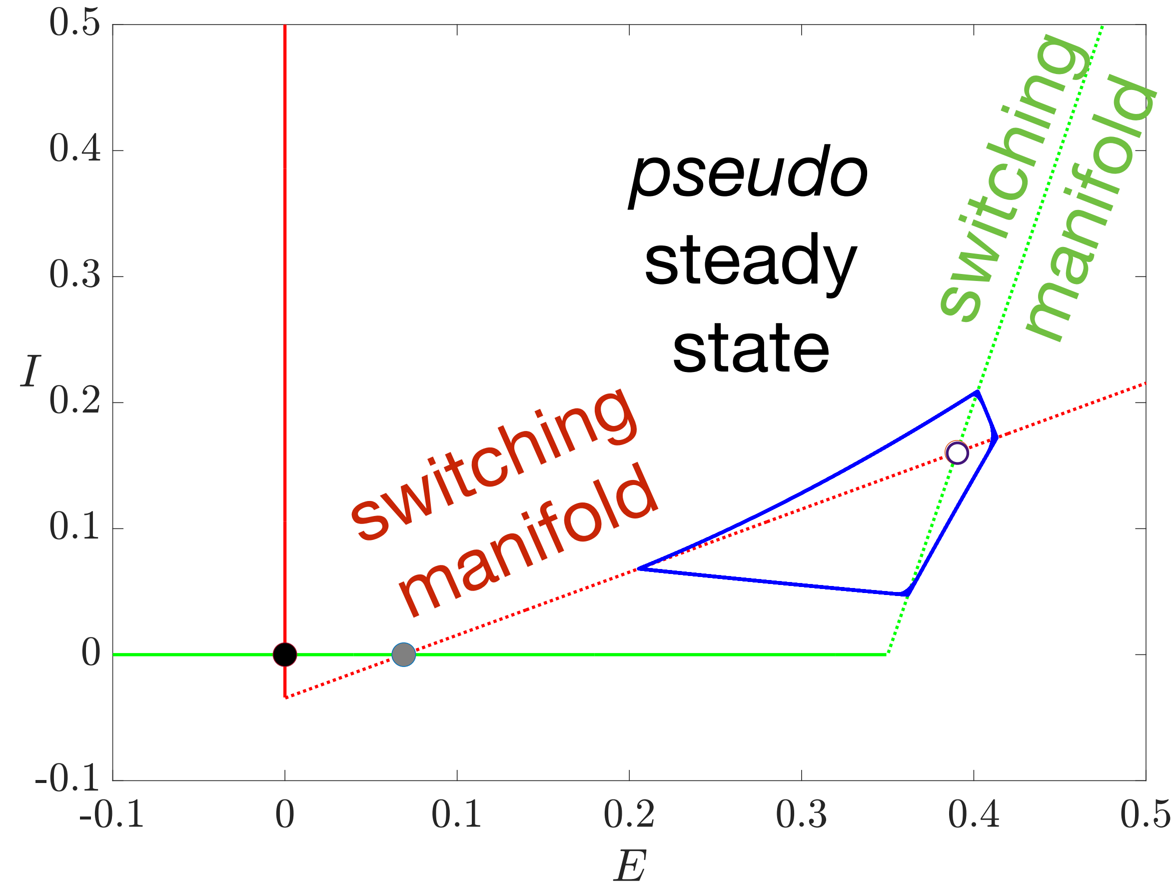
Pen & Paper for periodic orbits?



Heaviside limit:

$$H(u) = \lim_{\beta \to \infty} f(u)$$





Harris J and Ermentrout B 2015 Bifurcations in the Wilson-Cowan Equations with Nonsmooth Firing Rate, SIAM J Appl Dyn Syst 14 43–72

