Intermittent Interactions: Dynamics of Oscillator Networks with Dead Zones

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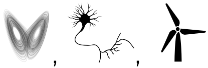
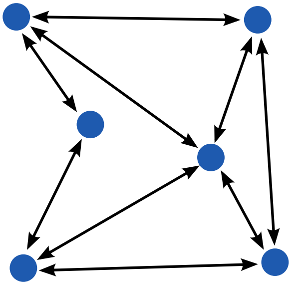
Threshold Networks — July 23, 2019

Network Dynamical Systems

Network of **dynamical units**: $\bullet \in \{$  $, \ldots \}$



Network structure (topology):

Who interacts with whom?

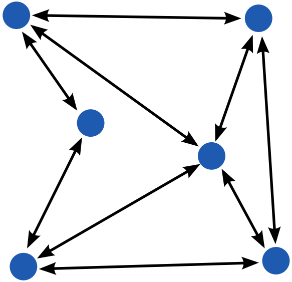
Network interaction:

How does one oscillator influence the other?

Q: How do **structure** and **interactions** shape the **network dynamics**?

Network Dynamical Systems

Network of **oscillatory units**: ● $\in \{$ , , $\ldots \}$



Network structure (topology):

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Network interaction:

How does one oscillator influence the other?

Q: How do network **structure**, **interactions**, and **dynamics** interact?

Oscillator Networks

Weak coupling approximation: Phase oscillators, $\theta_k \in \mathbf{T} = \mathbb{R}/2\pi\mathbb{Z}$ with

$$\dot{\theta}_k = \omega + \sum_{j=1}^{N} g(\theta_j - \theta_k)$$

Network properties

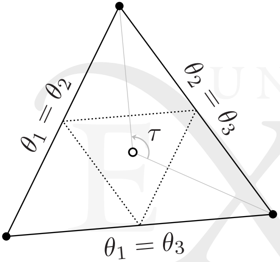
Network structure (topology):

all-to-all, identical

Network interaction:

2π-periodic coupling function $g : \mathbf{T} \to \mathbb{R}$

Three oscillators



Types of Coupling Functions

Single harmonic: Kuramoto–Sakaguchi coupling

$$g(\phi) = \sin(\phi + \alpha)$$

Multiple harmonics: Daido and co.

$$g(\phi) = \sum_{h=1}^{m} A_h \sin(\phi + \alpha_h)$$

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Coupling function with dead zones

Definition

A **dead zone** is a (maximal) open interval $U \subset \mathbf{T}$ such that $g(U) \equiv 0$.
Let ☠(g) denote the union of all dead zones of g. ($\mathbf{T} \smallsetminus$ ☠ live zone.)

Effective Interactions

Definition

The **effective coupling graph** $\mathcal{G}_g(\theta) = \mathcal{G}(\theta)$ for N fully symmetric phase oscillators with coupling function g at $\theta \in \mathbf{T}^N$ is the graph on N vertices with edges

$$E(\mathcal{G}_g(\theta)) = \left\{ (j, k) \;\middle|\; \theta_j - \theta_k \notin \skull(g) \right\}.$$

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Properties

- The symmetry action on $\mathbf{T}^N$ permuting the phases yields a symmetry action on the set of effective coupling graphs.
- The effective coupling graph inherits the symmetries of θ:

$$\theta \in \{\theta_1 = \cdots = \theta_N\} \quad \Longrightarrow \quad \mathcal{G}(\theta) \in \{\text{full graph}, \text{empty graph}\}$$

Three Questions

(1) Realizing effective coupling graphs

Theorem

Let θ be a generic point on $\mathbf{T}^N$ and H a graph on N vertices. Then there exists a coupling function g such that $\mathcal{G}(\theta) = H$.

(2) Stably realizing effective coupling graphs

Theorem

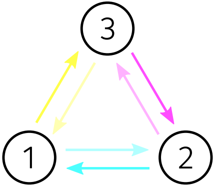
Let θ be a generic point on $\mathbf{T}^N$ and H a graph on N vertices. Under some additional assumption on H there exists a coupling function g such that $\mathcal{G}(\theta) = H$ and θ is a stable (relative) equilibrium of the system.

