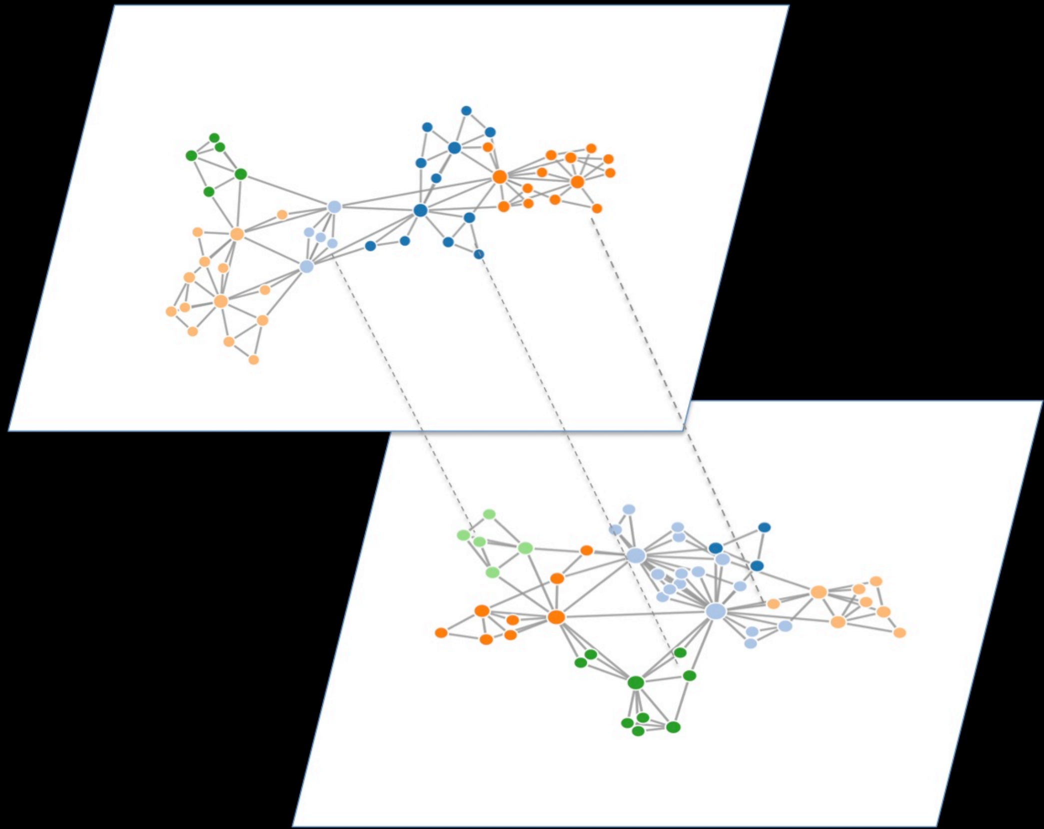


Threshold Networks

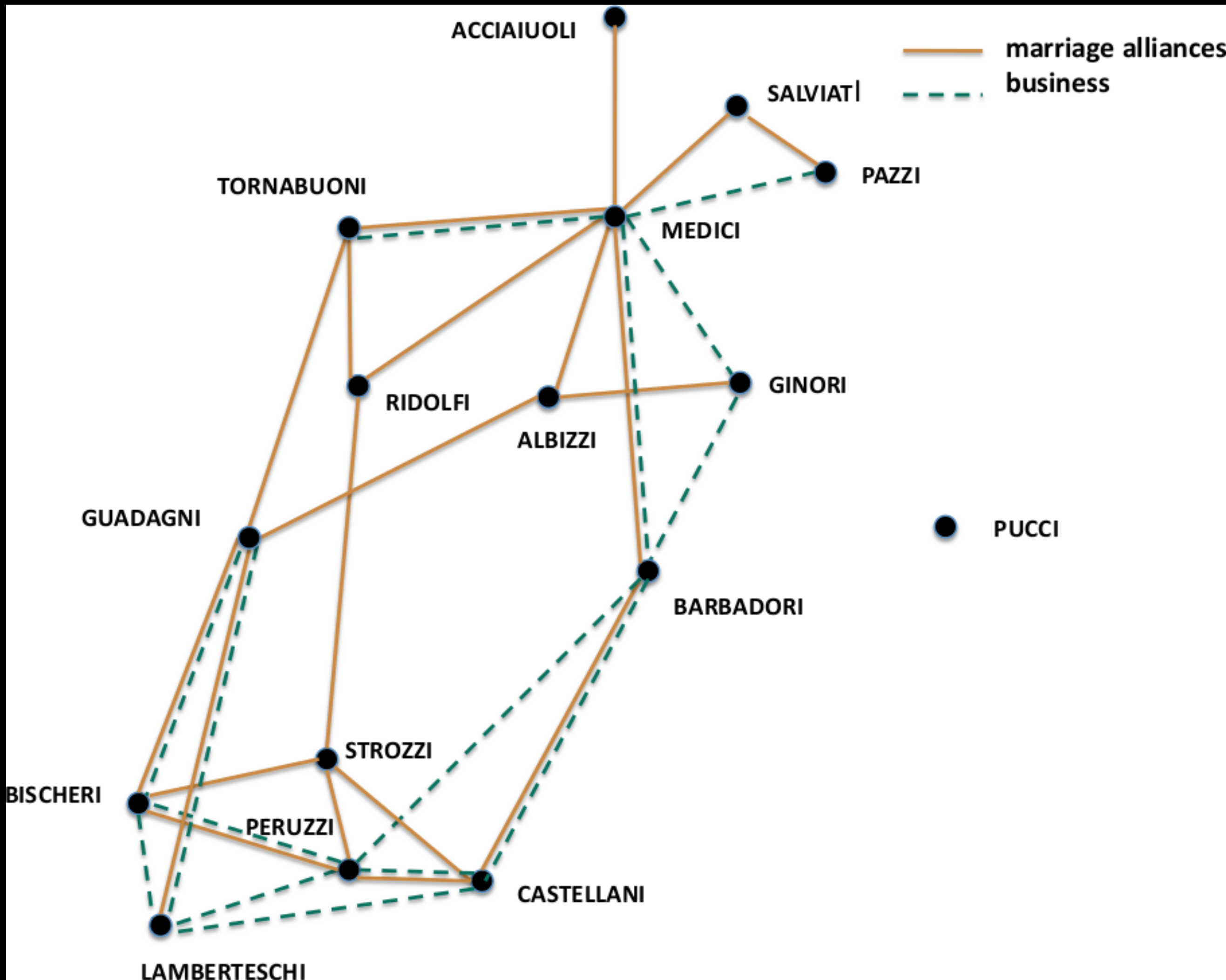
23 July 2019

MULTILAYER NETWORKS

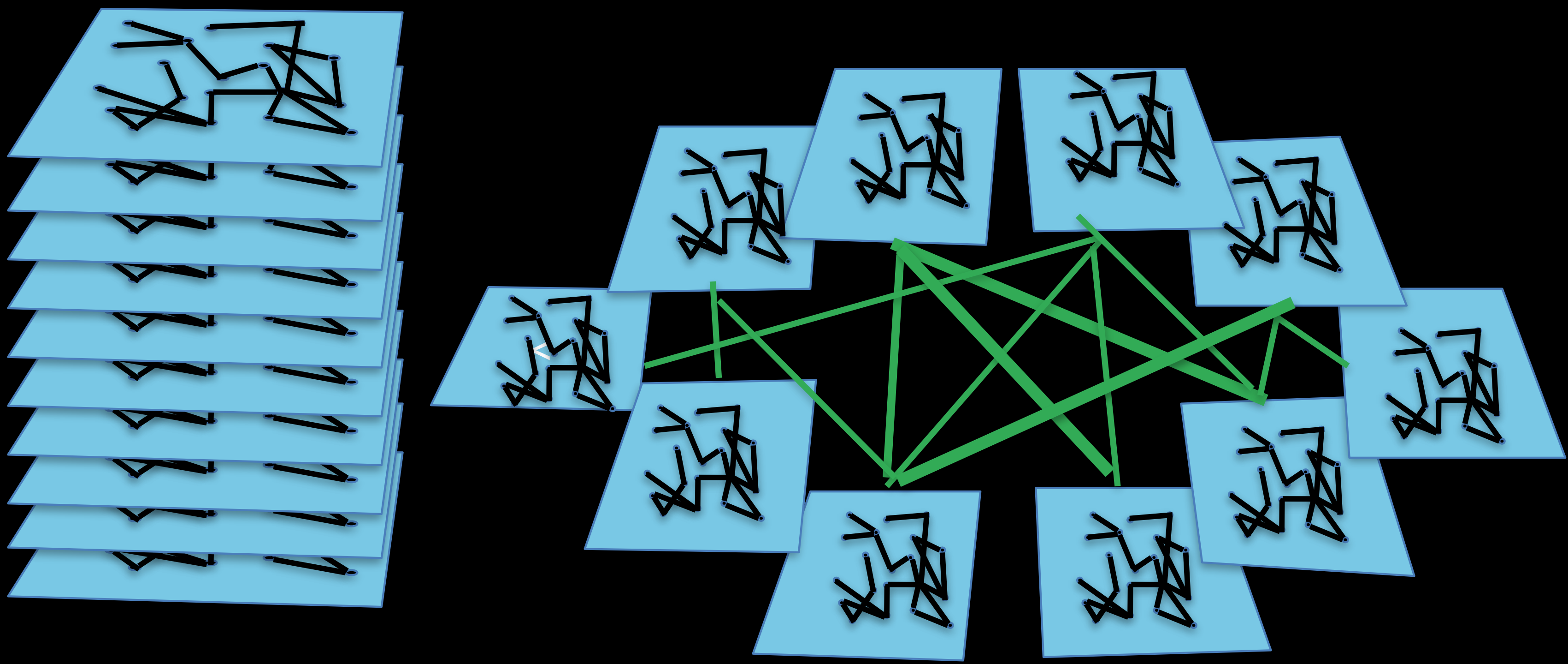
Ginestra Bianconi

School of Mathematical Sciences, Queen Mary University of London, UK

The Florentine Families

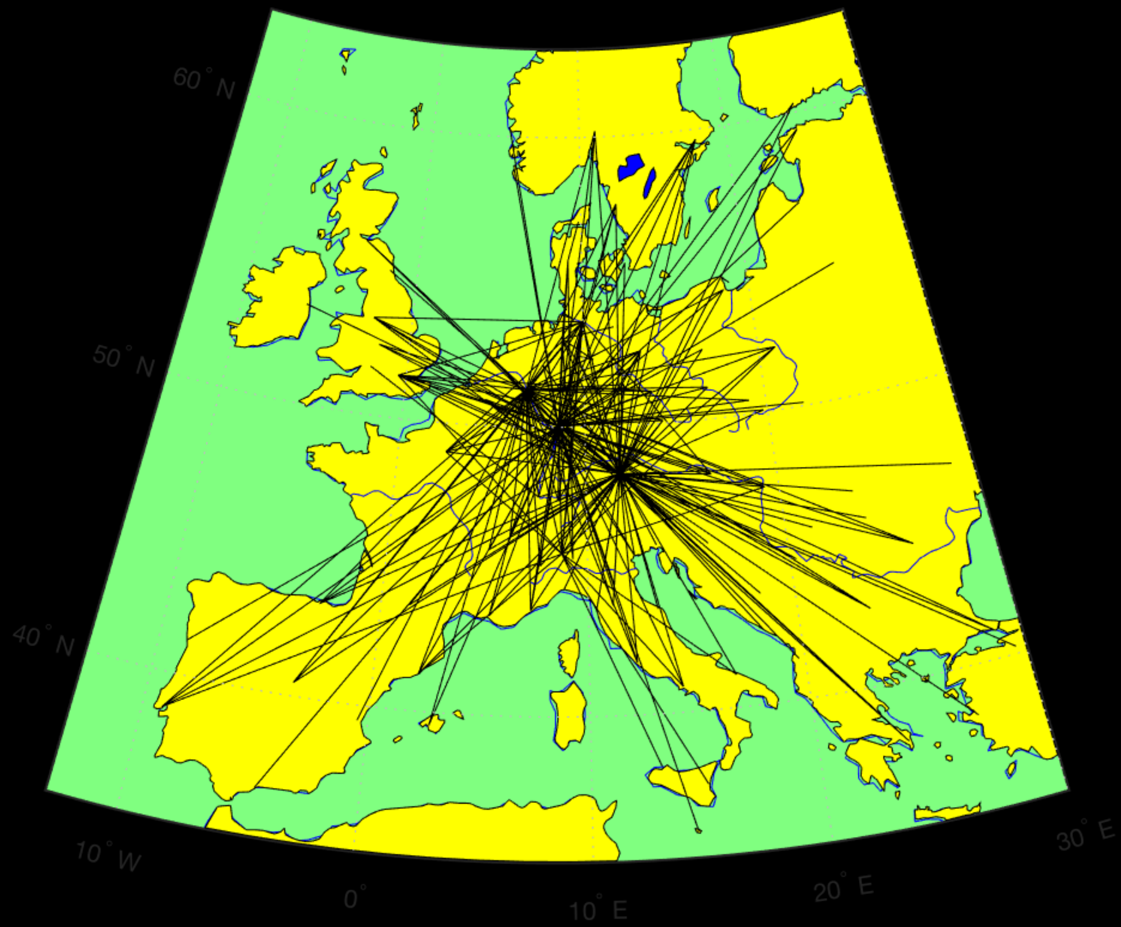


Large Scientific Collaboration Networks

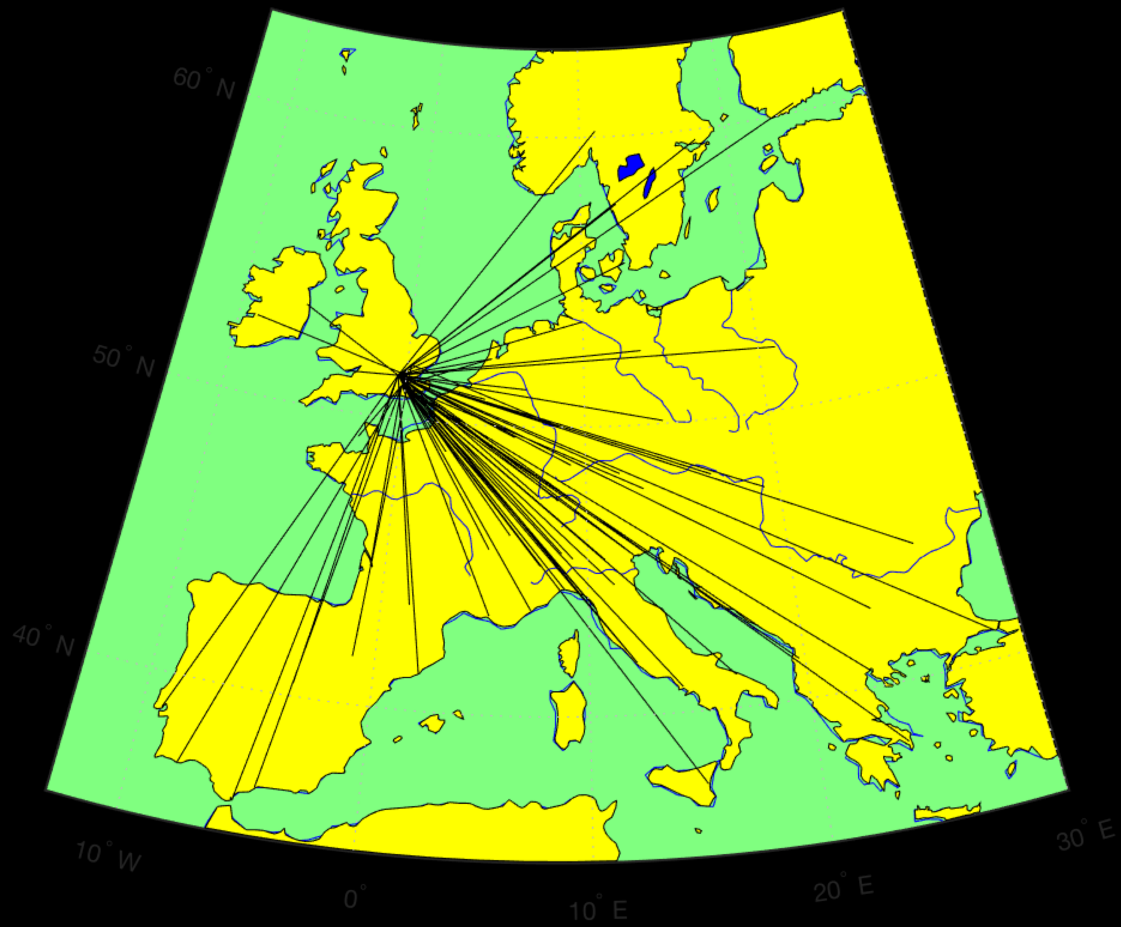


Jacovacci, Wu, Bianconi (2015)