Filippov convention

Switching manifolds described with an indicator function: e.g.,

$$h(E, I) = 0; \qquad h(E, I) = W_{EE}E - W_{EI}I + \theta_E$$

Convex differential inclusion: e.g.,

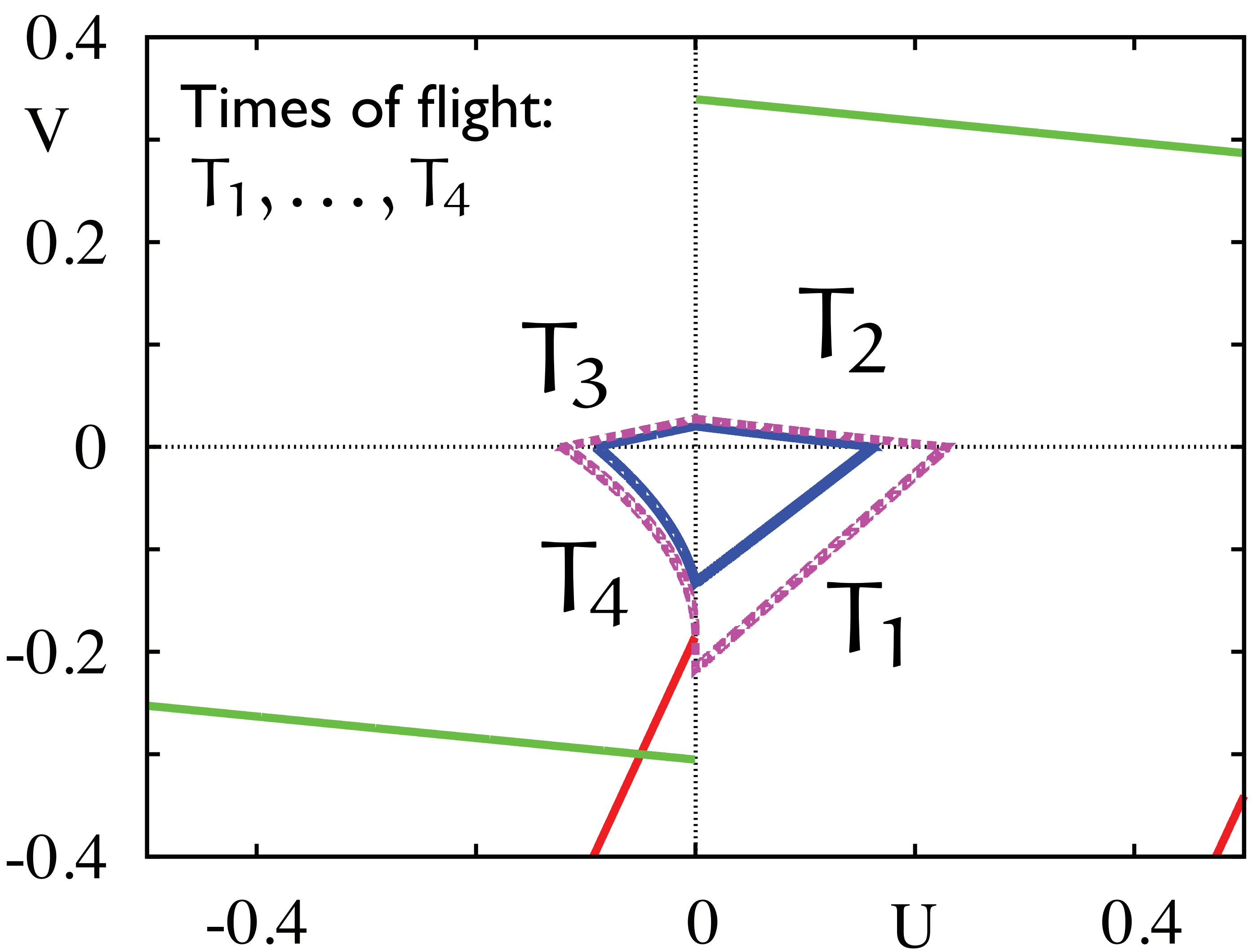
$$\frac{d}{dt}\begin{bmatrix} E \\ I \end{bmatrix} \in F(E, I) = \begin{cases} F_+(E, I) & \text{on one side} \\ \overline{\text{co}}(\{F_+, F_-\}, \kappa) & \text{on the switch} \\ F_-(E, I) & \text{on other side} \end{cases}$$

$$\overline{\text{co}}(\{f, g\}, \kappa) = \kappa f + (1 - \kappa)g, \qquad \kappa \in [0, 1]$$

Choose κ so that $\dot{h} = \nabla h \cdot F|_{\text{on switch}} = 0$

New variables (linear transformation) $A = -WJW^{-1}$

$$\frac{d}{dt}\begin{bmatrix} U \\ V \end{bmatrix} = A \begin{bmatrix} U - \theta_E \\ V - \theta_I \end{bmatrix} + WJ \begin{bmatrix} H(U) \\ H(V) \end{bmatrix} \qquad J = \begin{bmatrix} \tau_E^{-1} & 0 \\ 0 & \tau_I^{-1} \end{bmatrix}$$



Switching manifolds

$$h_1(U, V) = U = 0$$

$$h_2(U, V) = V = 0$$

$$\begin{bmatrix} U(t) \\ V(t) \end{bmatrix} = e^{At} \begin{bmatrix} U(0) \\ V(0) \end{bmatrix} + (I_2 - e^{At}) \left[\begin{bmatrix} \theta_E \\ \theta_I \end{bmatrix} - A^{-1} WJ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right]$$

Use matrix exponentials to patch a trajectory

Period: $\Delta = T_1 + T_2 + T_3 + T_4$

… or use Fourier series (if averse to expm)