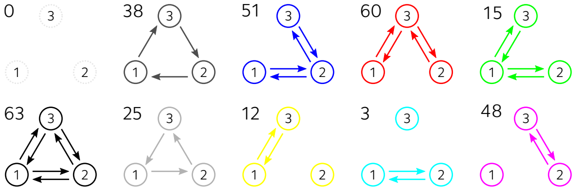
(3) Interplay of structure and dynamics

...

Awesome, Coloring Graphs!

Effective coupling graphs for $N = 3$ oscillators





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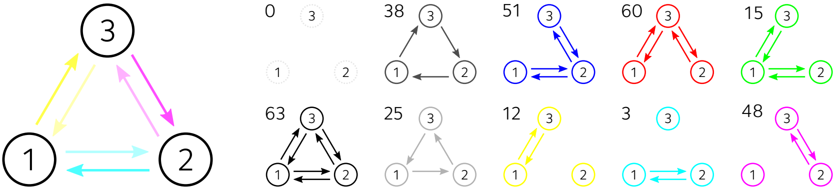
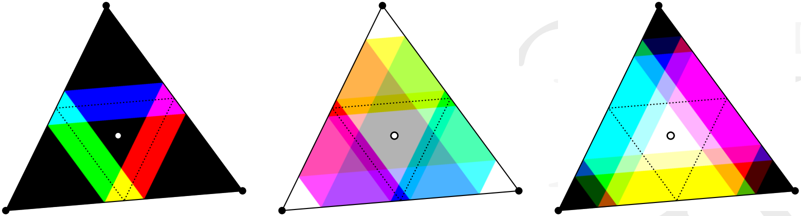
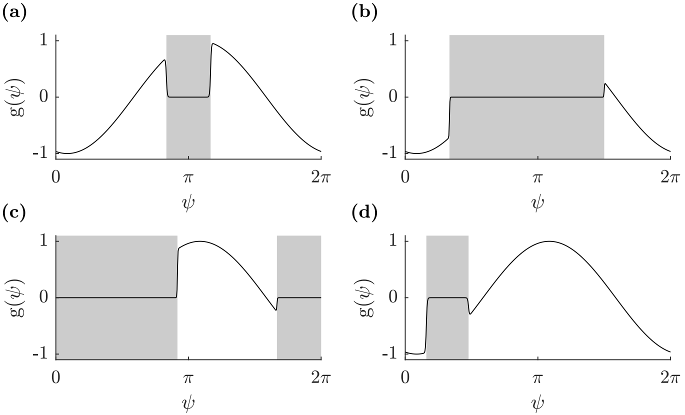


Phase space

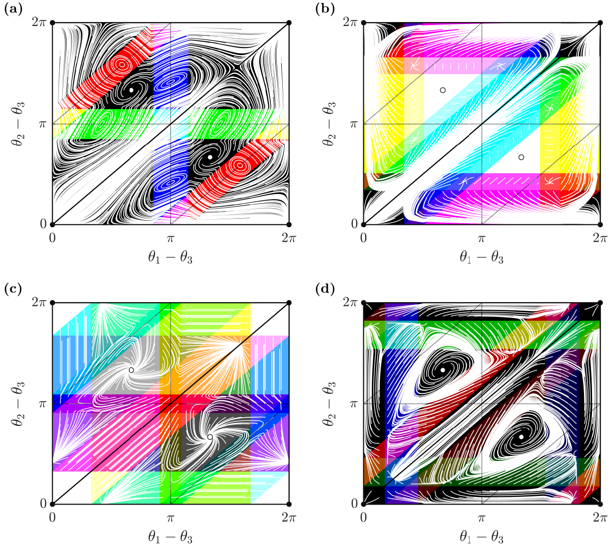


Almost Kuramoto–Sakaguchi Coupling

Kuramoto–Sakaguchi with one dead zone



More Colorful Pictures!



Properties of Asynchronous Networks

Local clocks. No global network clock is assumed and nodes may evolve independent of each other.

Variable network topology. Changes in connection structure may depend on the state of the system or be given by a stochastic process.

Event driven dynamics. Synchronization events associated with stopping or waiting states of nodes.

Nonsmooth dynamics. Dynamics is only piecewise smooth and there may be a mix of continuous and discrete dynamics.