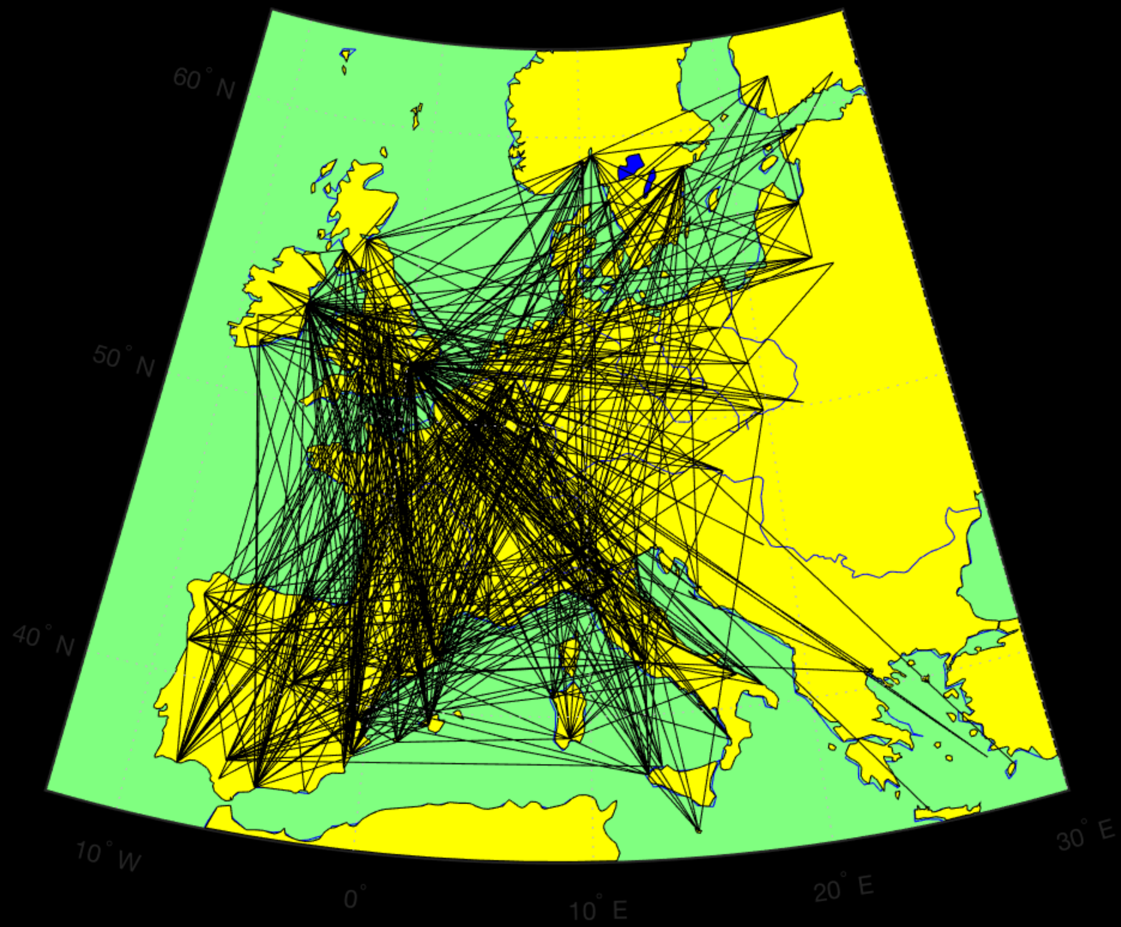
Multilayer Air Transportation Network



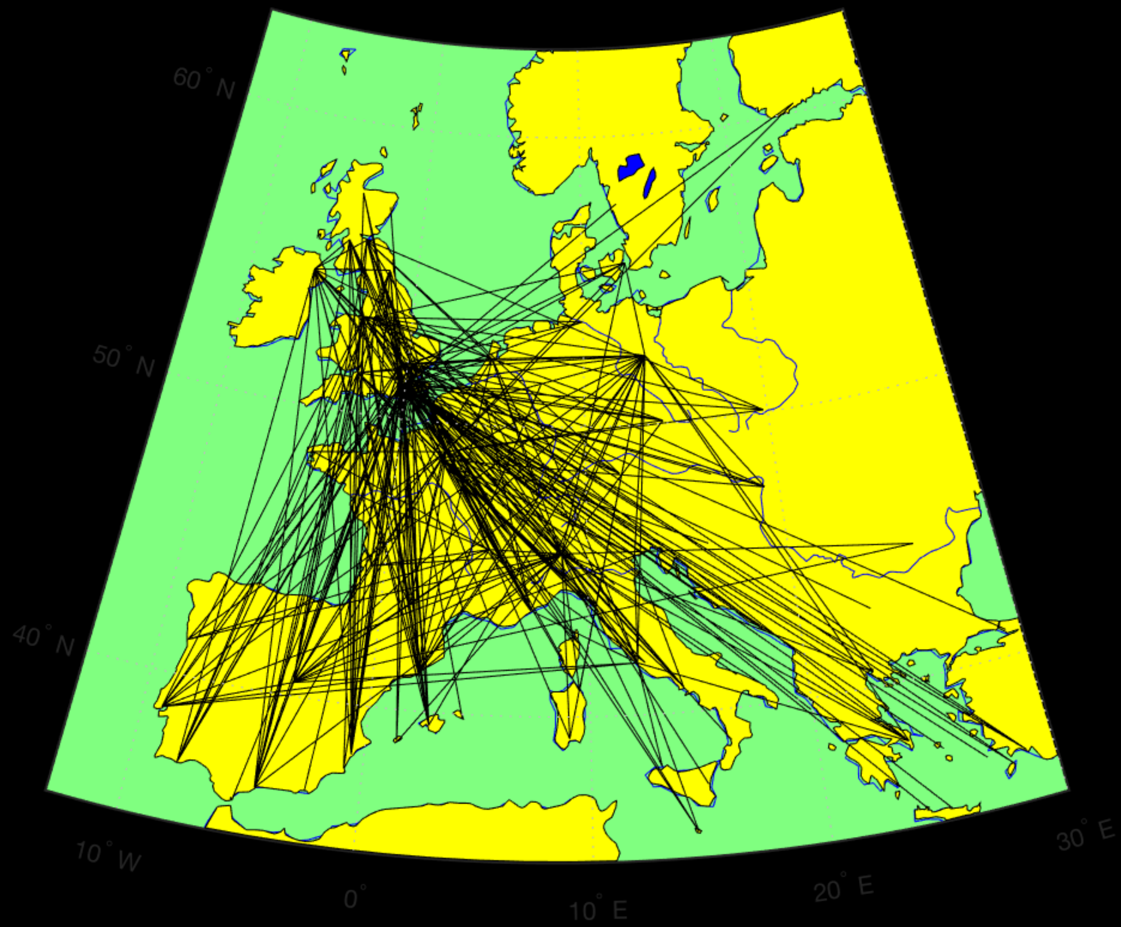
Multilayer Air Transportation network



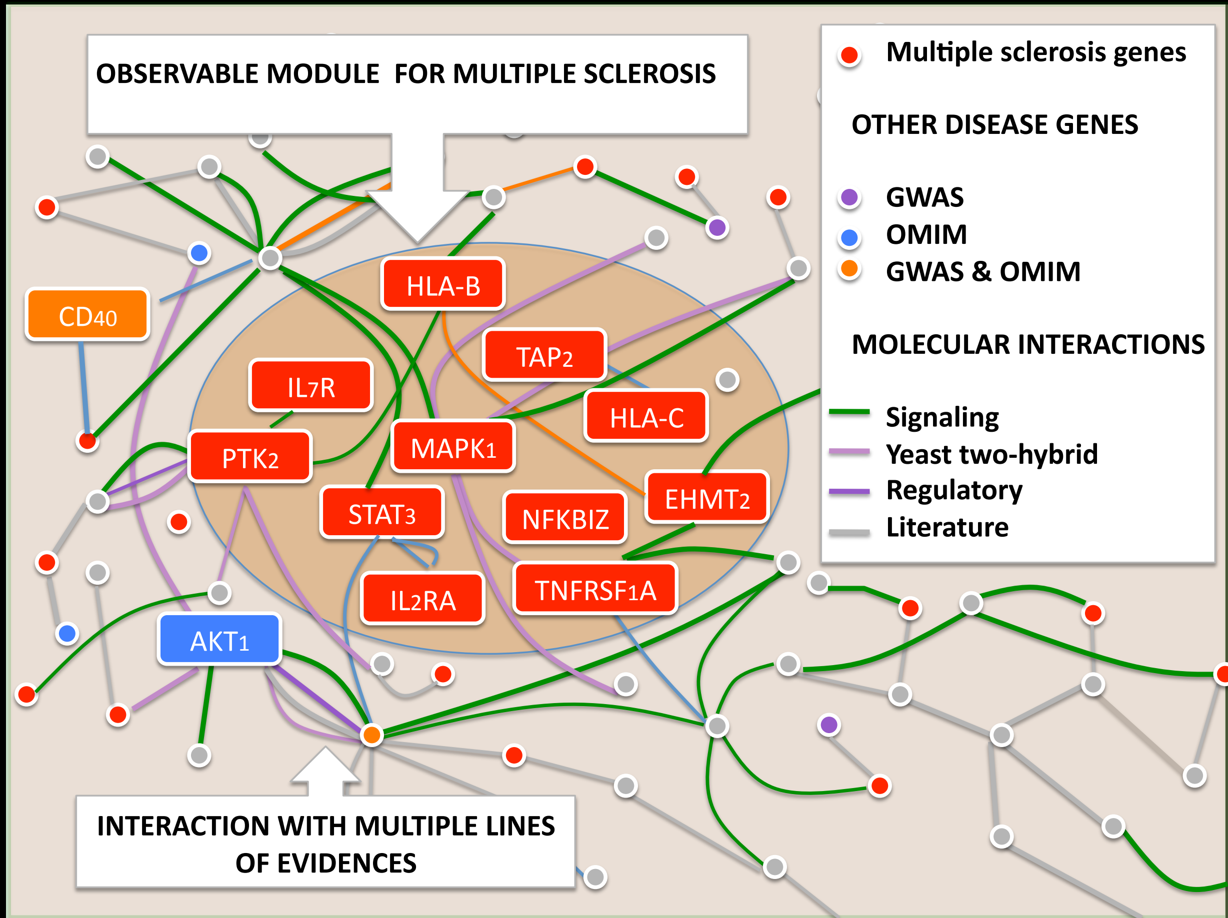
Multilayer Air Transportation Network



Multilayer Air Transportation Network

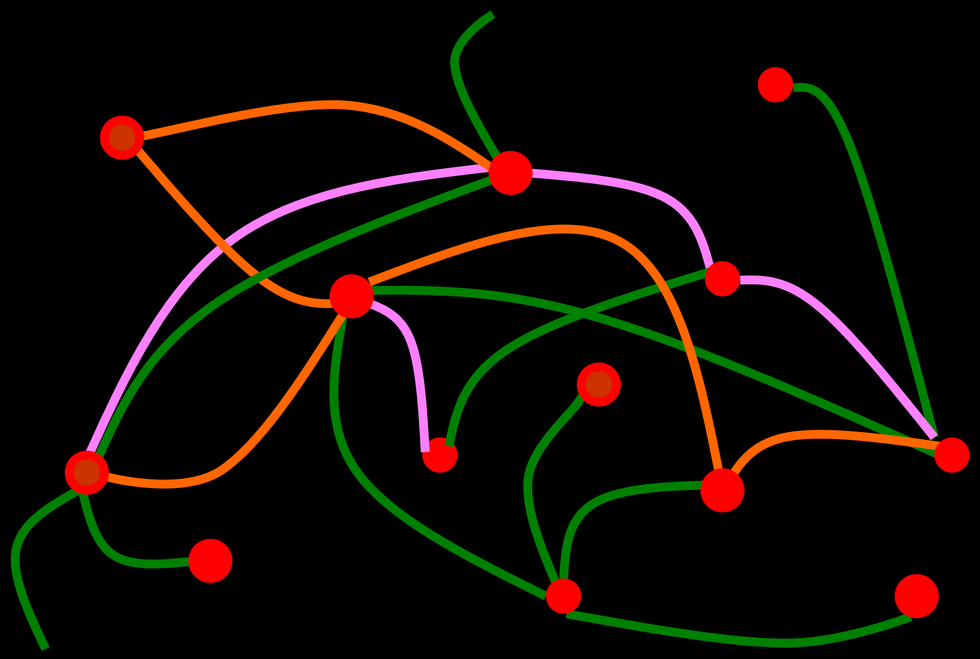


The Interactome

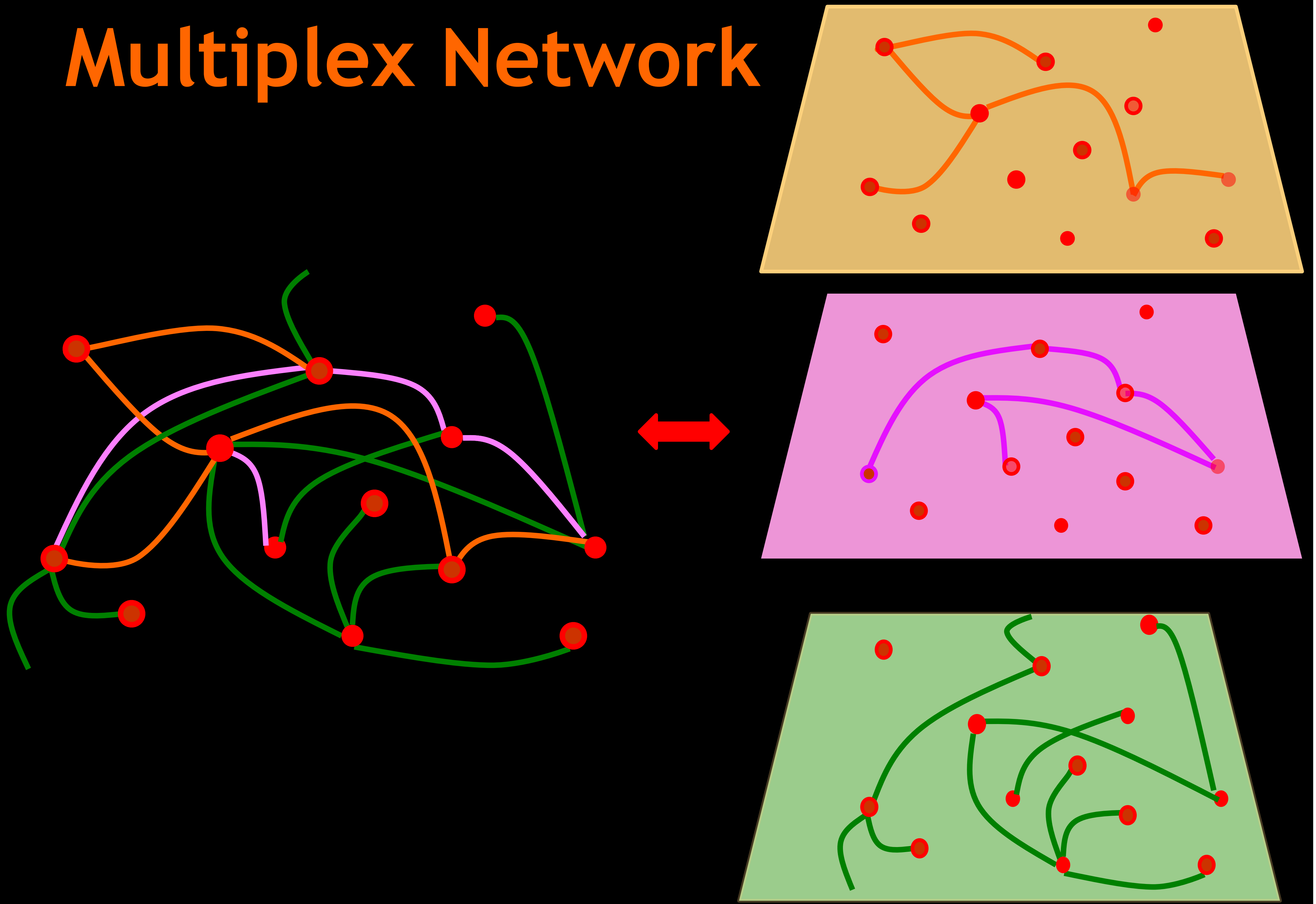


Menche et al.

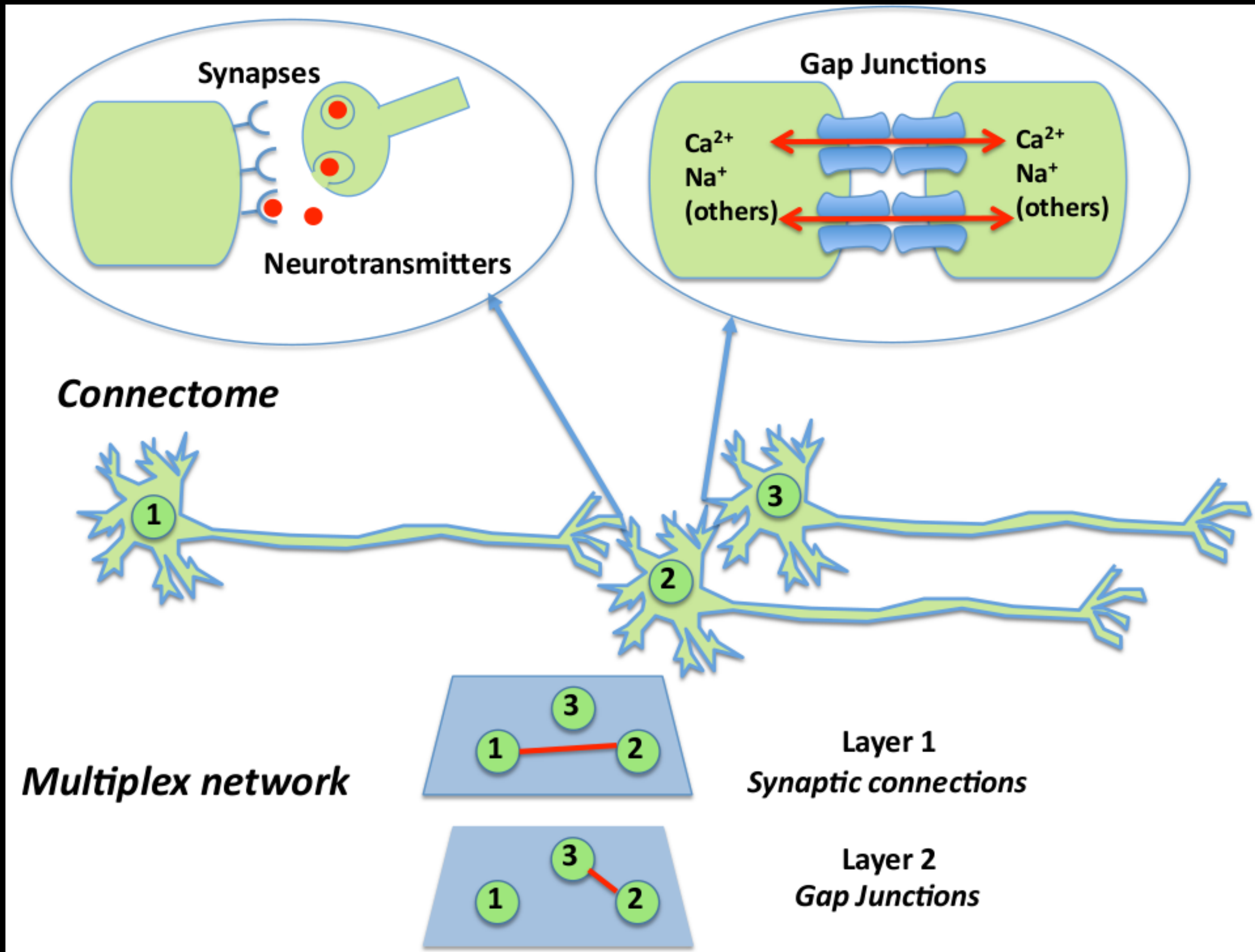
Multiplex Network



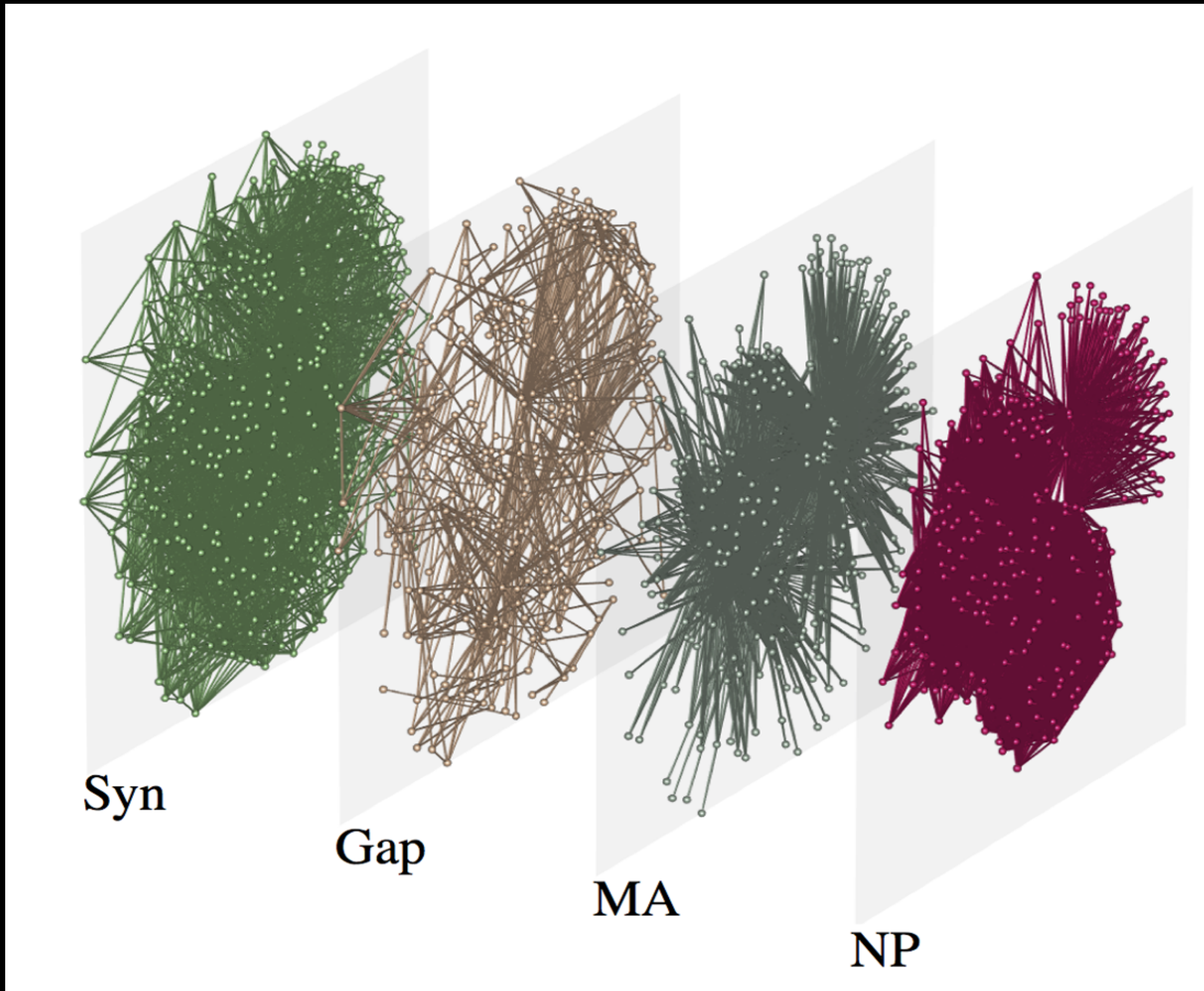
Multiplex Network



C.Elegans neuronal network



“Wireless” C.Elegans neuronal network



Multilayer connectome of c.elegans, Bentley et al (2016)