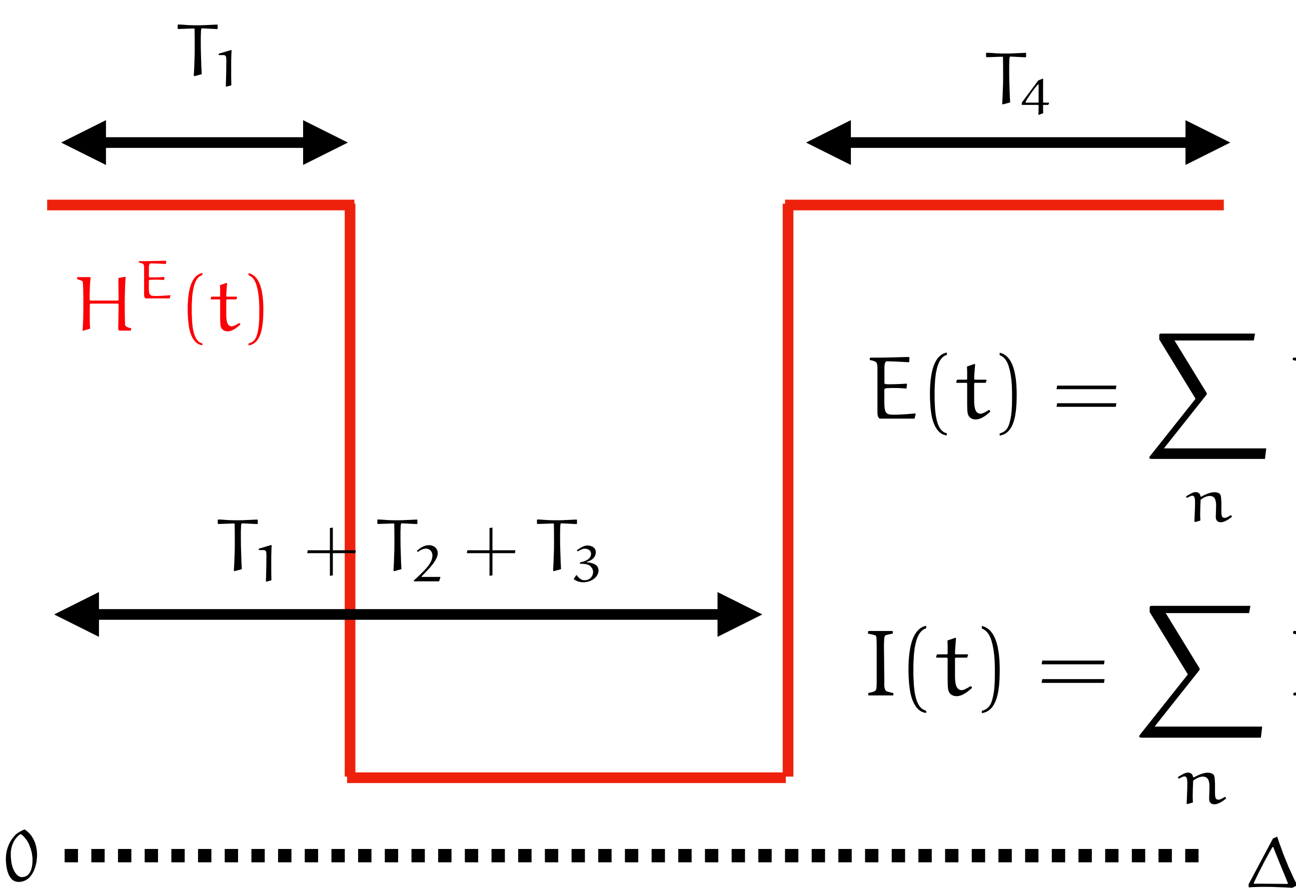
$$\tau_E \frac{d}{dt} E = -E + H^E$$
$$\tau_I \frac{d}{dt} I = -I + H^I$$

$$H^E(t) = H\left(W_{EE}E(t) - W_{EI}I(t) + \theta_E\right) = \sum_n H_n^E e^{2\pi i n t/\Delta}$$

$$H^I(t) = H\left(W_{IE}E(t) - W_{II}I(t) + \theta_I\right) = \sum_n H_n^I e^{2\pi i n t/\Delta}$$

$$H_n^I = \frac{1}{2\pi i n}\left[1 - e^{-2\pi i n (T_1+T_2)/\Delta}\right]$$

$$H_n^E = \frac{1}{2\pi i n}\left[1 - e^{-2\pi i n T_1/\Delta} + e^{-2\pi i n (T_1+T_2+T_3)/\Delta}\left(1 - e^{-2\pi i n T_4/\Delta}\right)\right]$$