How to formalize the notion of an asynchronous network?

CB and M J Field (2017), Nonlinearity, 30(2), 558–594.

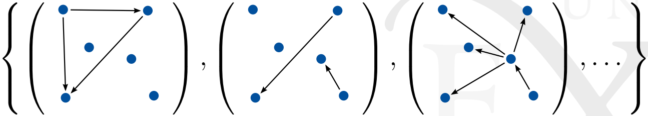
A Model of an Asynchronous Network

Definition

- A **network** $\mathcal{N}$ has **nodes**, $N_1, \ldots, N_k$ with states in manifolds M_j. Set $\mathbf{M} = \prod M_j$ is the **network phase space**. Foliations $\mathbb{L}_j$ of M_j describe **constraints**. Assume a **constraining node** N_0 in $\mathcal{N}$ (no phase space).

- A graph α (restrictions apply) determines a **connection structure** on $\mathcal{N}$ ($\emptyset$ denotes no connections). A collection $\mathcal{A} = \{\alpha, \beta, \ldots\}$ is a **generalized connection structure**.

$\mathcal{A} =$



A Model of an Asynchronous Network

Definition (Continued)

- For each $\alpha \in \mathcal{A}$ take a **network vector field** $\mathbf{f}^\alpha = (f_1^\alpha, \ldots, f_k^\alpha) \in \mathcal{F}$ on $\mathbf{M}$ such that
  - f_j^α depends nontrivially on $\mathbf{x}_\ell \in M_\ell$ if and only if $N_\ell \to N_j$,
  - f_j^α is tangent to the leaves of $\mathbb{L}_j$ if $N_0 \to N_j$.
- The **event map** $\mathcal{E} : \mathbf{M} \to \mathcal{A}$ defines events. Event sets $E_\alpha = \mathcal{E}^{-1}(\alpha)$ partition $\mathbf{M}$.

An **asynchronous network** is a tuple $\mathfrak{N} = (\mathcal{N}, \mathcal{A}, \mathcal{F}, \mathcal{E})$.

CB and M J Field (2017), Nonlinearity, 30(2), 558–594.

Dynamics of an Asynchronous Network?

An asynchronous network defines a *state dependent* dynamical system through

$$\mathbf{F}(\mathbf{X}) = \mathbf{f}^{\mathcal{E}(\mathbf{X})}(\mathbf{X})$$

Solution Curves

Subject to some assumptions, an asynchronous network $\mathfrak{N}$ gives rise to a well-defined semiflow $\Phi_{\mathfrak{N}}$ on $\mathbf{M}$.

1. Let $\mathbf{X}^0$ denote the initial conditon.
2. Given $\mathbf{X}^k \in E_\alpha = \mathcal{E}^{-1}(\alpha)$ find minimal $0 < T \leq \infty$ such that $\Phi_\alpha(\mathbf{X}, t) \in E_\beta$ for $t < T$ and $\Phi_\alpha(\mathbf{X}, T) \in E_\beta$, $\beta \neq \alpha$.
3. Set $\mathbf{X}^{k+1} = \Phi_\alpha(\mathbf{X}, T)$.

The resulting semiflow $\Phi_{\mathfrak{N}}$, continuous in time t but not necessarily in $\mathbf{X}$.

(3) Interplay of structure and dynamics

Theorem

A phase oscillator network with coupling function g such that ☠$(g) \neq \emptyset$ naturally defines an asynchronous network.

$$
\begin{aligned}
\text{node state} &= \text{phase in } \mathbf{T} \\
\text{connection structure} &= \text{effective coupling graph} \\
\text{event map } \mathcal{E}(\mathbf{X}) &= \mathcal{G}_g(\theta) \\
&\vdots
\end{aligned}
$$

Some More Questions

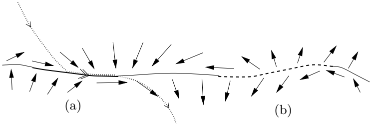
Questions about Asynchronous Networks

- Are there good conditions on the event sets that ensure **existence of trajectories**?
- Asynchronous networks are **related to Filippov systems** and other piecewise smooth systems.
- Products, when does an asynchronous network **decompose** into a product?