Multilayer networks

In order to

characterize, model and predict

complex systems

we need to reveal the interplay between

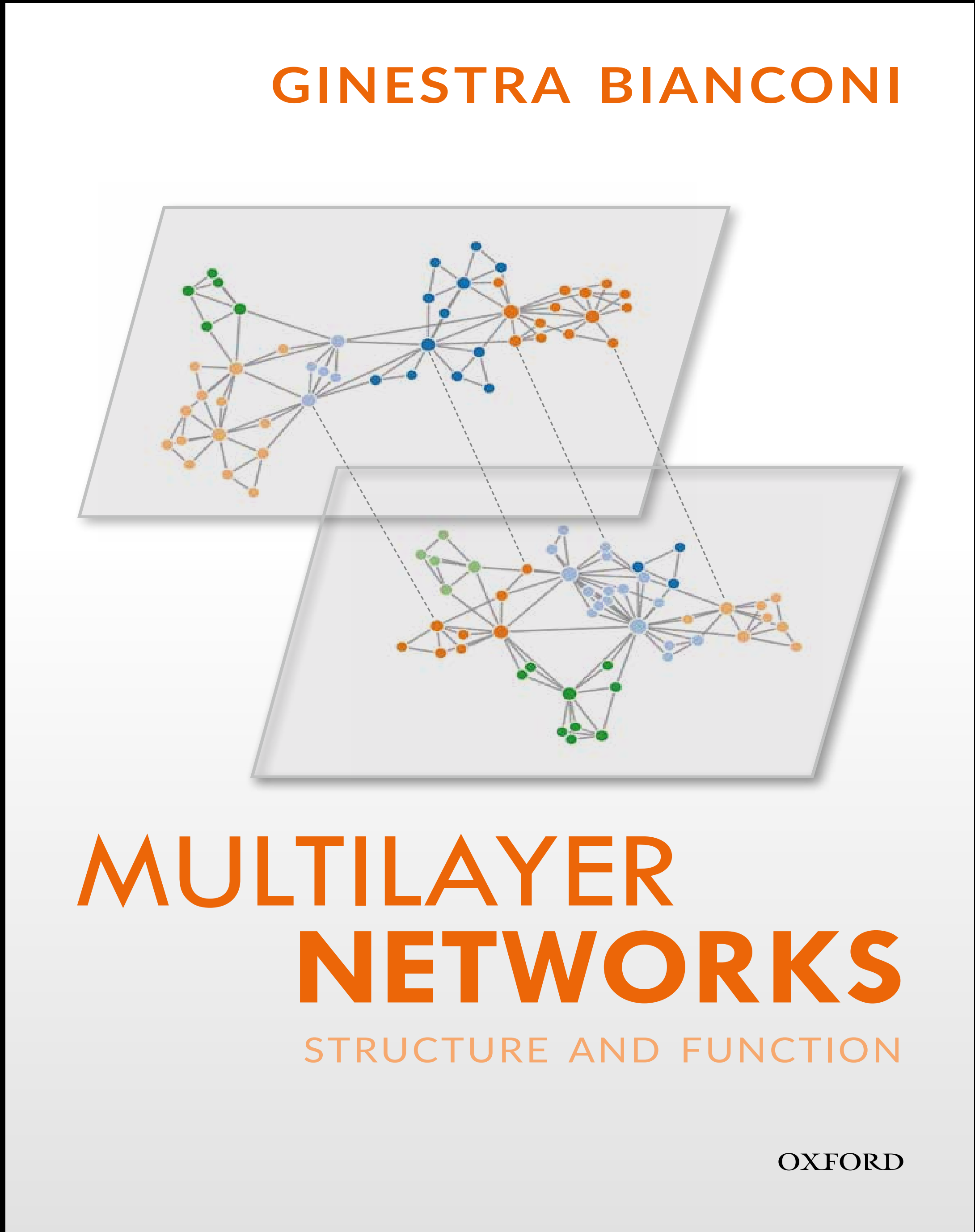
the structure

and the

the function

of

multilayer networks



MULTILAYER NETWORKS
Structure and Function
by
Ginestra Bianconi

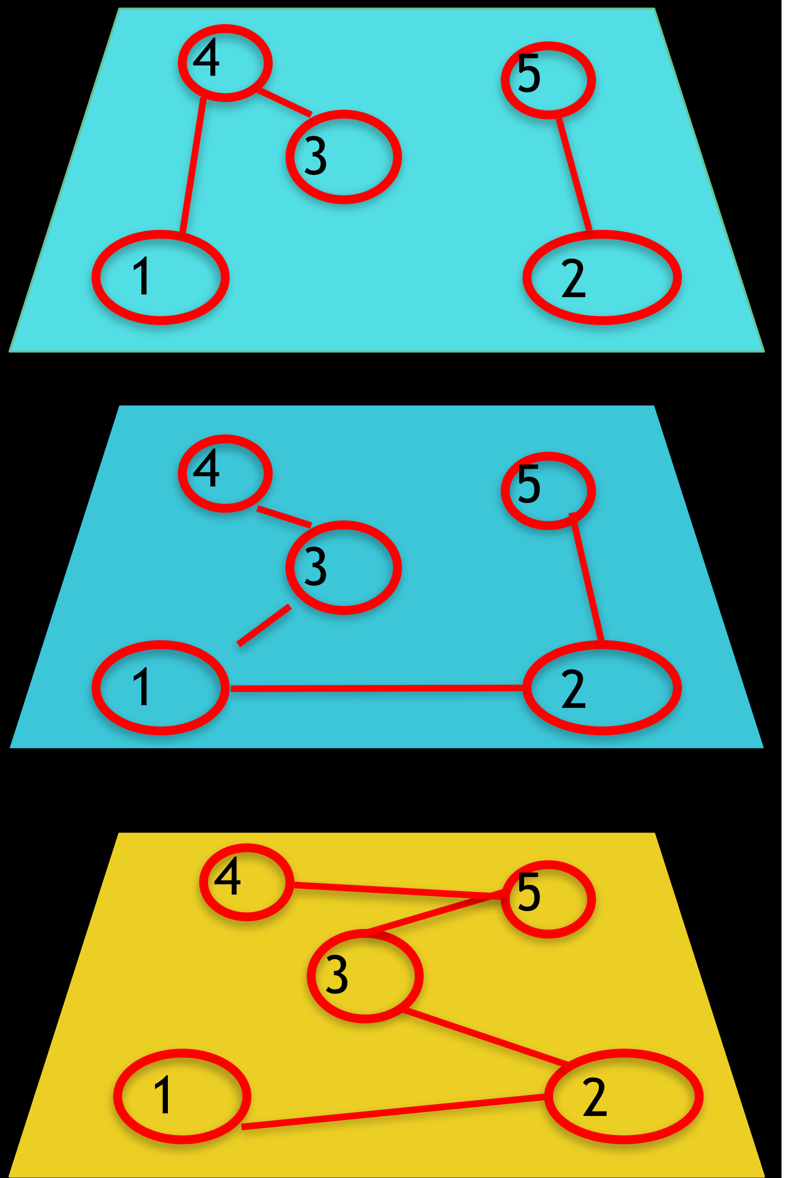
- Pedagogical presentation
- Discussion of general concepts in terms of their impact on interdisciplinary applications

Multiplex Networks
And
Information

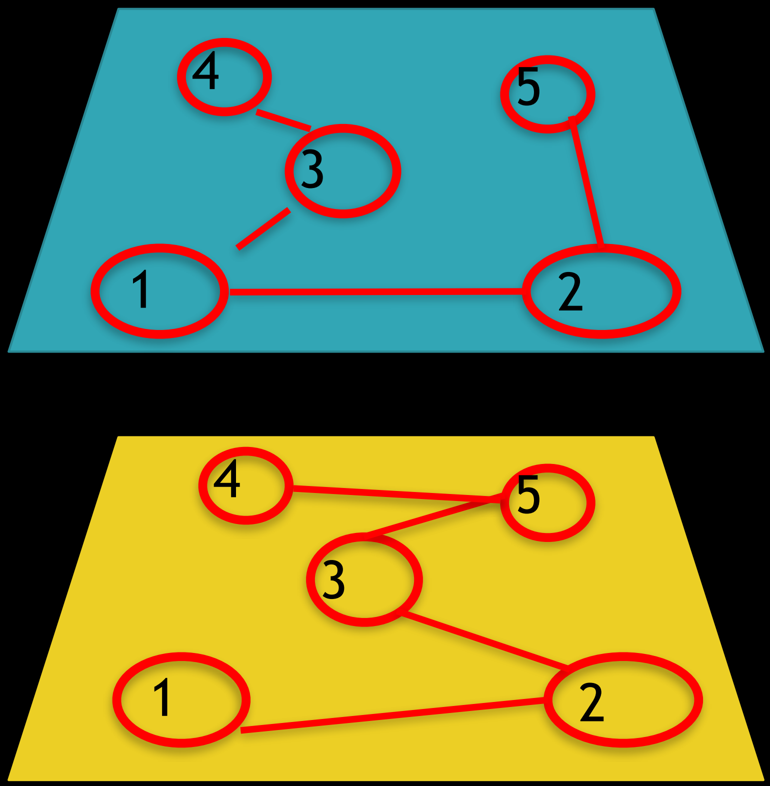
Why multiplex networks encode
more information
than their layers taken in isolation

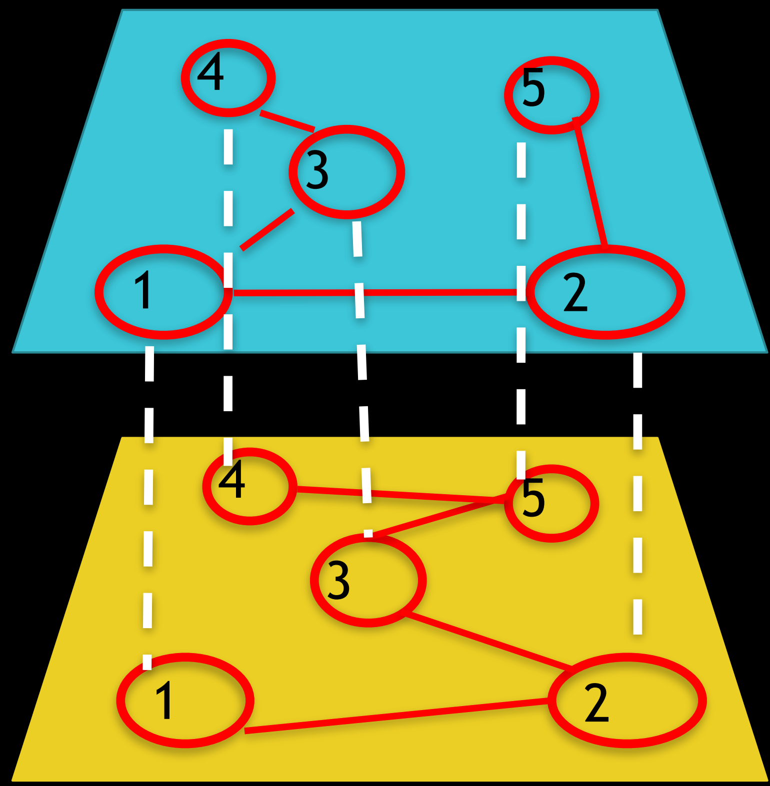
Multiplex Networks

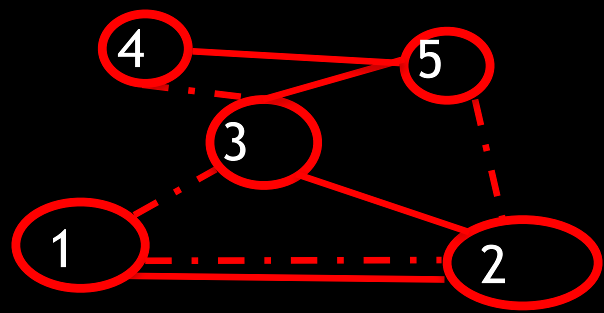
- A multiplex is formed by M layers
- and by a set of N nodes each one having a "replica node" in every layer
- Each layer is a network



Different representations of the same multiplex network







Multiplex networks

A multiplex network of N nodes and M layers is an ordered set of networks

$$\vec{G} = (G^{[1]}, G^{[2]}, \ldots, G^{[M]})$$

where

$$G^{[\alpha]} = (V, E^{[\alpha]})$$

indicates the α network among the N nodes in the set V

Representation of a multiplex

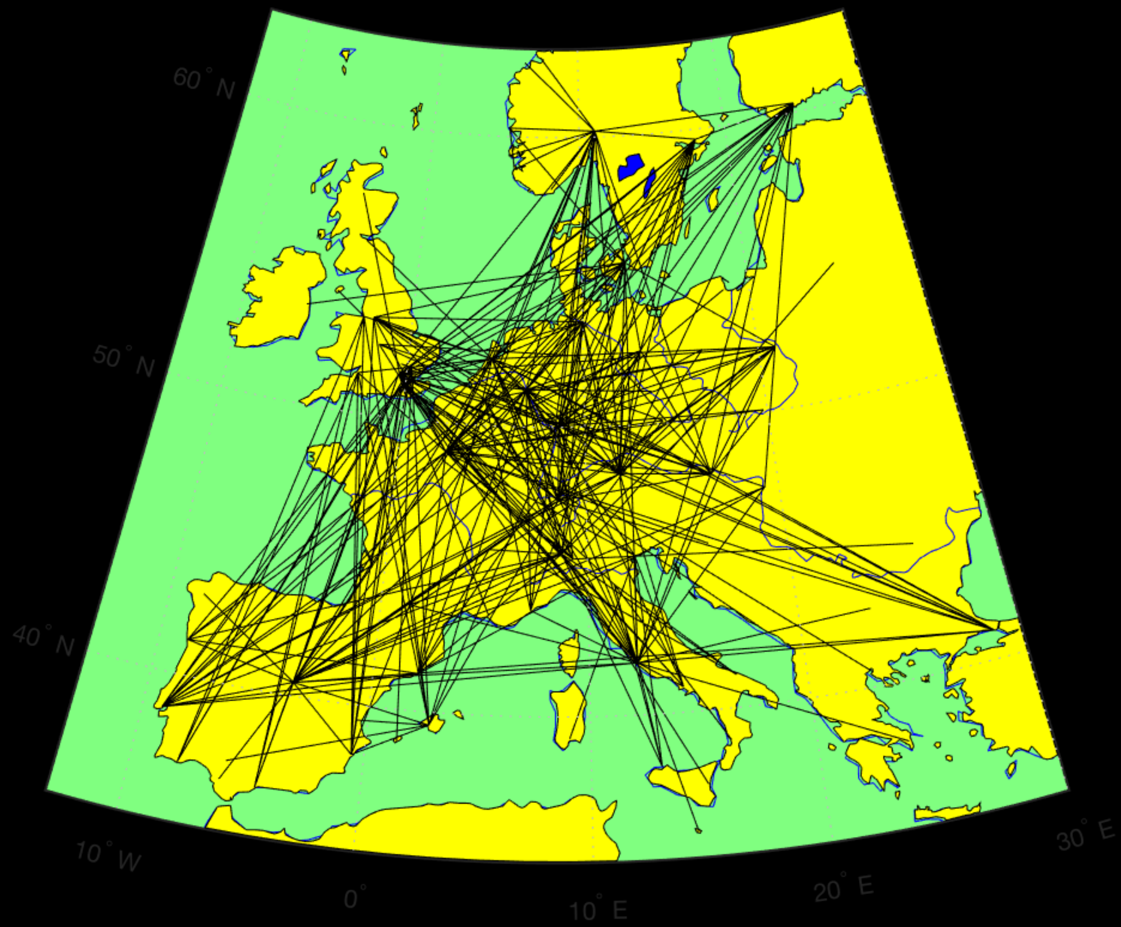
A multiplex network of N nodes formed by M layers
is fully specified by
M adjacency matrices

$$\mathbf{a}^{[\alpha]}$$

with α=1, 2, ... M
of matrix elements

$$a_{ij}^{[\alpha]} = \begin{cases} 1 \text{ if node } i \text{ is linked to node } j \text{ in layer } \alpha \\ 0 \text{ otherwise} \end{cases}$$