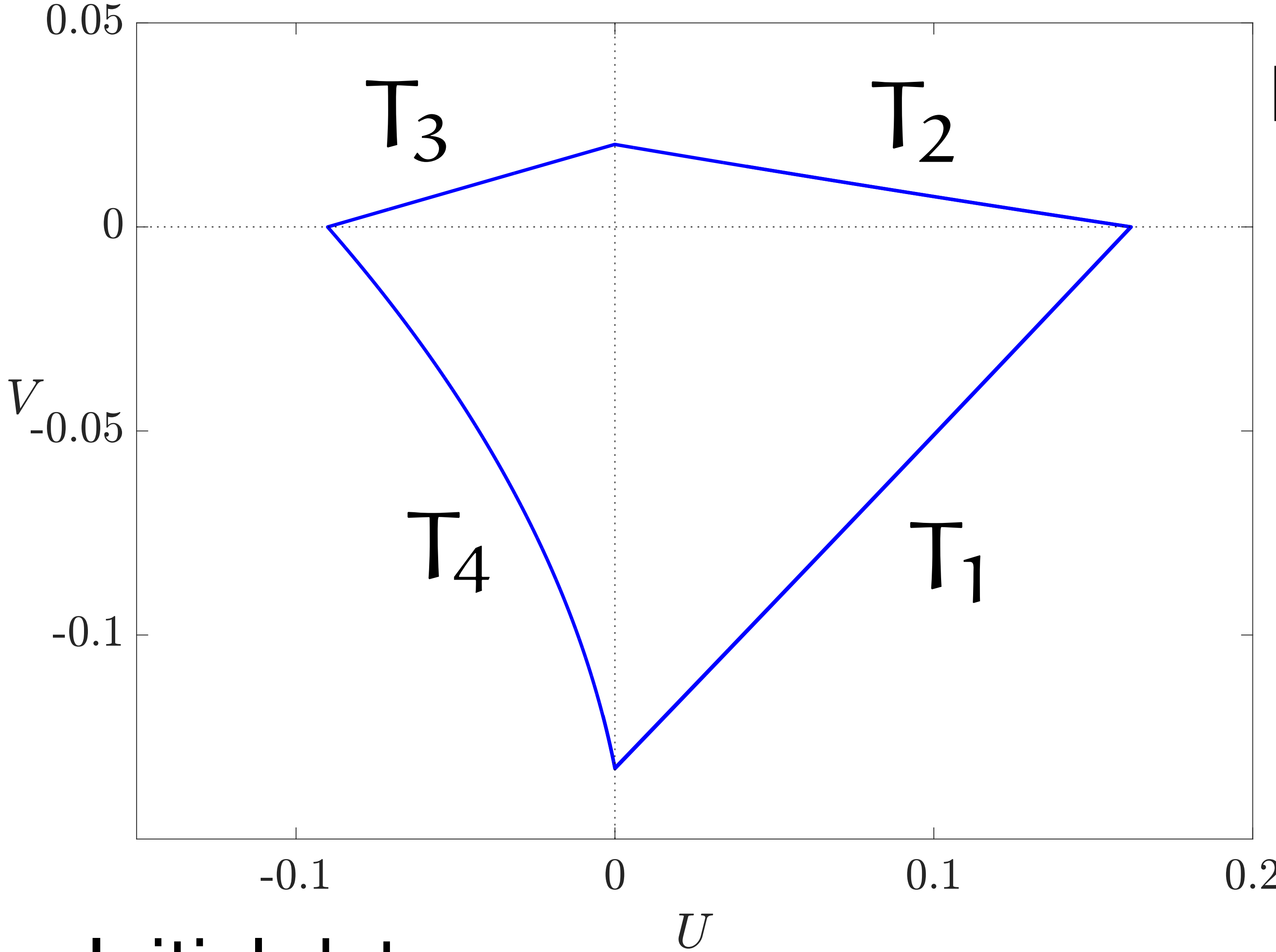
$$E(t) = \sum_n E_n e^{2\pi i n t/\Delta}$$

$$I(t) = \sum_n I_n e^{2\pi i n t/\Delta}$$

$$E_n = \frac{H_n^E}{1 + \frac{2\pi i n}{\Delta}\tau_E}$$

$$I_n = \frac{H_n^I}{1 + \frac{2\pi i n}{\Delta}\tau_I}$$

Self consistent periodic orbit



Four switching conditions:

$$V(T_1) = 0$$
$$U(T_2) = 0$$
$$V(T_3) = 0$$
$$U(T_4) = 0$$

Initial data:

$(0, V_0)$

Five unknowns:

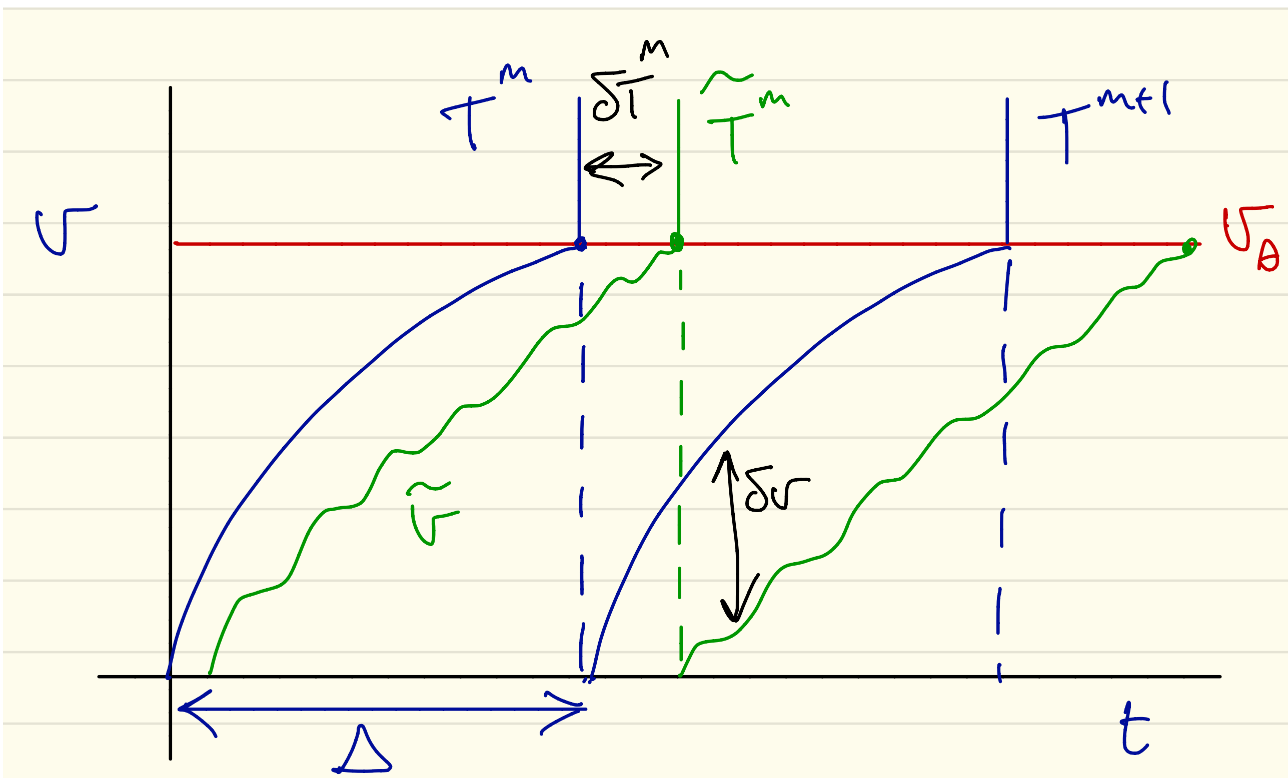
$(V_0, T_1, T_2, T_3, T_4)$

One periodicity condition:

$$V_0 = V(\Delta)$$

Linear stability (non-smooth)

Propagation of perturbations through switching manifolds - periodic IF example



$$\dot{v} = -v + I$$

Indicator function : $h(v) = v - v_\theta$

$$h(v(T^m)) = 0 = h(\tilde{v}(\tilde{T}^m))$$

Taylor expand :

$$\delta T^m = -\left.\frac{\delta v(t)}{\dot{v}(t)}\right|_{t=m\Delta^-}$$

$$\delta v(m\Delta + \delta T^m) \simeq \delta v(m\Delta) + [\dot{\tilde{v}}(m\Delta^+) - \dot{v}(m\Delta^+)]\delta T^m$$

$$\simeq \left(\frac{\dot{v}(m\Delta^+)}{\dot{v}(m\Delta^-)}\right)\delta v(m\Delta)$$

Saltation

Makes sense … and generalises

Floquet multiplier is unity :

Linearised flow

$$\delta v(\Delta) = \left(\frac{\dot{v}(m\Delta^+)}{\dot{v}(m\Delta^-)}\right) e^{-\Delta} \delta v(0) = \frac{I}{-v_\theta + I} e^{-\Delta} = 1.\delta v(0)$$

Saltation

In general

$$\delta z^+ = K(T)\delta z^-$$

Reset : $z \to g(z)$

$$K(T) = Dg(z(T^-)) + \frac{[\dot{z}(T^+) - Dg(z(T^-))\dot{z}(T^-)][\nabla_z h(z(T^-))]^\top}{\nabla_z h(z(T^-)) \cdot \dot{z}(T^-)}$$