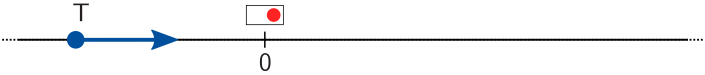
CB and M J Field (2017), Nonlinearity, 30(2), 558–594.

Dynamics of an Asynchronous Network!

The dynamics can be qualitatively different than for example the dynamics given by a Filippov system.

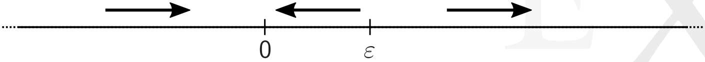


Asynchronous Network



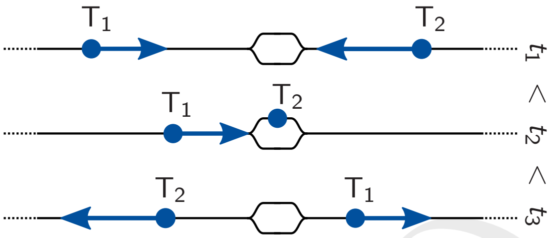
Network $\mathcal{N} = \{N_0, N_1\}$ with $M_1 = \mathbb{R}$, $\mathcal{A} = \{\emptyset, \alpha = N_0 \to N_1\}$. Vector fields $\mathbf{f}^{\emptyset}(\mathbf{X}) = v > 0$, $\mathbf{f}^{\alpha}(\mathbf{X}) = 0$ and $\mathcal{F} = \left\{\mathbf{f}^{\emptyset}, \mathbf{f}^{\alpha}\right\}$. Event map $\mathcal{E}(\mathbf{X}) = \alpha$ if $\mathbf{X} = 0, \mathcal{E}(\mathbf{X}) = \emptyset$ otherwise.

Filippov Approximation on $[-\infty, 0]$



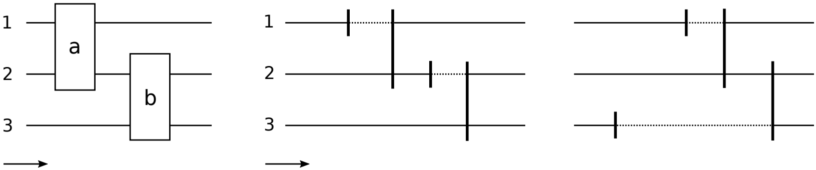
di Bernardo *et al* (2008). Piecewise-smooth Dynamical Systems.
CB (2015). Local Representation of Asynchronous Networks by Filippov Systems, Dynamical Equivalence and Approximation. In Prep.

Example: Two Trains in Passing Loop

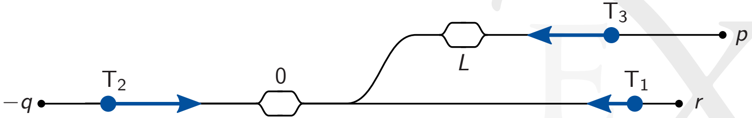


Dynamical Decomposition

The dynamics can be decomposed into basic networks a, b equivalent to a single passing loop.



Dotted lines: stopped nodes, vertical bars: events. Temporal evolution depends on initializations; stopping events may occur in any order in contrast to the restarting events.



CB and M J Field (2017), Nonlinearity, 30(2), 595–621.

Decomposition Result

Theorem

We can decompose a (functional) asynchronous network into spatiotemporal building blocks.

CB and M J Field (2017), Nonlinearity, 30(2), 595–621.

Conclusions and Outlook

Conclusions

- Coupling functions with **dead zones** induce asynchronous networks.
- We can potentially understand the dynamics by looking at **transitions between effective network structures**.
- Asynchronous networks provide a framework to **describe dynamical phenomena** in science and engineering including **stopping and restarting** events.
- Nonsmooth nature of networks allows for a **reductionist approach** (factorization).

Outlook (i.e., More Questions)

- Understand the dynamics even for small networks?
- Larger networks and more complicated structural network?
- Approximation theorems for ε-☠?

Thank you for your attention!

References

- P Ashwin, CB, and C Poignard (2019), arXiv:1904.00626.
- CB and M J Field (2017), Nonlinearity, 30(2), 558–594.
- CB and M J Field (2017), Nonlinearity, 30(2), 595–621.

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