Link Overlap in Multilayer Air Transportation Networks



Link Overlap

The total overlap $O^{[\alpha\alpha']}$
between layer α and layer α'
is given by'

$$O^{[\alpha,\alpha']} = \sum_{i<j} a_{ij}^{[\alpha]} a_{ij}^{[\alpha']}$$

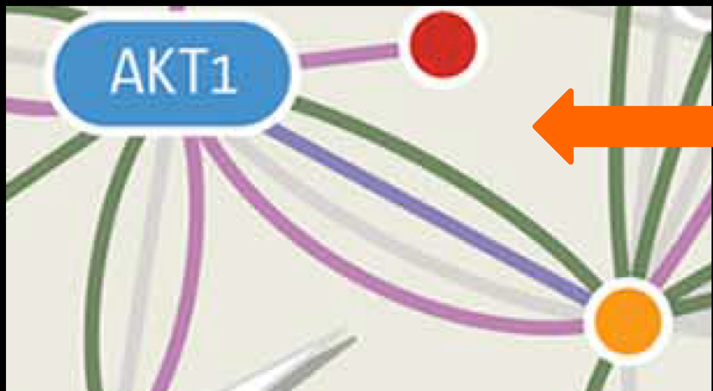
The local overlap $o_i^{[\alpha,\alpha']}$
of node i between layer α and layer α'
is given by

$$o_i^{[\alpha,\alpha']} = \sum_{j=1}^{N} a_{ij}^{[\alpha]} a_{ij}^{[\alpha']}$$

Multiplicity of link overlap

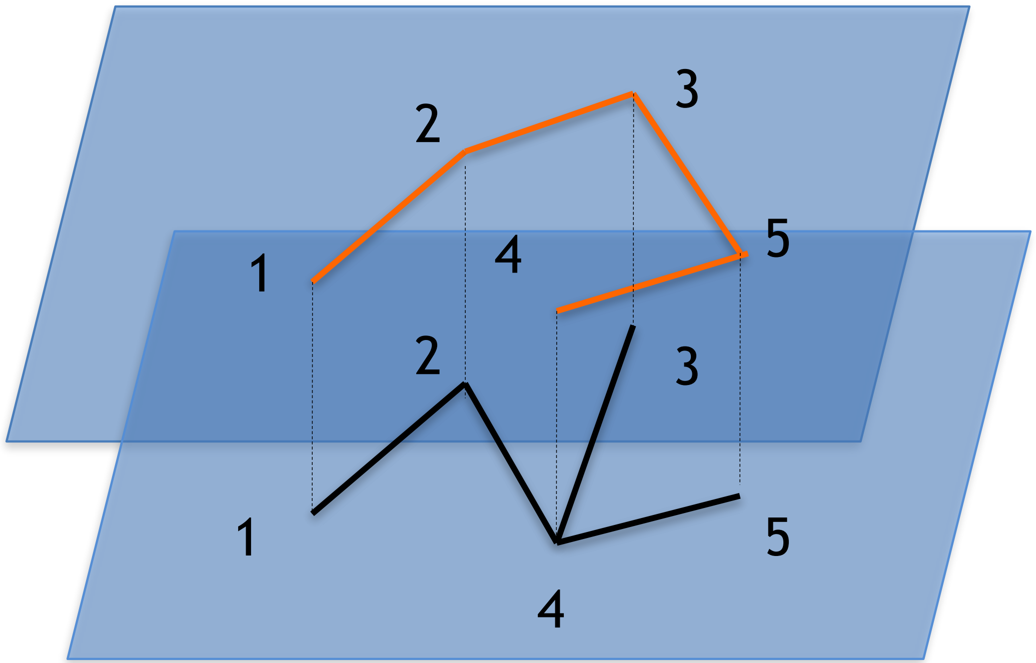
The multiplicity of link overlap is the number of layers in which a given link is present

$$\mu_{ij} = \sum_{\alpha=1}^{M} a_{ij}^{[\alpha]}$$

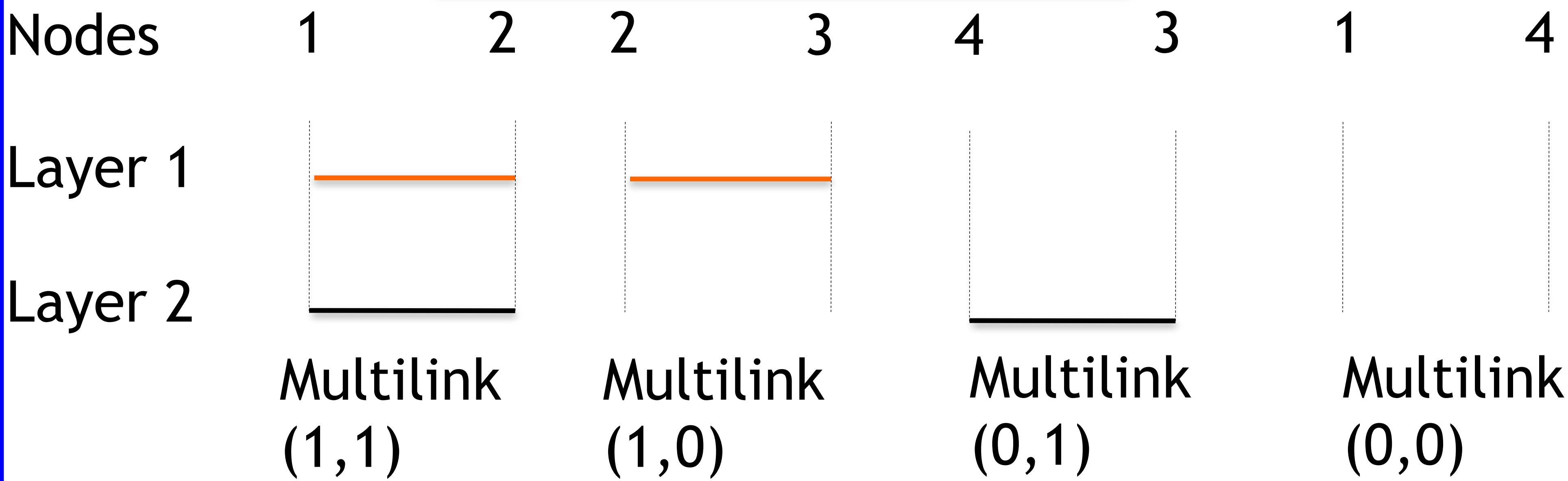


$$\mu_{ij} = 4$$

Multilinks



G. Bianconi
PRE (2013)



Multidegree

The multidegree $k_i^{\vec{m}}$ of a node i is defined as the number of multilinks

$$\vec{m} = \left(m^{[1]}, m^{[2]}, \ldots, m^{[M]}\right)$$

incident to it

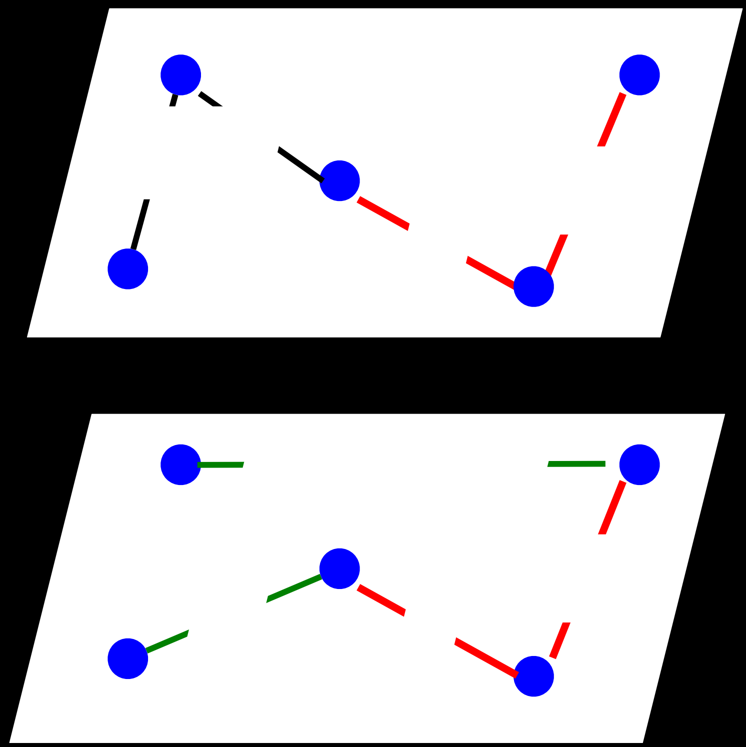
In the case of two layers we have

$$k_i^{(1,0)} = \sum_{j=1}^{N} a_{ij}^{[1]}(1 - a_{ij}^{[2]})$$

$$k_i^{(0,1)} = \sum_{j=1}^{N} (1 - a_{ij}^{[1]})a_{ij}^{[2]}$$

$$k_i^{(1,1)} = \sum_{j=1}^{N} a_{ij}^{[1]}a_{ij}^{[2]}$$

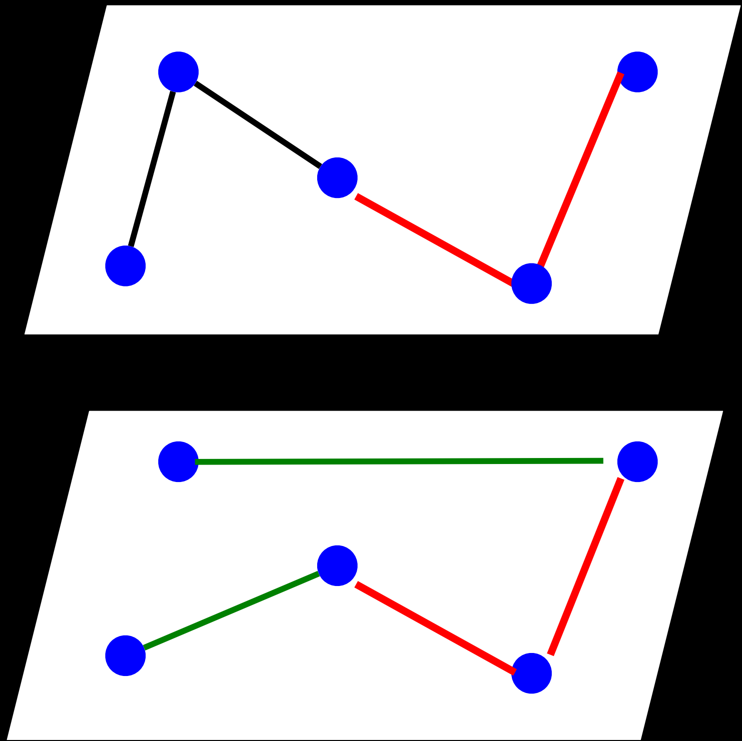
Configuration model with multidegree



G. Bianconi PRE (2013)

$$P(\overrightarrow{G}) = \frac{1}{Z}\prod_{i=1}^{N}\delta\left(k_i^{(1,0)} - \sum_{j=1}^{N} A_{ij}^{(1,0)}\right)\delta\left(k_i^{(0,1)} - \sum_{j=1}^{N} A_{ij}^{(0,1)}\right)\delta\left(k_i^{(1,1)} - \sum_{j=1}^{N} A_{ij}^{(1,1)}\right)$$

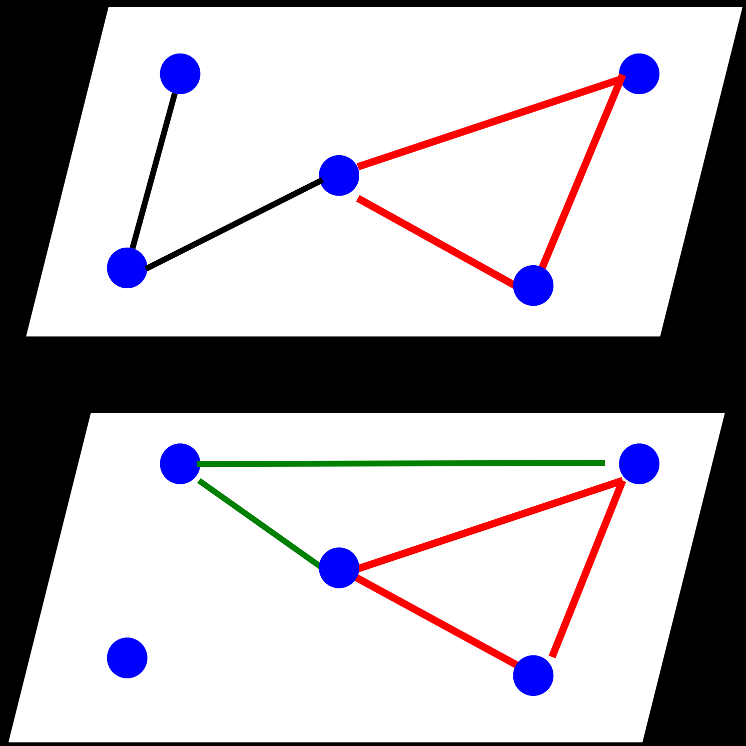
Configuration model with multidegrees



G. Bianconi PRE (2013)

$$P(\vec{G}) = \frac{1}{Z}\prod_{i=1}^{N}\delta\left(k_i^{(1,0)} - \sum_{j=1}^{N} A_{ij}^{(1,0)}\right)\delta\left(k_i^{(0,1)} - \sum_{j=1}^{N} A_{ij}^{(0,1)}\right)\delta\left(k_i^{(1,1)} - \sum_{j=1}^{N} A_{ij}^{(1,1)}\right)$$

Exponential random graph with expected multidegrees



Constructive algorithm

For every pair of nodes (i,j)

Draw a multilink $\vec{m}$

with probability $p_{ij}^{\vec{m}}$,

i.e. put a link in every layer

where $m^{[\alpha]} = 1$

G. Bianconi PRE (2013)

Marginals

The marginal probability that two nodes i and j are connected my a multilink $\vec{m}$ is given by

$$p_{ij}^{\vec{m}} = \frac{e^{-\lambda_i^{\vec{m}} - \lambda_j^{\vec{m}}}}{1 + \sum_{\vec{m}' \neq \vec{0}} e^{-\lambda_i^{\vec{m}'} - \lambda_j^{\vec{m}'}}}$$

Probability of multilinks in duplex networks with structural cutoff

Probabilities of the multilinks

$$p_{ij}^{(1,0)} = \frac{k_i^{(1,0)} k_j^{(1,0)}}{\langle k^{(1,0)} \rangle N}$$

$$p_{ij}^{(0,1)} = \frac{k_i^{(0,1)} k_j^{(0,1)}}{\langle k^{(0,1)} \rangle N}$$

$$p_{ij}^{(1,1)} = \frac{k_i^{(1,1)} k_j^{(1,1)}}{\langle k^{(1,1)} \rangle N}$$

Structural cutoff

$$k_i^{(0,1)} < \sqrt{\langle k^{(0,1)} \rangle N}$$

$$k_i^{(1,0)} < \sqrt{\langle k^{(1,0)} \rangle N}$$

$$k_i^{(1,1)} < \sqrt{\langle k^{(1,1)} \rangle N}$$

Entropy of correlated multiplex ensembles

Entropy of a canonical multiplex ensemble
with expected multidegree sequence

$$S = -\sum_{i<j} \sum_{\vec{m}} p_{ij}^{\vec{m}} \ln p_{ij}^{\vec{m}}$$

Entropy of a microcanonical multiplex ensemble
with given multi degree sequence

$$\Sigma = \ln \mathcal{N}$$

G. Bianconi PRE (2013)

Non equivalence of the two ensembles

The two ensembles are non-equivalent

We have

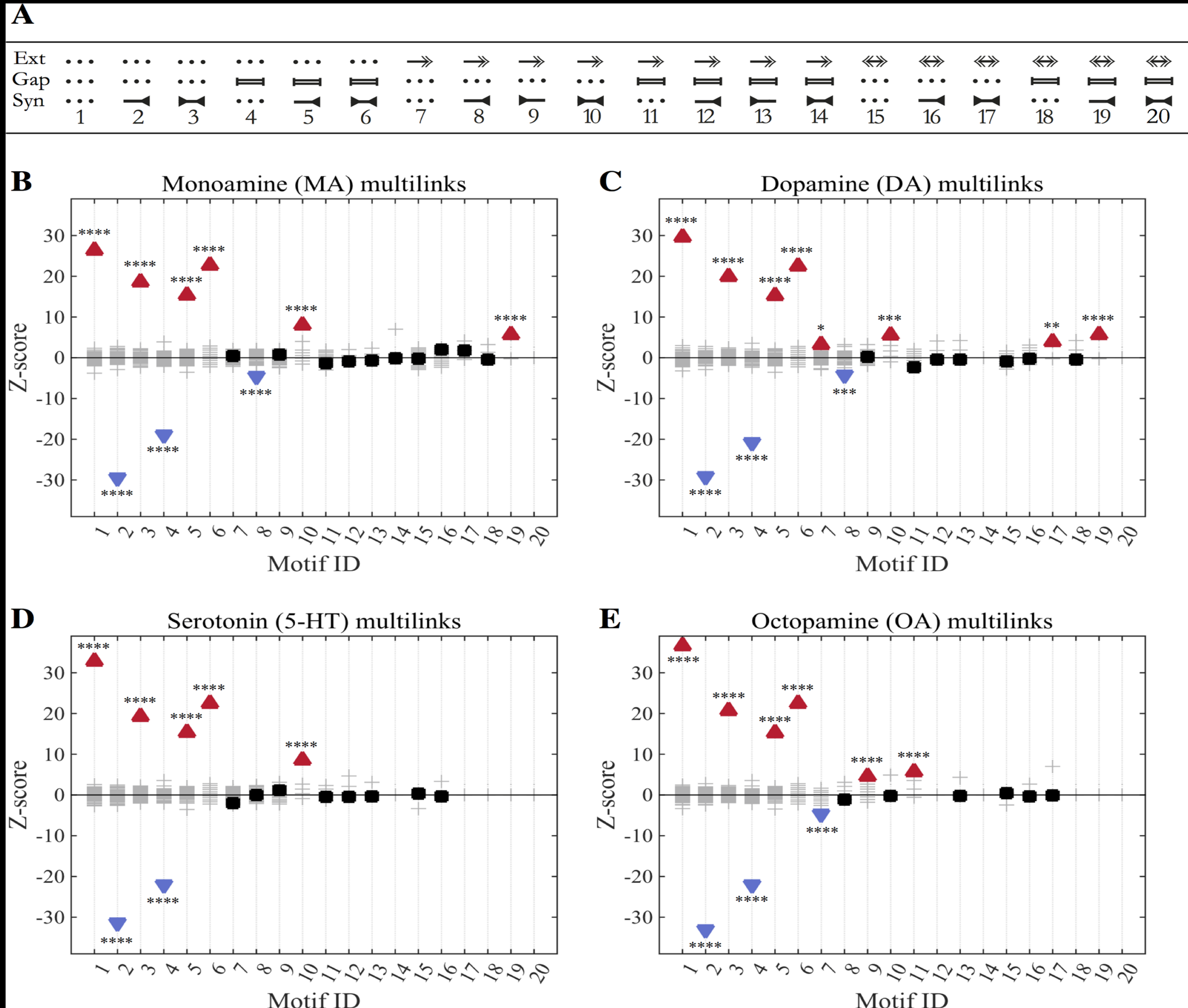
$$\Sigma = S - \Omega$$

and in the sparse multiplex network limit

$$\Omega = -\sum_{\vec{m}} \sum_{i=1}^{N} \ln \left[\frac{(k_i^{\vec{m}})^{k_i^{\vec{m}}} e^{-k_i^{\vec{m}}}}{k_i^{\vec{m}}!} \right]$$

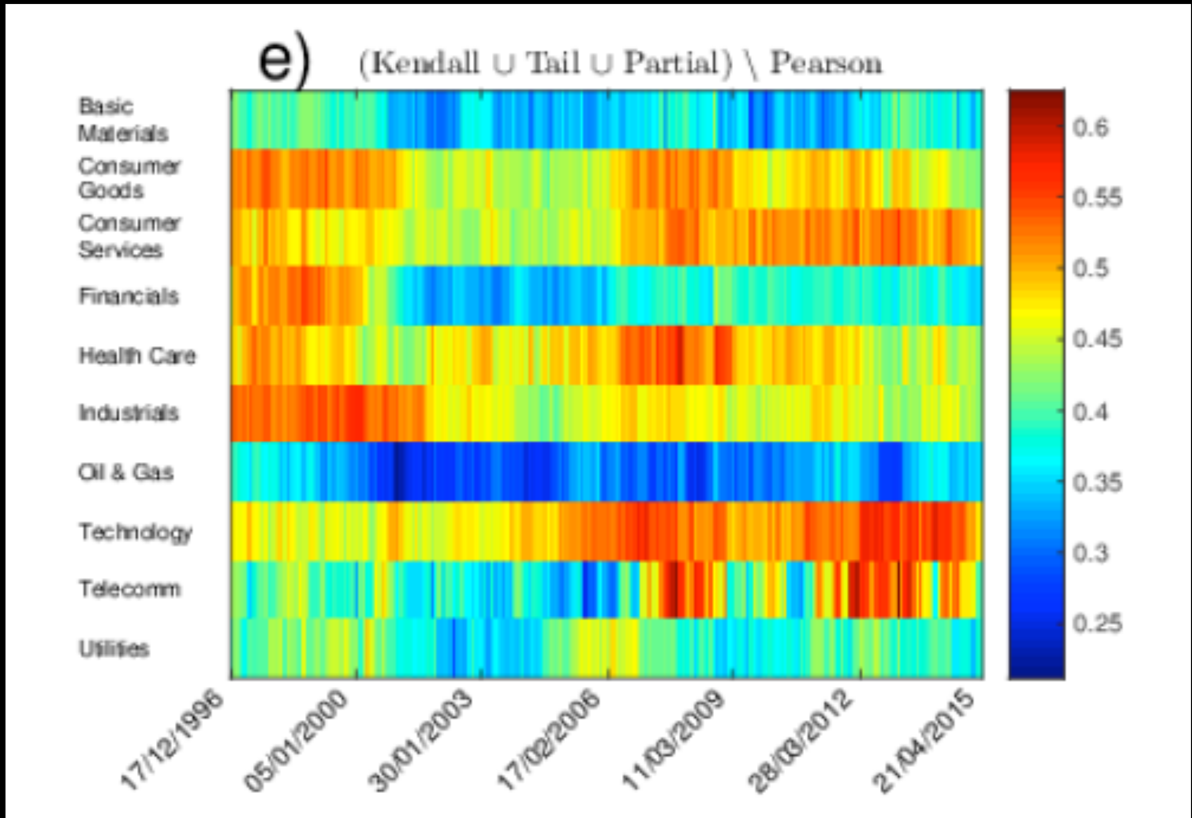
G. Bianconi PRE (2013)