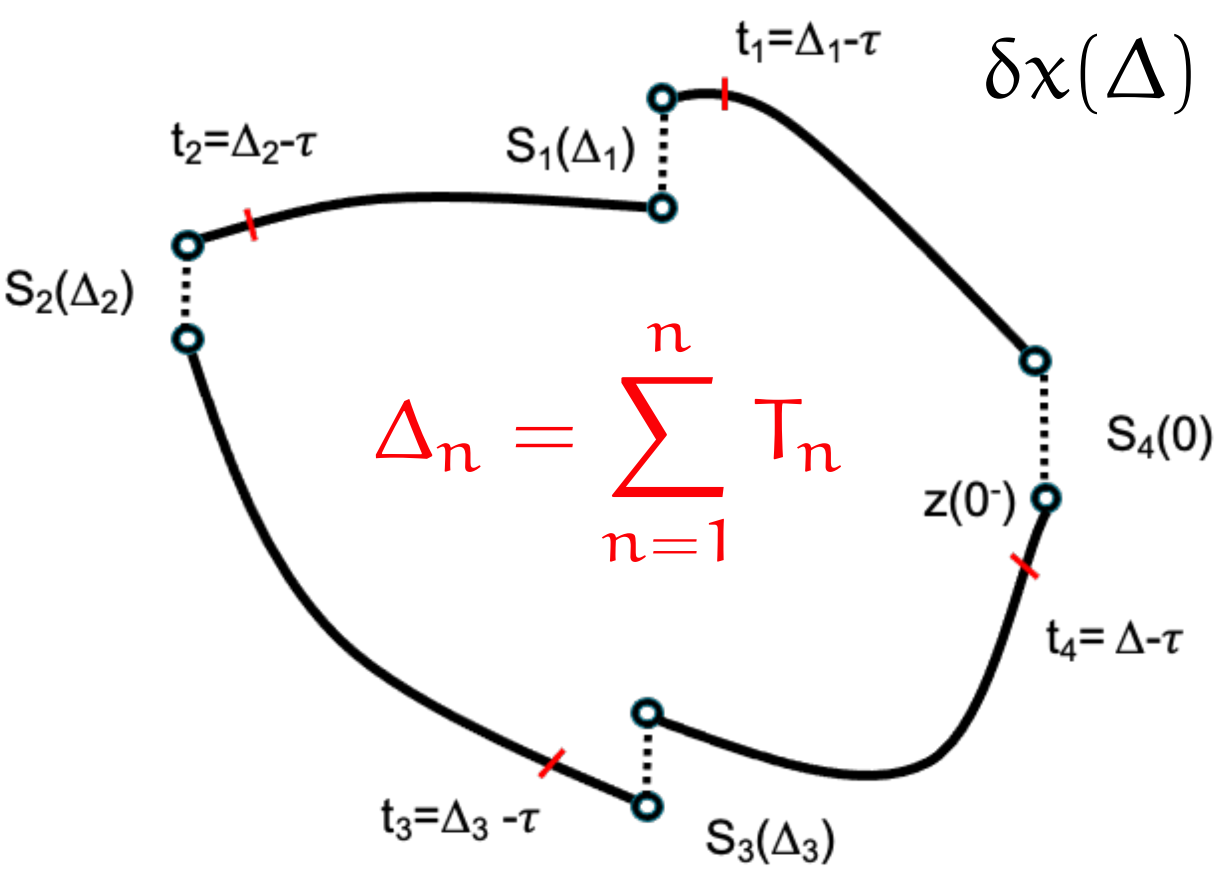
Saltation *matrix*

Nicks, Chambon, Coombes 2018 Clusters in nonsmooth oscillator networks PRE 97, 032213

Nonsmooth Floquet theory

Away from switching $\dot{\delta x} = A\delta x$ (matrix exp solu)