Multilinks as motifs



Multilayer connectome of c.elegans, Bentley et al (2016)

Multidegrees of the financial multiplex network reveal the different role of financial sectors

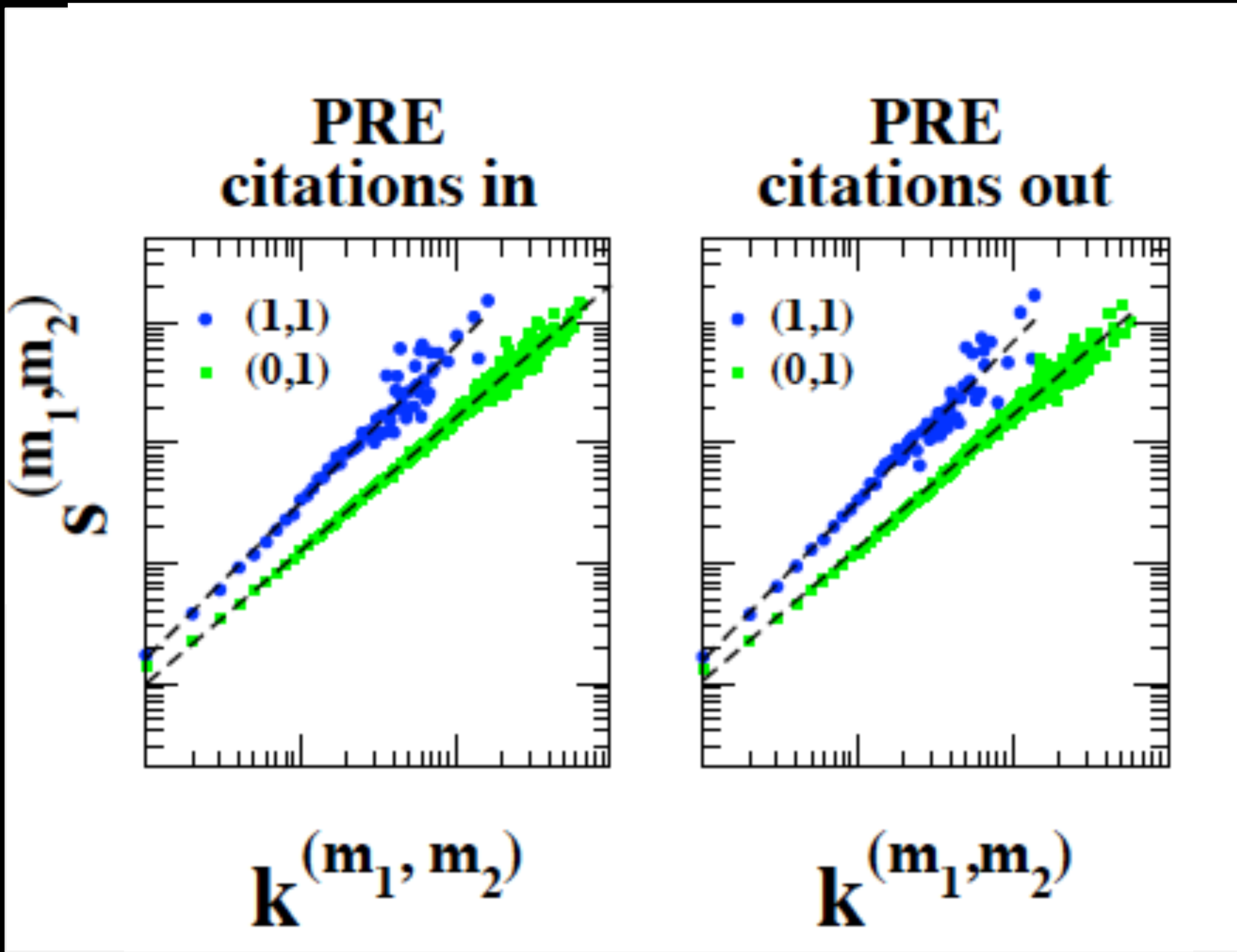


Musmeci et al. (2016)

Multistrength vs multidegree in the citation layer of the citation/collaboration duplex network

The way scientists cite their collaborators is different from the way they cite the other scientists.

People tend to cite more the hubs with whom they have collaborated.



G. Menichetti et. al. Plos One (2014)

Multiplex networks encode more information than single layers

Multiplex networks are not equivalent to a larger single network

Different types of links describe different types of interactions, therefore multiplex networks encode more information than their single layers taken in isolation

Influences of multilinks

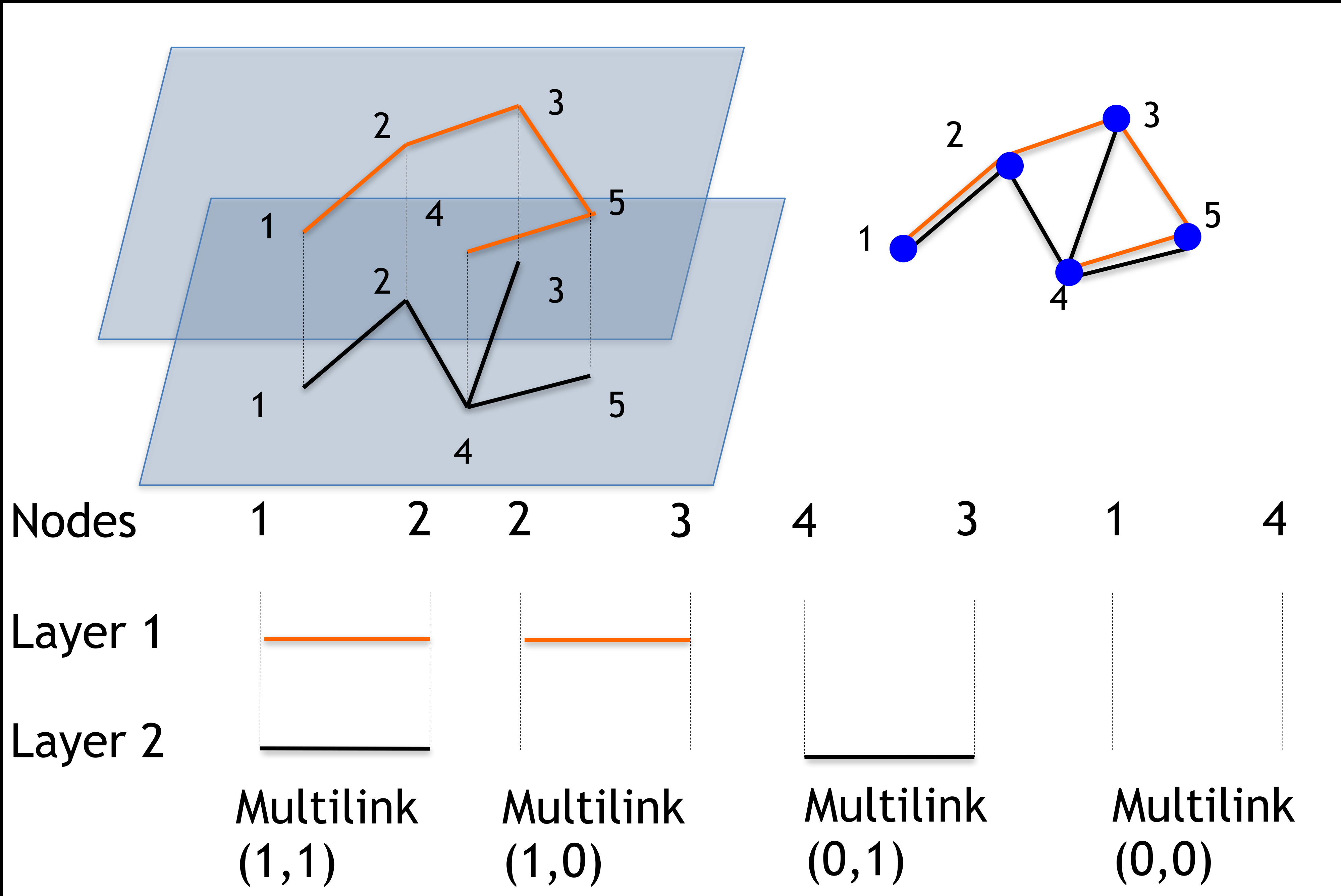
In a multiplex network different pattern of connections might contribute differently to the centrality of a node

The influence of a multilink
is indicated by

$$\vec{m}$$

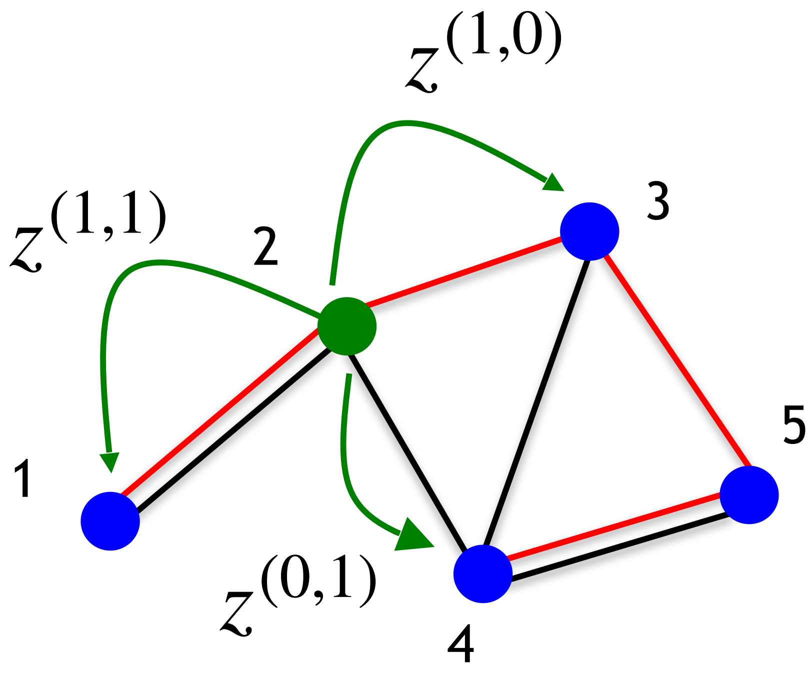
J. Iacovacci, et al. 2016

Multilinks



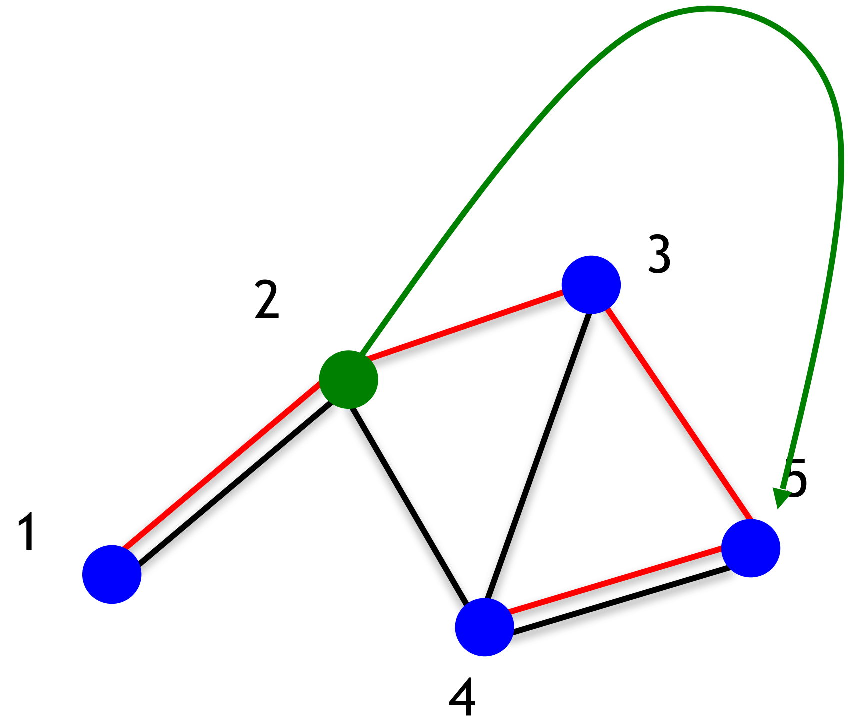
Nodes 1 2 2 3 4 3 1 4
Layer 1
Layer 2
Multilink (1,1)
Multilink (1,0)
Multilink (0,1)
Multilink (0,0)

Functional Multiplex PageRank



(a)

The random walker can jump to a neighbor node, with a probability proportional to the influence of the corresponding multilink



(b)

The random walker can jump to a random node (teleportation)

Functional Multiplex PageRank

The centrality of a node i

is a function

$$X_i(\mathbf{z})$$

depending on the values of the influences **z** attributed to multilinks

For a duplex network

$$\mathbf{z} = \left(z^{(1,0)}, z^{(0,1)}, z^{(1,1)}\right)$$

Non-linear effects due to the overlap of the links

The Functional Multiplex PageRank allows for the inclusion of strong non-linear effects due to the overlap of the links.

For example, in a duplex network we can have

$$z^{(1,1)} \neq z^{(1,0)} + z^{(0,1)}$$

and we can weight multilinks (1,1) much more or much less than the sum of the weight of multilinks (0,1) and (1,0).

Absolute Multiplex PageRank

From the
Functional Multiplex PageRank
we can extract the
Absolute Multiplex PageRank
given by

$$X_i^{\star} = \max_{\mathbf{z}} X(\mathbf{z})$$

which can provide an overall ranking of the nodes
of the multiplex network

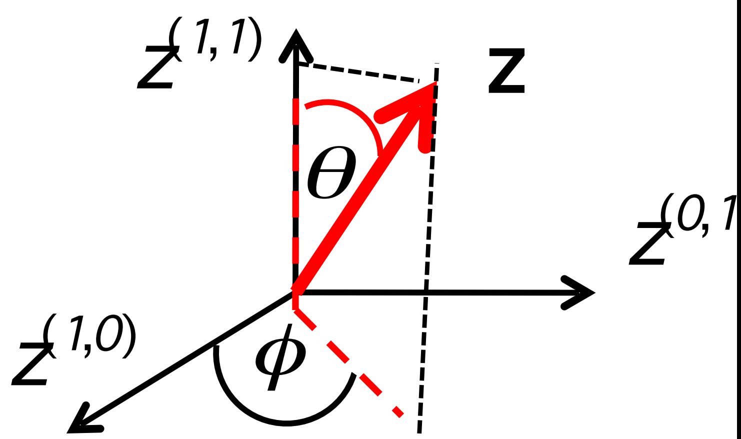
The case of a duplex network (M=2)

The Functional Multiplex PageRank, depends only of the direction of the vector of influences **z**, therefore we take

$$z^{(1,0)} = \sin\theta\cos\phi$$
$$z^{(0,1)} = \sin\theta\sin\phi$$
$$z^{(1,1)} = \cos\theta$$

with

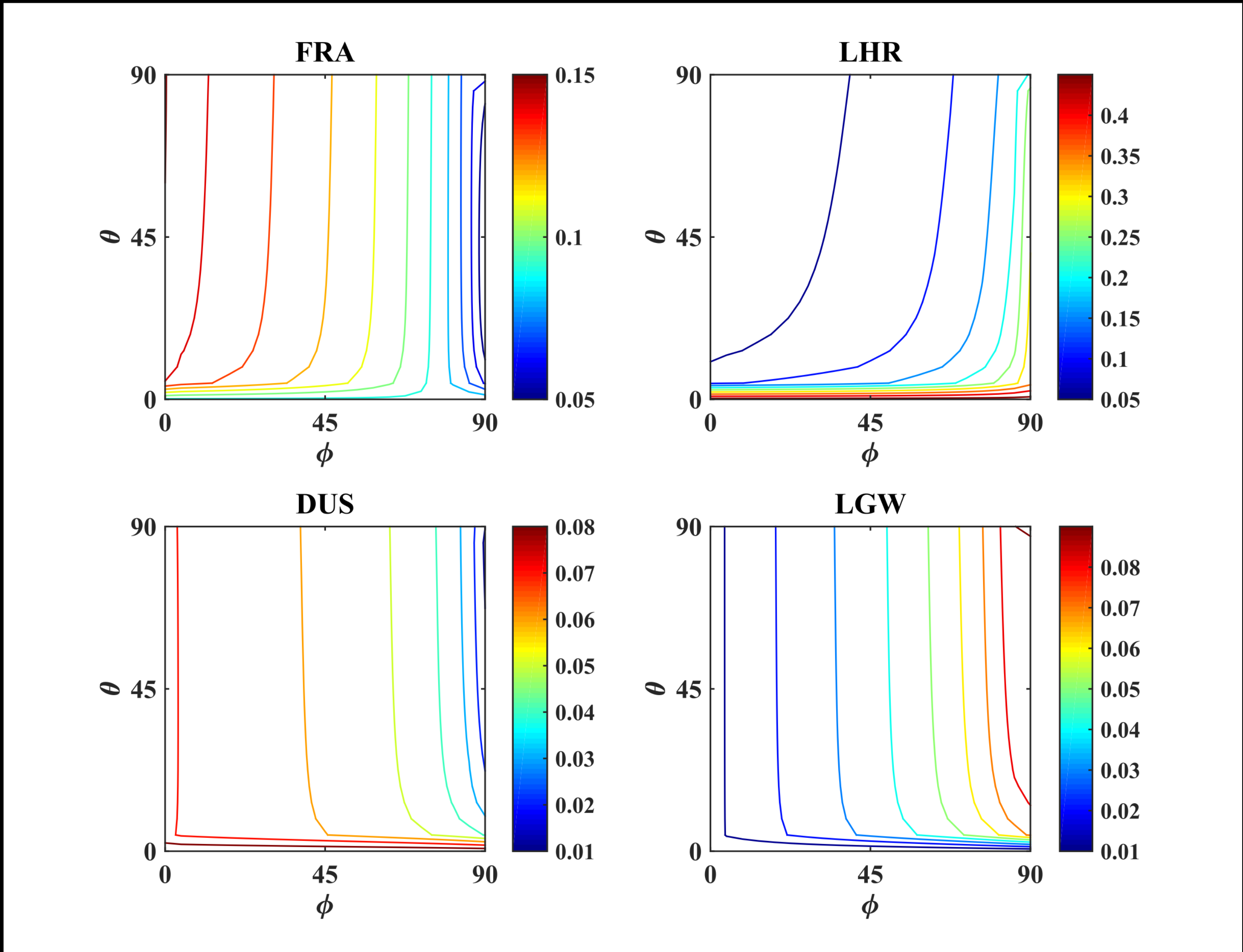
$$\theta, \phi \in [0, \pi/2]$$



Top ranked airports I in the duplex Lufthansa/British Airways network according to the Absolute Multiplex PageRank

Rank	Airport
1	Heathrow Airport (LHR)
2	Munich Airport (MUC)
3	Frankfurt Airport (FRA)
4	Gatwick Airport (LGW)

Different pattern to success of major airports



For $\phi=0^{o}$ $\theta=90^{o}$ multilinks (1,0) have major influence

For $\phi=90^{o}$ $\theta=90^{o}$ multilinks (0,1) have major influence

For $\theta=0^{o}$ multilinks (1,1) have major influence

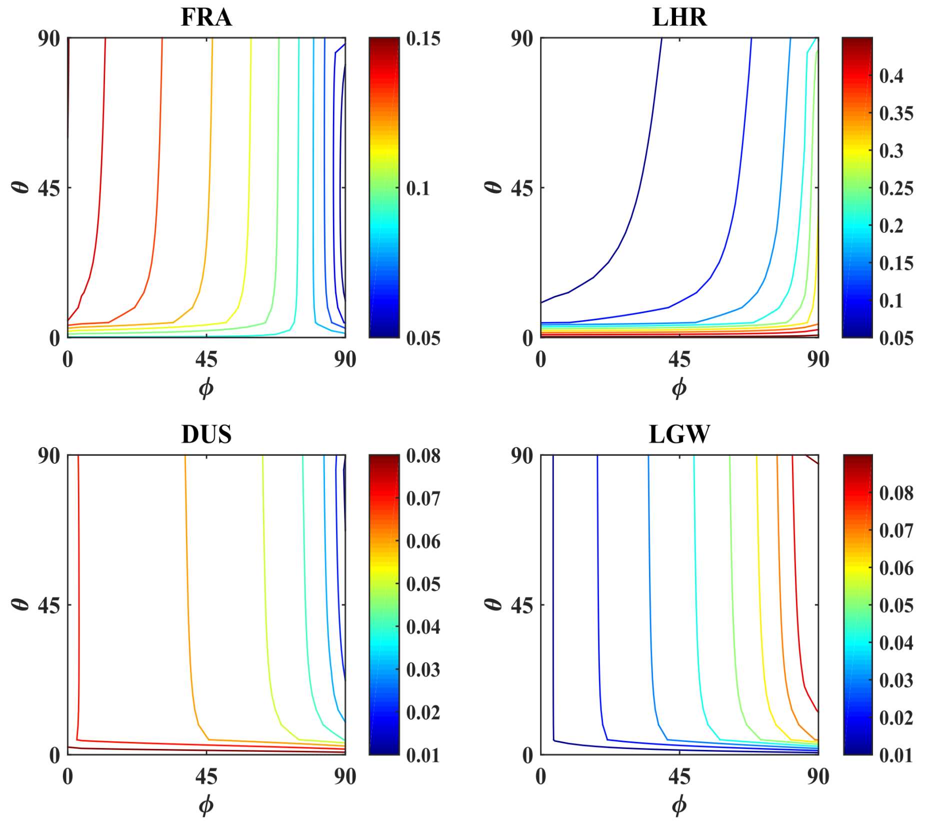
Correlations between the *pattern to success*

$$\rho_{ij} = \frac{\langle X_i X_j \rangle - \langle X_i \rangle \langle X_j \rangle}{\sigma_i \sigma_j}$$

where the average and the standard deviation are calculated on a grid (ϕ_r ,θ_s) with r=1,2,..,N_ϕ and s=1,2,...,N_θ and

$$\sigma_i = \langle X_i^2 \rangle - \langle X_i \rangle$$

Correlations between the pattern to success between major airports



ρ	LHR	FRA	LGW	DUS
LHR	1	-0.797	0.484	0.351
FRA	-0.797	1	-0.983	0.275
LGW	0.484	-0.983	1	-0.729
DUS	0.351	0.2758	-0.729	1

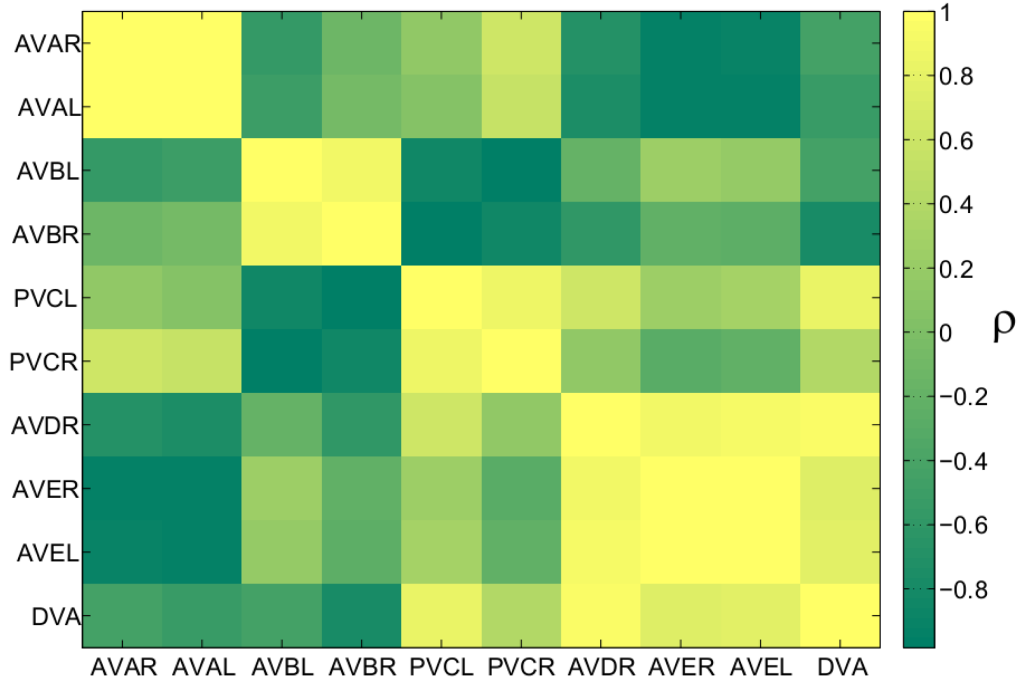
Functional Multiplex PageRank of the connectome of C.elegans

Top ranked neurons

Rank	Neuron	Rank	Neuron
1	AVAR	6	PVCR
2	AVAL	7	AVDR
3	AVBL	8	AVER
4	AVBR	9	AVEL
5	PVCL	10	DVA

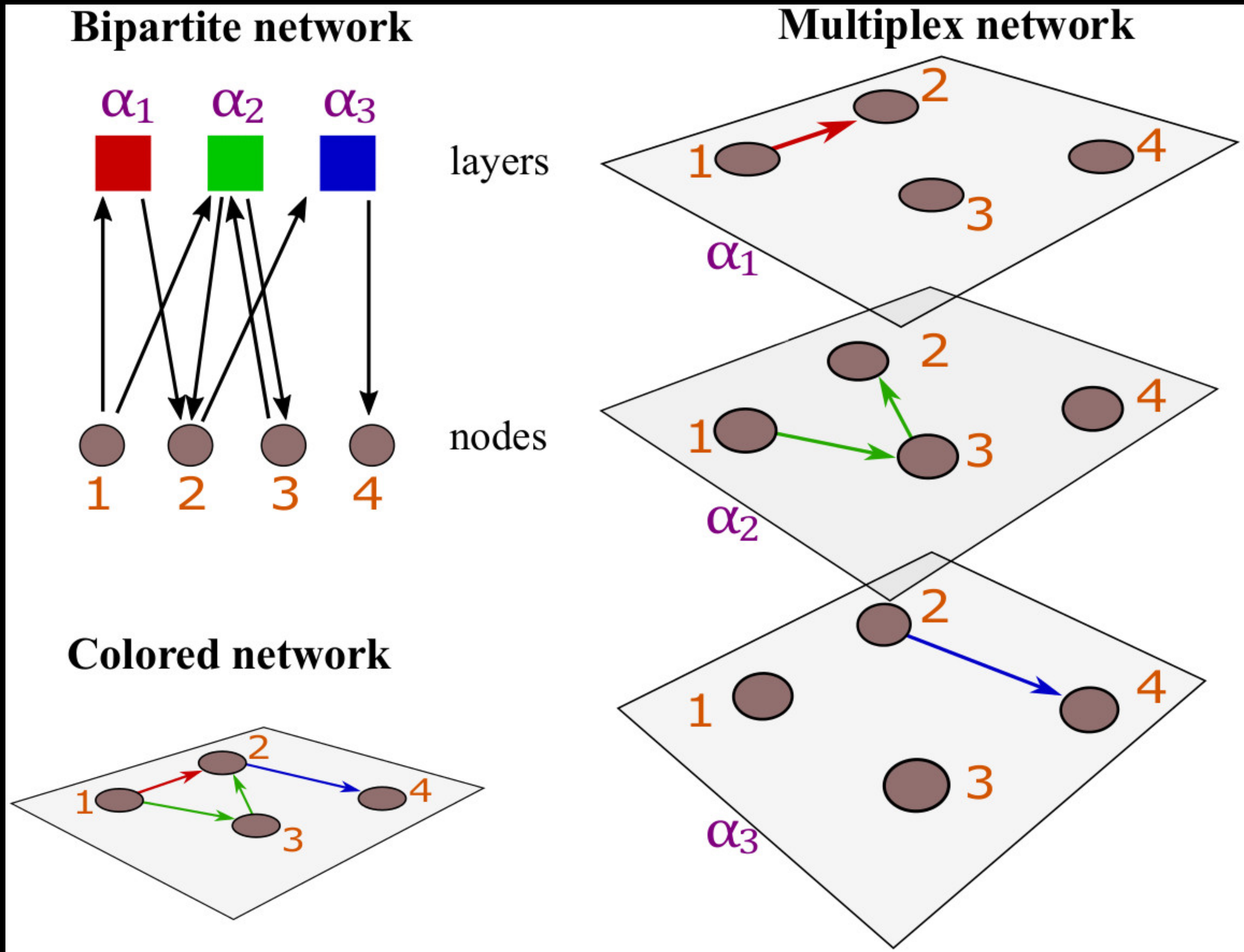
Similar neurons types have
Correlated *Multiplex PageRank*

Pearson correlations



J. Iacovacci et al. (2016)

Multiplex network representation



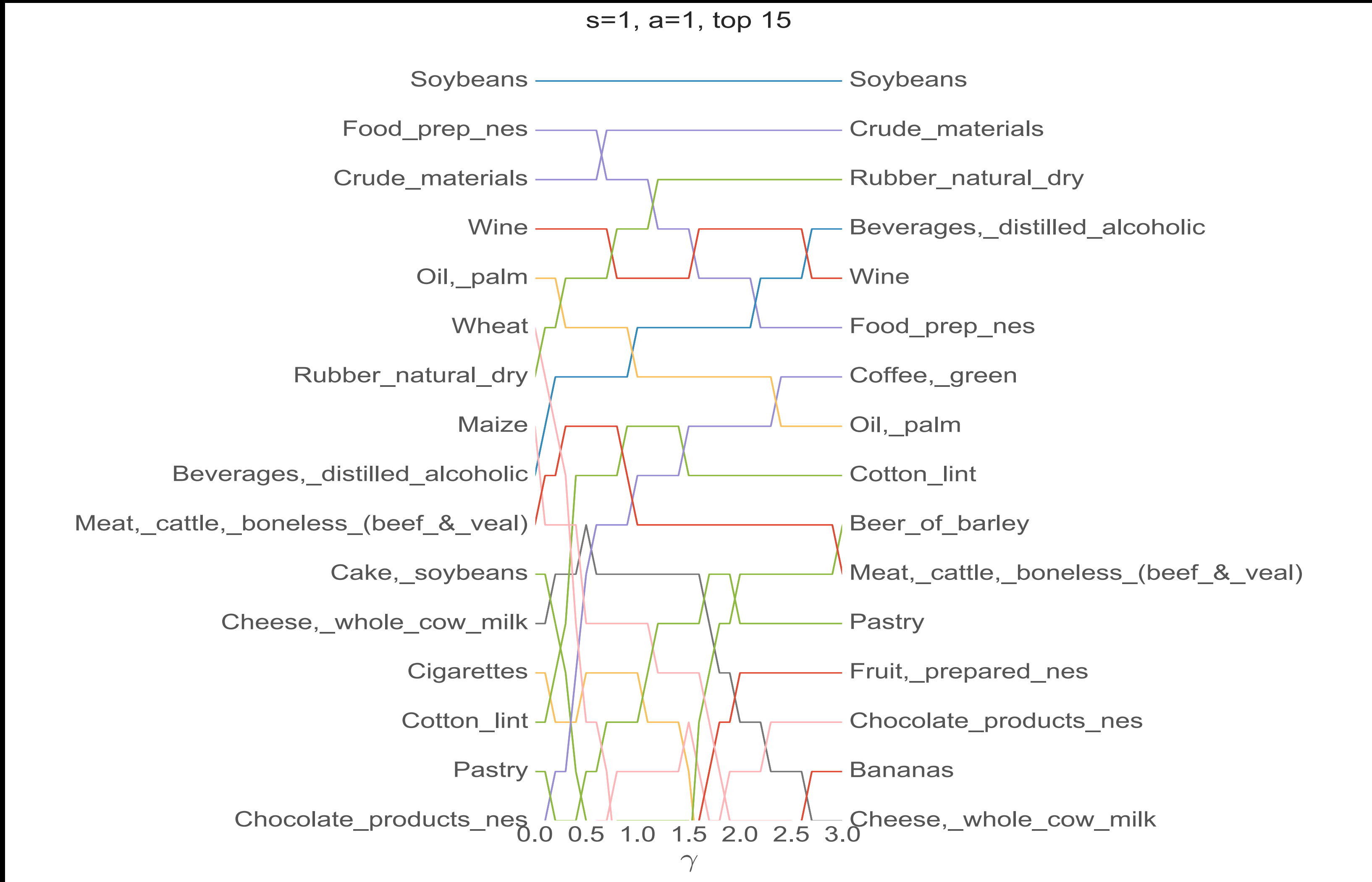
Rahmede et al. (2018)

MultiRank

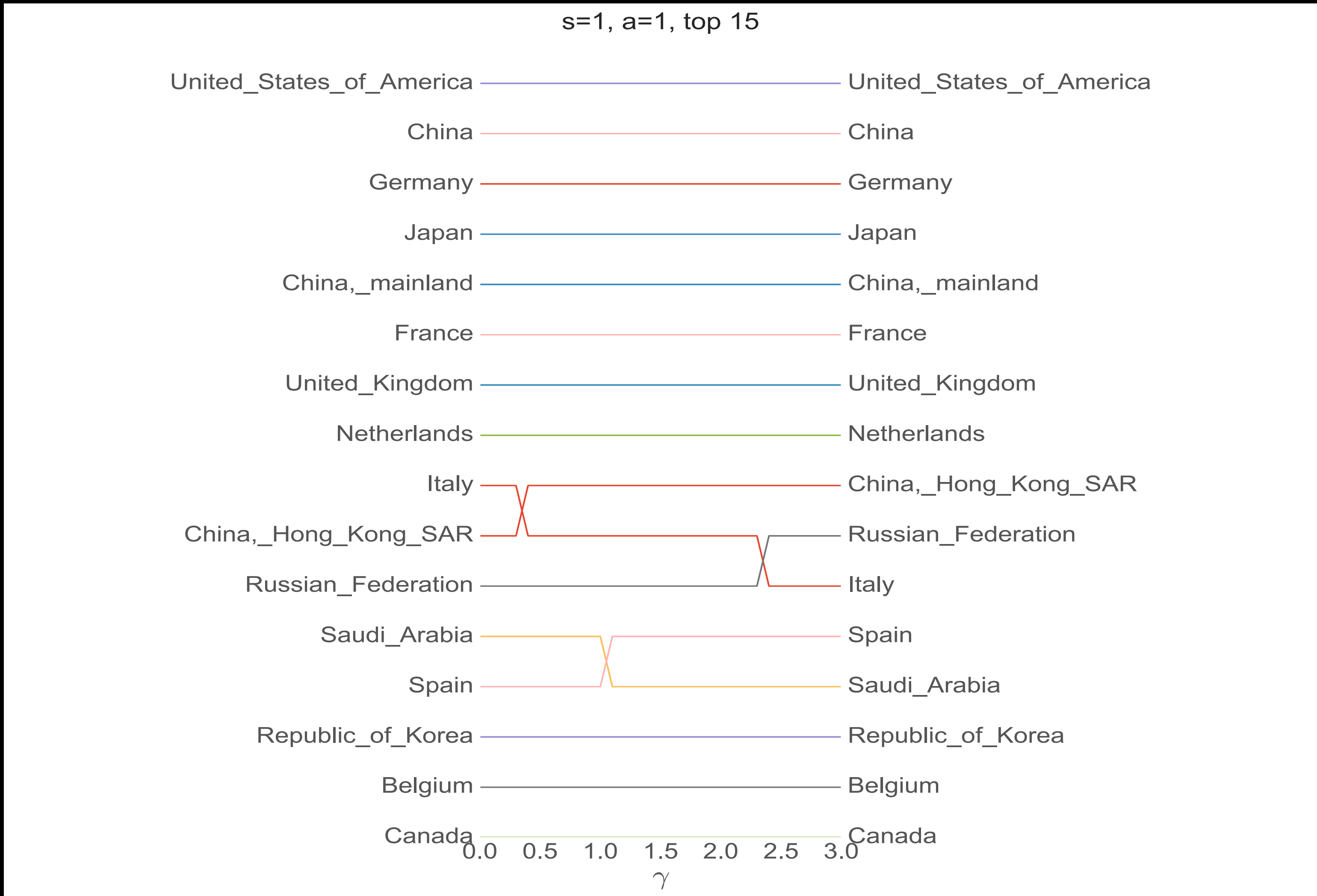
Node are central
if already central nodes point to them in very influent layers
Layers are more influent
if very central nodes are active in it

The parameter γ allows to weight differently the
contribution of different nodes to the influence of
the layers

Centrality of countries in the FAO Multiplex Trade Networks



Influences of layers in the FAO Multiplex Trade Networks



Rahmede et al. Journal Complex Networks (2018)

Multilayer networks
And Robustness

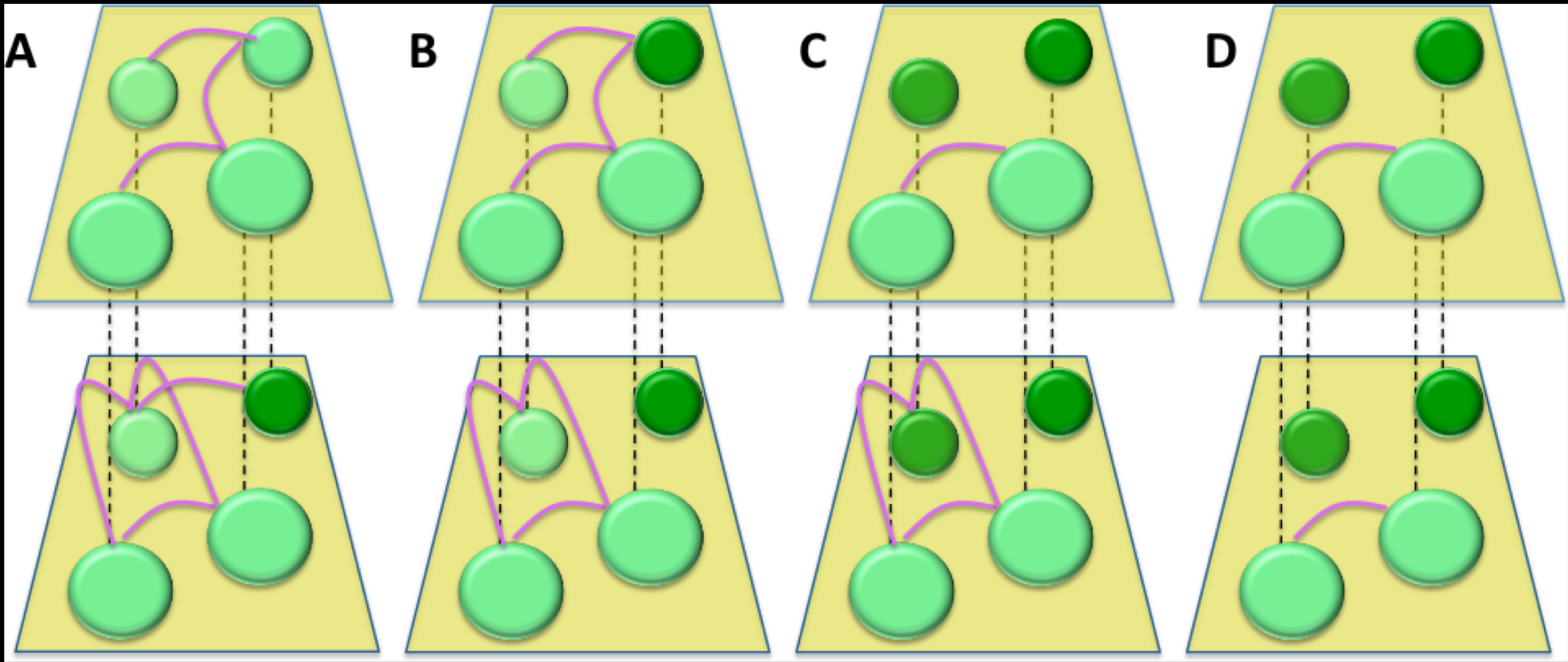
Interdependent multiplex networks

A multiplex network is
interdependent
if all the interlinks imply the interdependence of the connected replica nodes.