Saltation (jumps) at switching $\delta x^+ = K(T)\delta x^-$



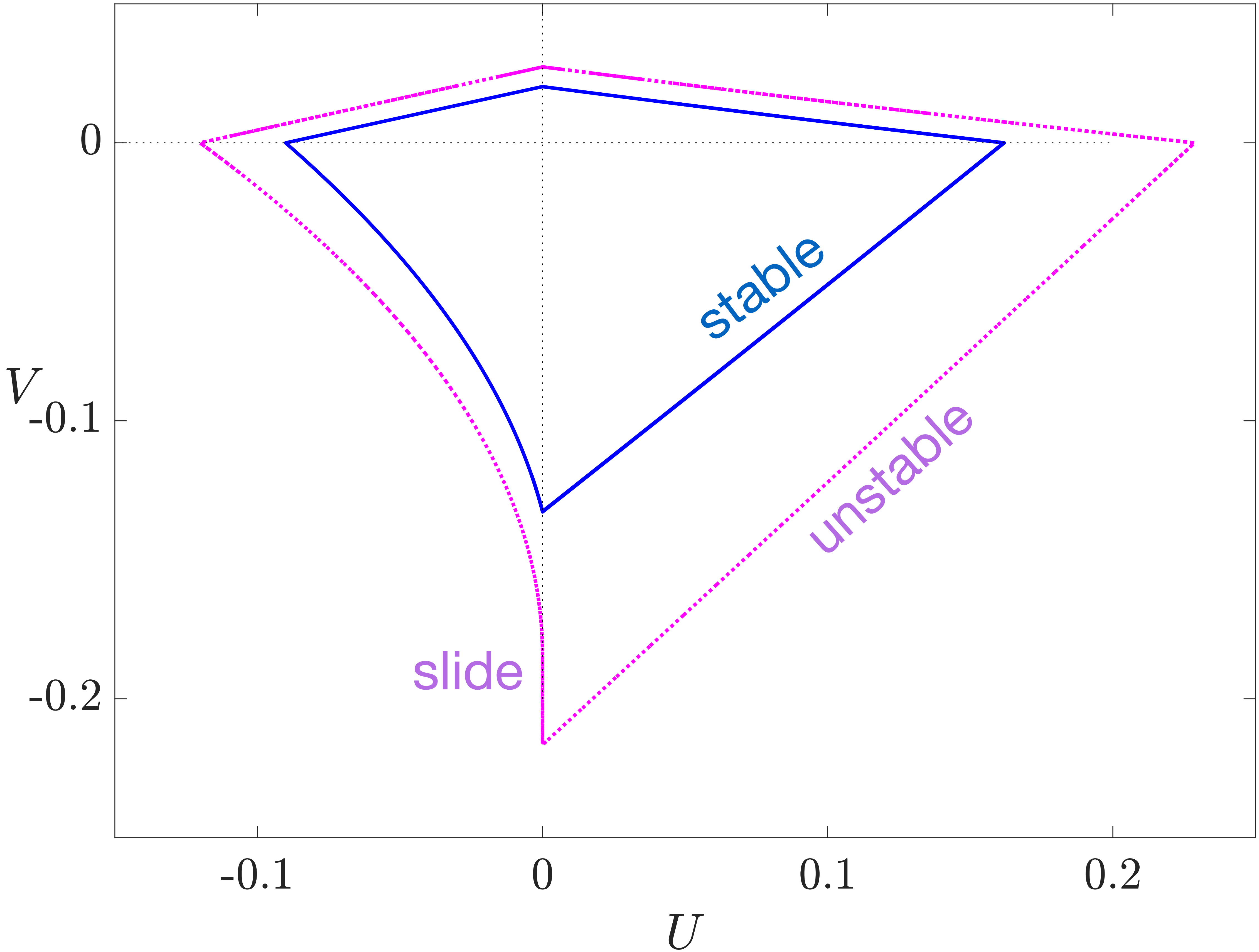
$$\Delta_n = \sum_{n=1}^{n} T_n$$

$$\delta x(\Delta) = K_4 e^{AT_4} K_3 e^{AT_3} K_2 e^{AT_2} K_1 e^{AT_1} \delta x(0)$$
$$\equiv e^{\sigma\Delta}\delta x(0)$$

cf

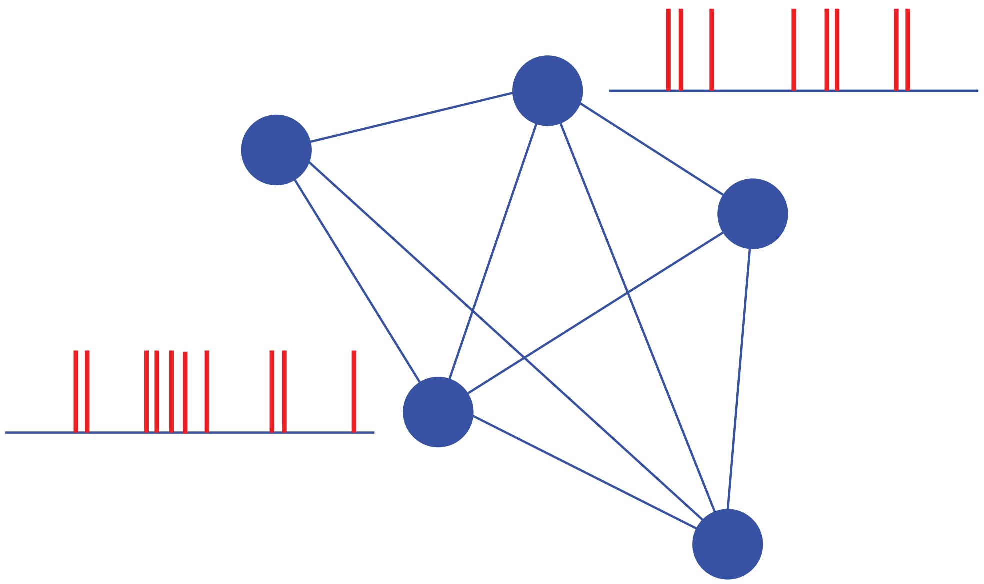
$$\sigma_{\text{smooth}} = \frac{1}{\Delta}\int_0^{\Delta} \text{Tr DF}(\text{orbit}(t))dt$$