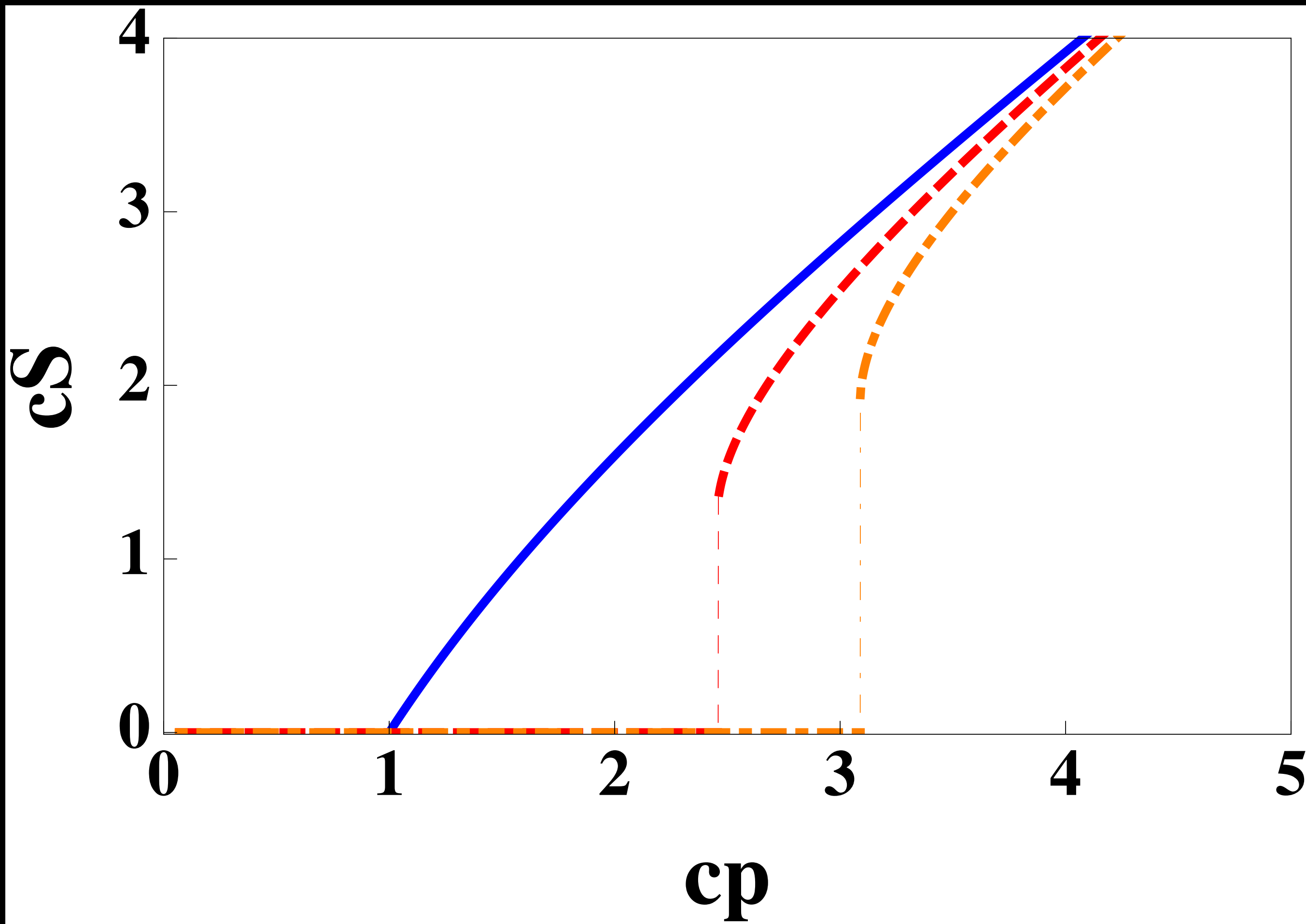
Two nodes are interdependent if the damage of one node implies the damage of the other interdependent node, independently on the rest of the network.

Mutually connected giant component

Any two nodes of the mutually connected giant component are connected by at least one path in each layer of the multiplex network



Mutual connected component of a Poisson multiplex network with no link overlap



Nature Physics News & Views

MULTILAYER NETWORKS

Dangerous liaisons?

Many networks interact with one another by forming multilayer networks, but these structures can lead to large cascading failures. The secret that guarantees the robustness of multilayer networks seems to be in their correlations.

Ginestra Bianconi

Natural complex systems evolve according to chance and necessity — trial and error — because they are driven by biological evolution. The expectation is that networks describing natural complex systems, such as the brain and biological networks within the cell, should be robust to random failure. Otherwise, they would have not survived under evolutionary pressure. But many natural networks do not live in isolation; instead they interact with one another to form multilayer networks — and evidence is mounting that random networks of networks are acutely susceptible to failure. Writing in *Nature Physics*, Saulo Reis and colleagues[1] have now identified the key correlations responsible for maintaining robustness within these multilayer networks.

In the past fifteen years, network theory[2,3] has granted solid ground to the expectation that natural networks resist failure. It has also extended the realm of robust systems to man-made self-organized networks that do not obey a centralized design, such as the Internet or the World Wide Web. In fact, it has been shown that many isolated complex biological, technological and social networks are scale free, meaning that their nodes

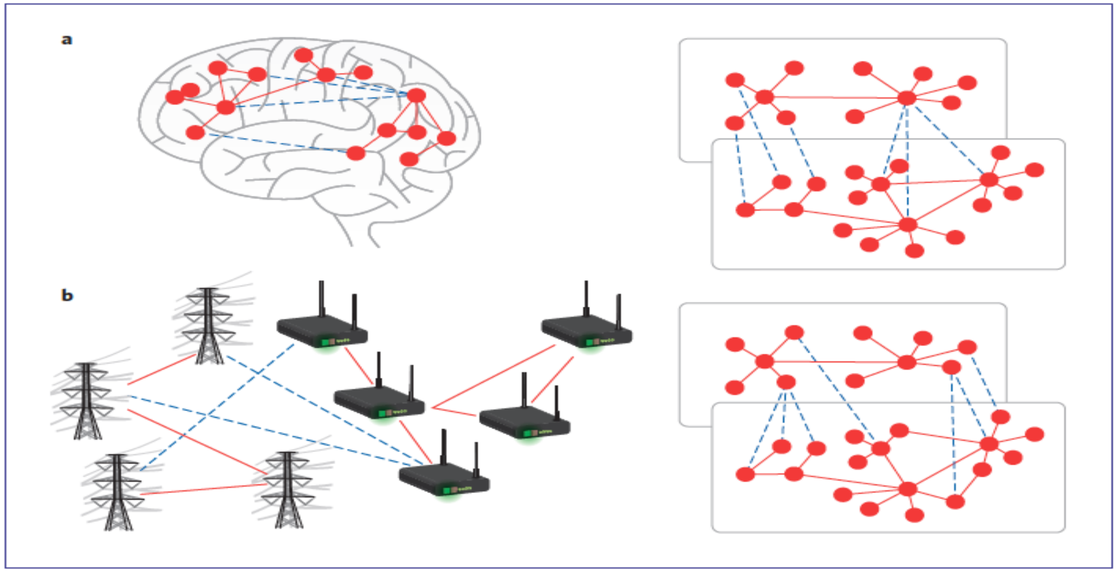


Figure 1 | Reis *et al.*[1] have shown that correlations between intra- (red) and interlayer (blue dotted) interactions influence the robustness of multilayer networks. **a**, In the brain, each network layer has multilayer assortativity and the hubs in each layer are also the nodes with more interlinks, so liaisons between layers are trustworthy. **b**, In complex infrastructures (such as power grids and the Internet), if the interlinks are random, the resulting multilayer network is affected by large cascades of failures[6], and liaisons can be considered dangerous.

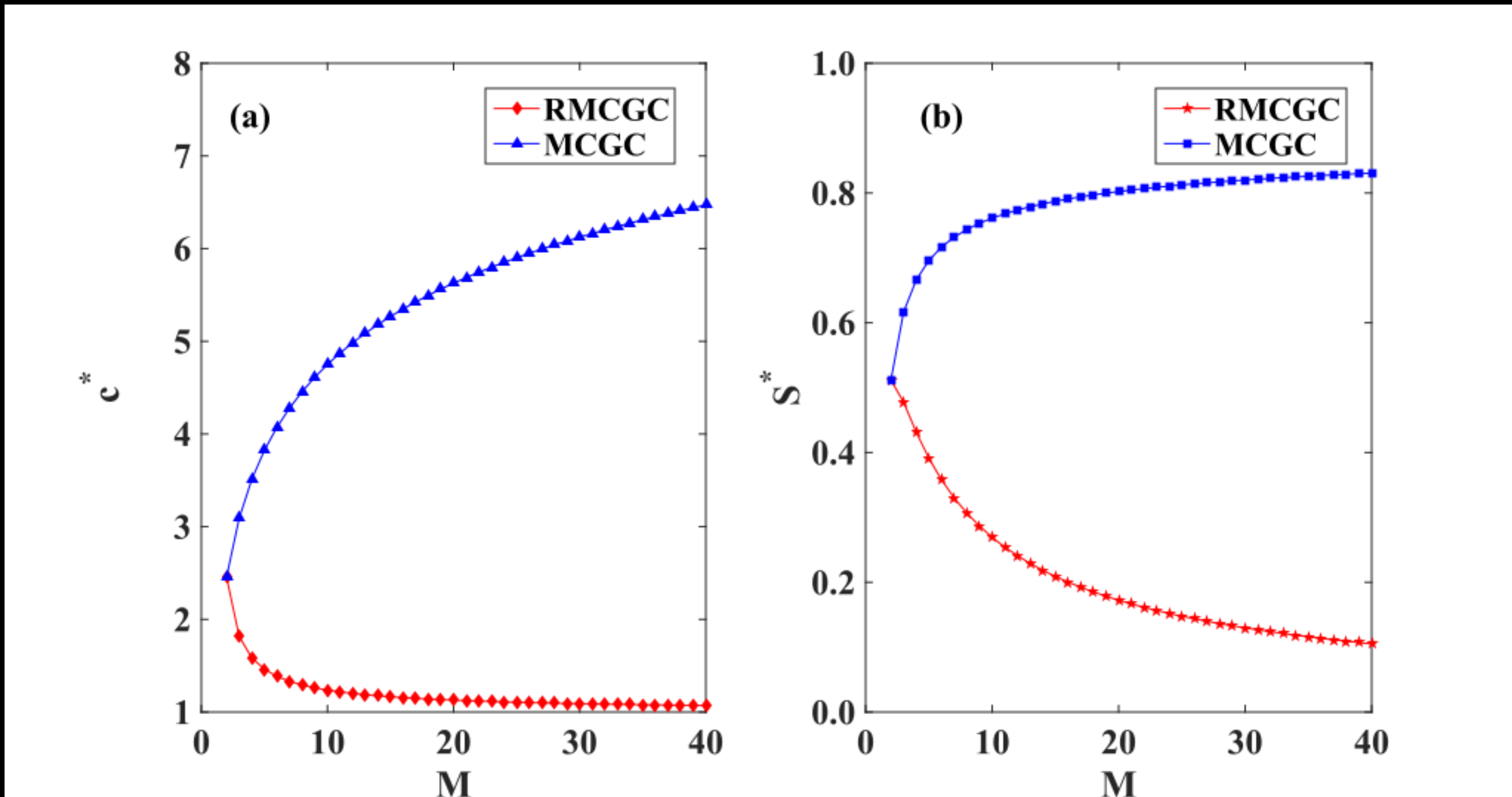
How to boost
the robustness of multiplex network?

Redundant interdependencies

Is a multiplex network more or less robust if we add new layers?

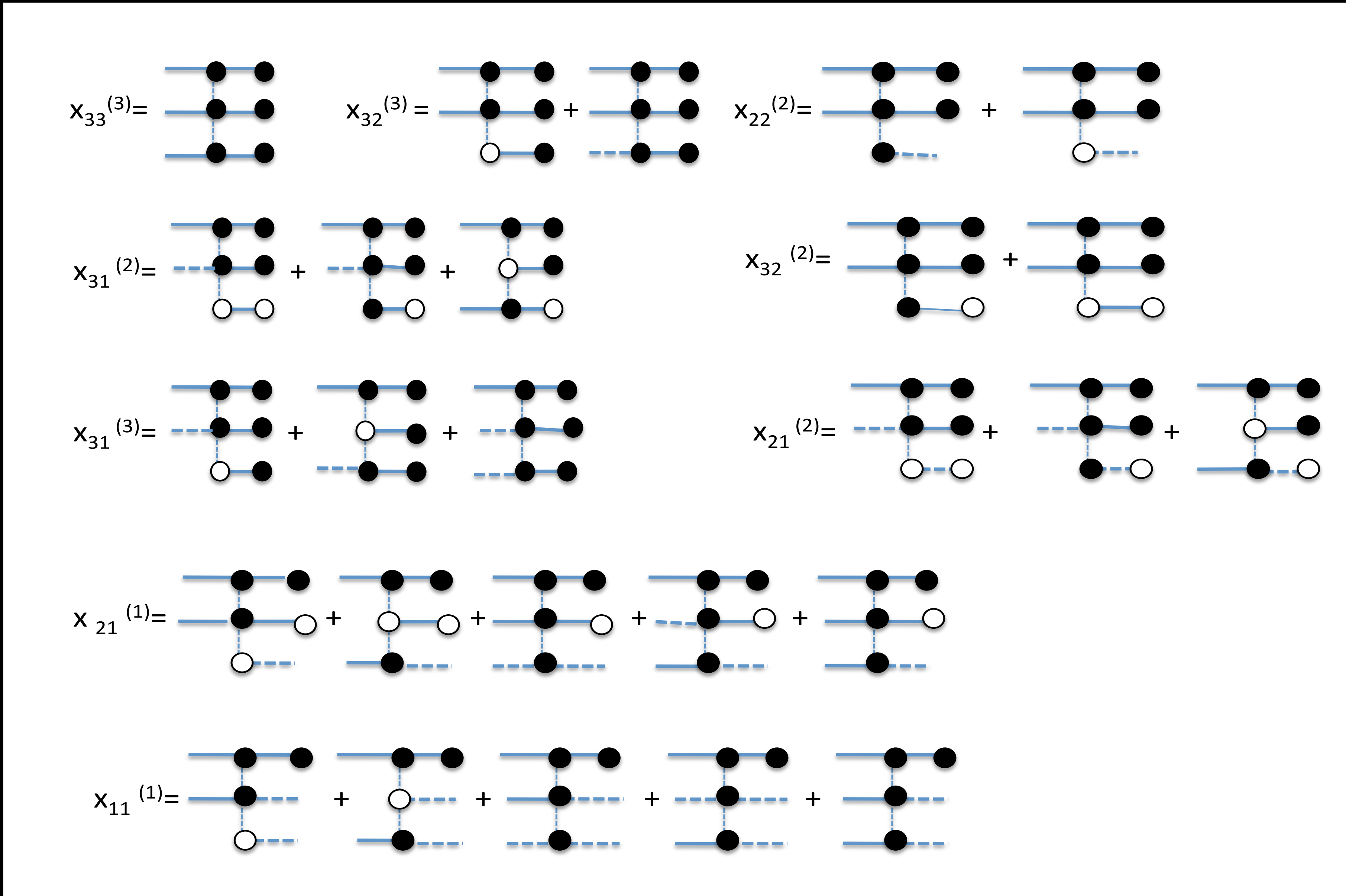
If interdependencies are redundant and a node can be in the Redundant MCGC as long as at least one replica node is active, then the more layers we add to the network the more robust it becomes

Redundant Mutually Connected Giant Component

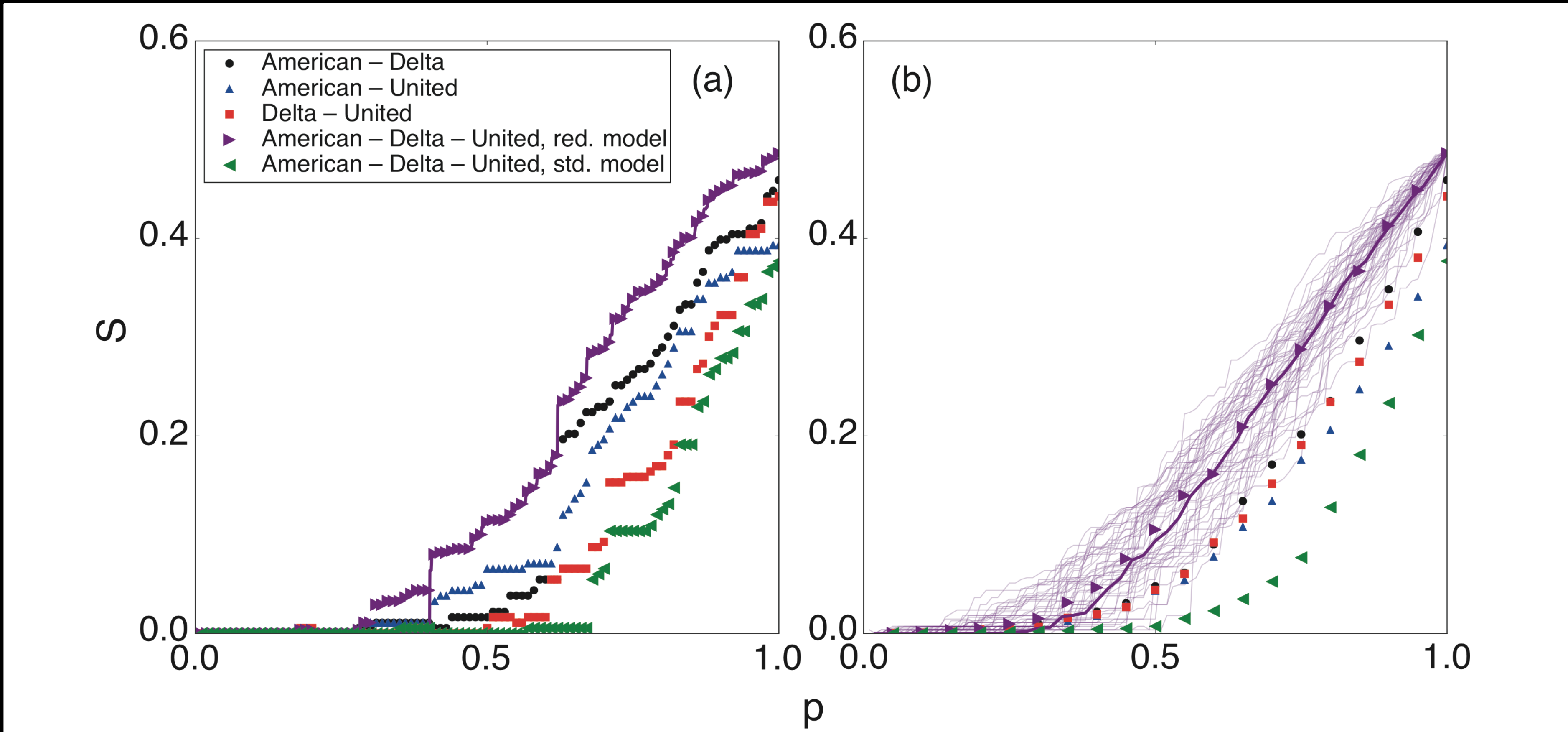


Radicchi Bianconi PRX (2017)