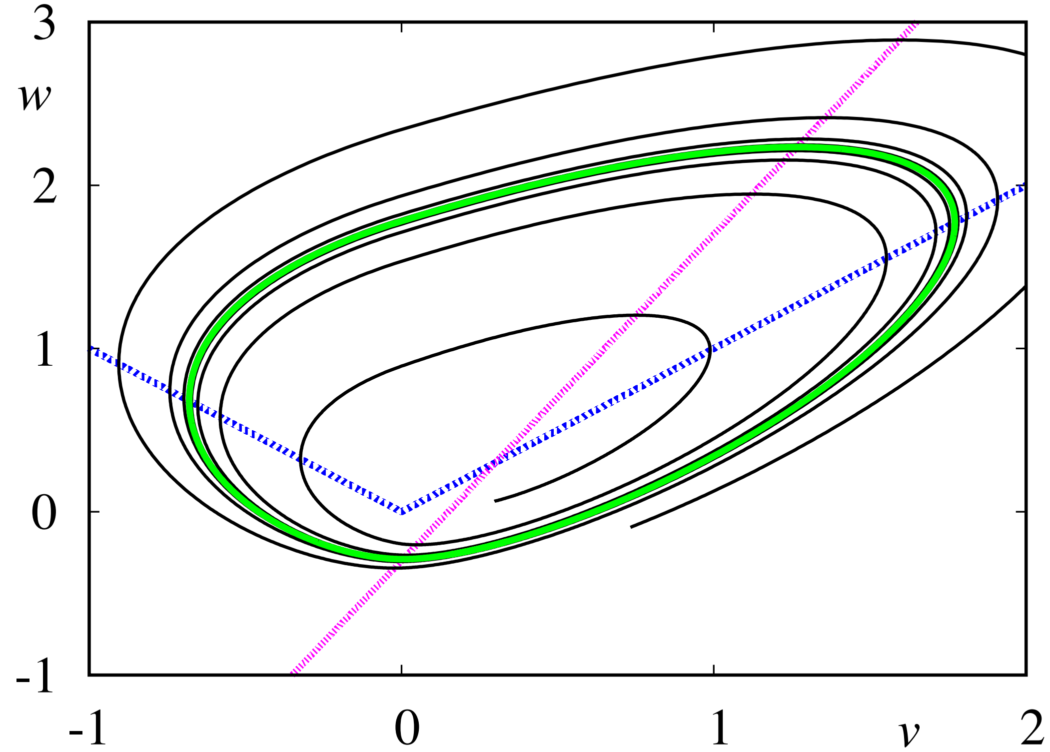
$$\sigma = -\left(\frac{1}{\tau_E} + \frac{1}{\tau_I}\right) + \frac{1}{\Delta}\log \frac{\dot{V}(\Delta_1^+)}{\dot{V}(\Delta_1^-)}\frac{\dot{U}(\Delta_2^+)}{\dot{U}(\Delta_2^-)}\frac{\dot{V}(\Delta_3^+)}{\dot{V}(\Delta_3^-)}\frac{\dot{U}(\Delta_4^+)}{\dot{U}(\Delta_4^-)}$$


0
-0.1
-0.2
V
-0.1
0
0.1
0.2
U
stable
unstable
slide

Piece-wise linear systems

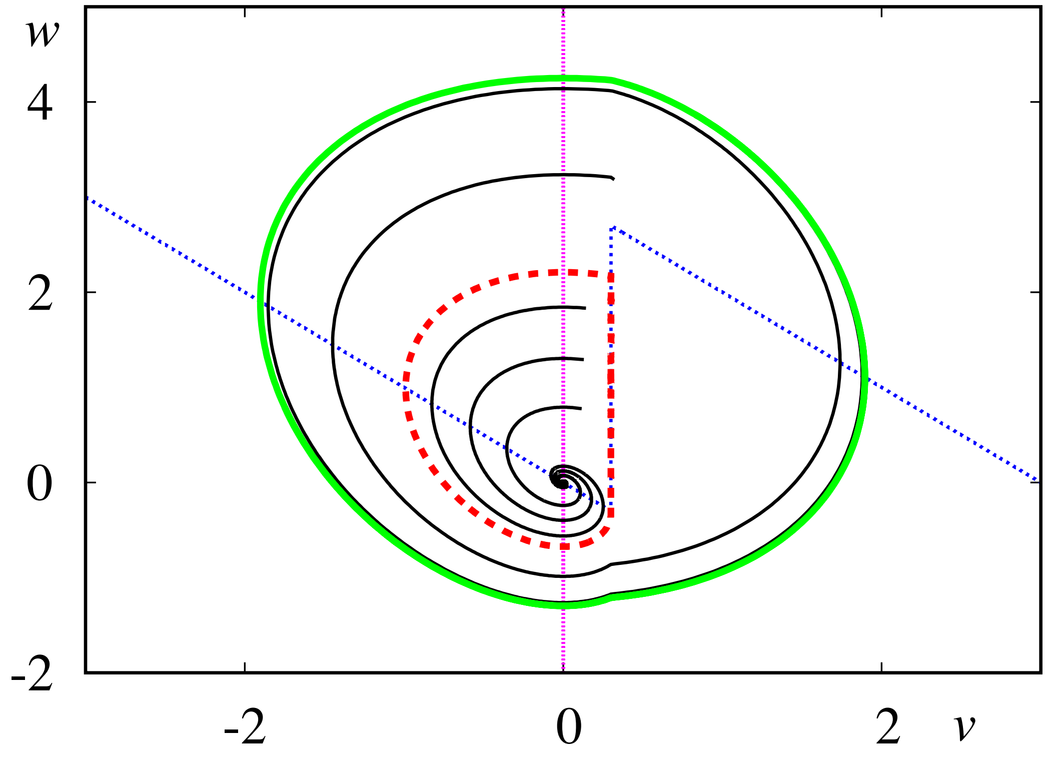
Canards, folded nodes and mixed-mode oscillations in piecewise-linear slow-fast systems M. Desroches, A. Guillamon, E. Ponce, R. Prohens, S. Rodrigues and A. E. Teruel. SIAM Review, 58(4), 653–691, 2016.





non-smooth Andronov -Hopf

saddle-node of limit cycles



A Tonnelier. The McKean's caricature of the FitzHugh-Nagumo model I. the space-clamped system. SIAM Journal on Applied Mathematics, 63:459–484, 2002.

Next: networks