Equations



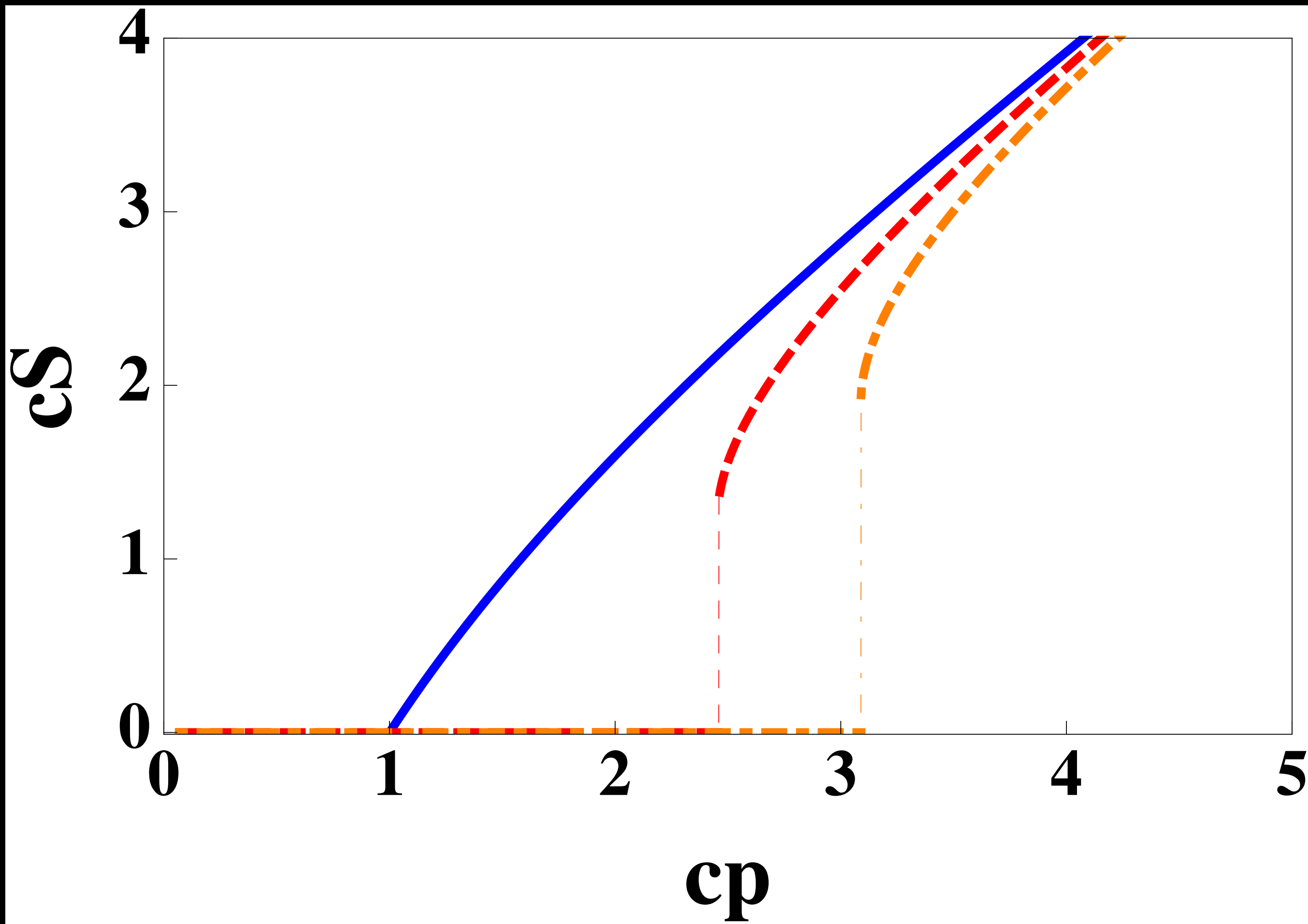
Redundant interdependencies boost the robustness of multilayer networks



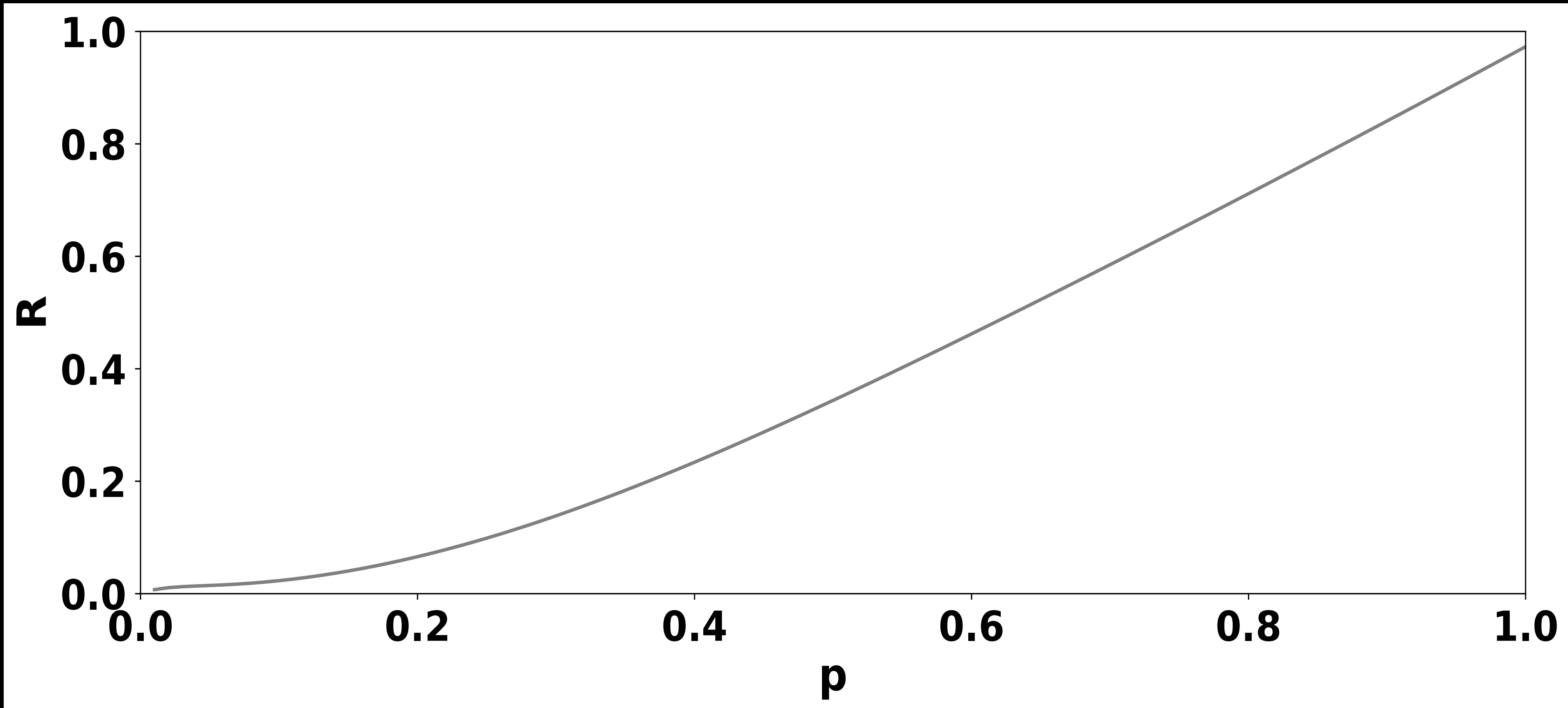
Radicchi Bianconi PRX (2017)

How can we assess the risk of dramatic failures in real finite multiplex networks?

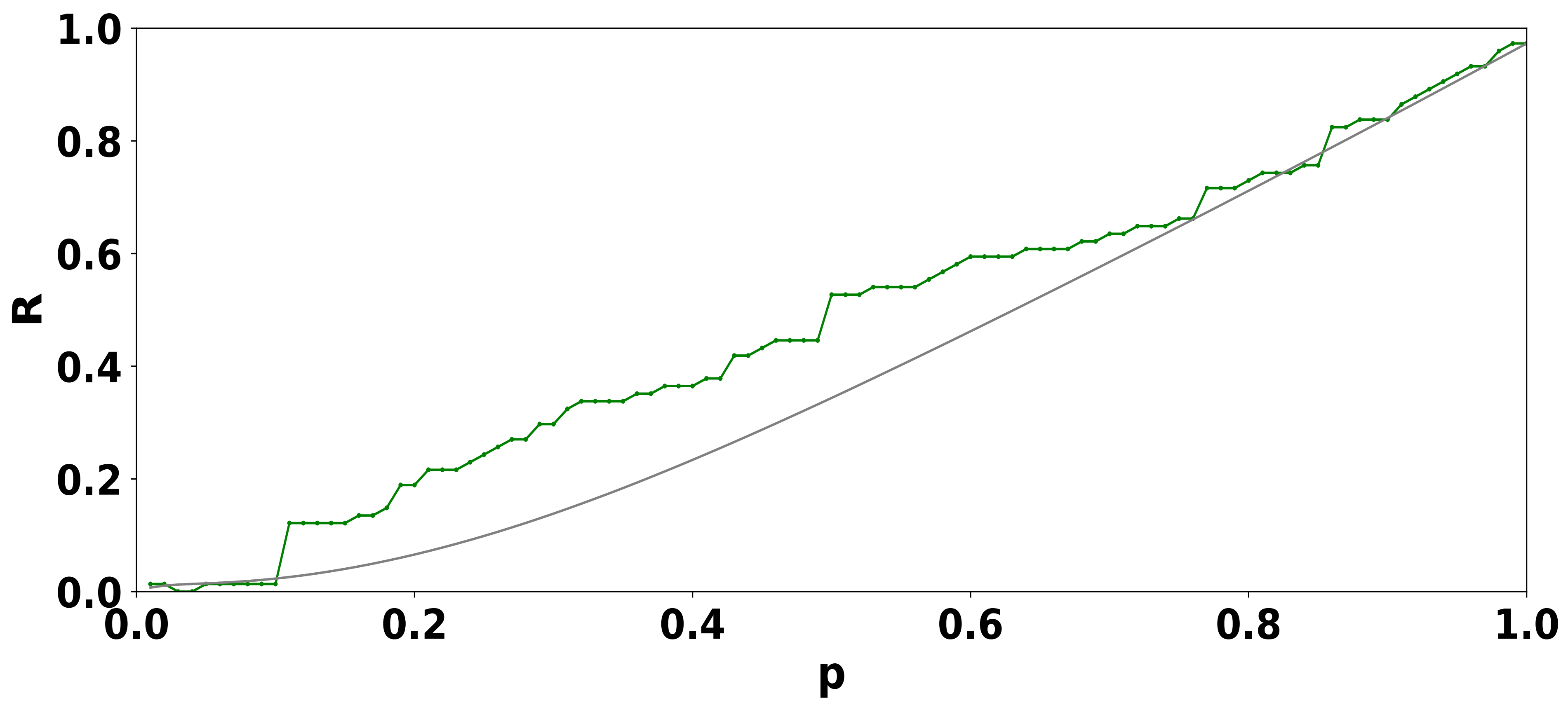
Mutual connected component of a Poisson multiplex network with no link overlap



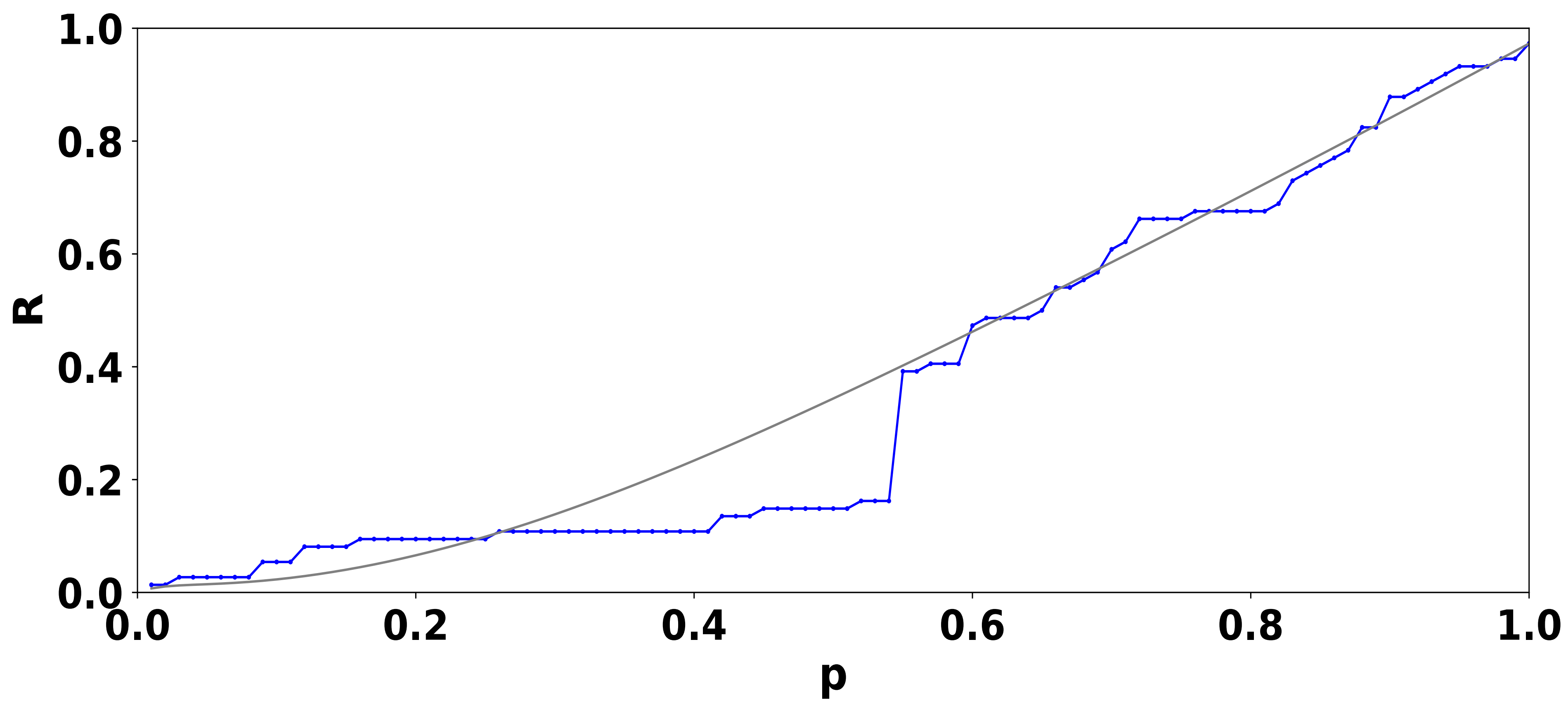
Mean-field approach to percolation



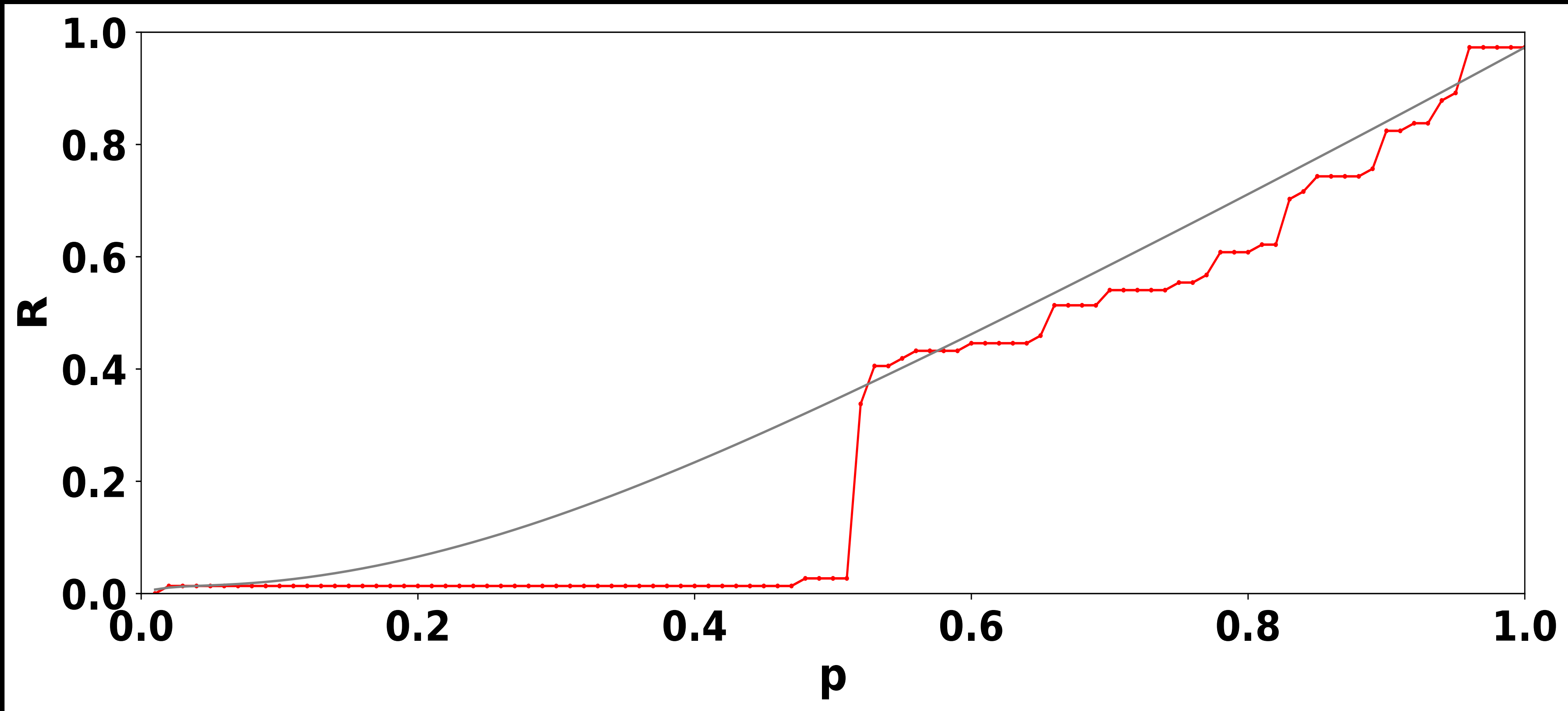
Single instance of percolation



Single instance of percolation



Single instance of percolation



Large deviation approach to percolation

The full distribution π (R) of the relative size R of the MCGC is measured for every value of p

$$\pi(R) = \frac{1}{Q\Delta R}\sum_{\mu=1}^{Q}\delta(R, R^{\mu})$$

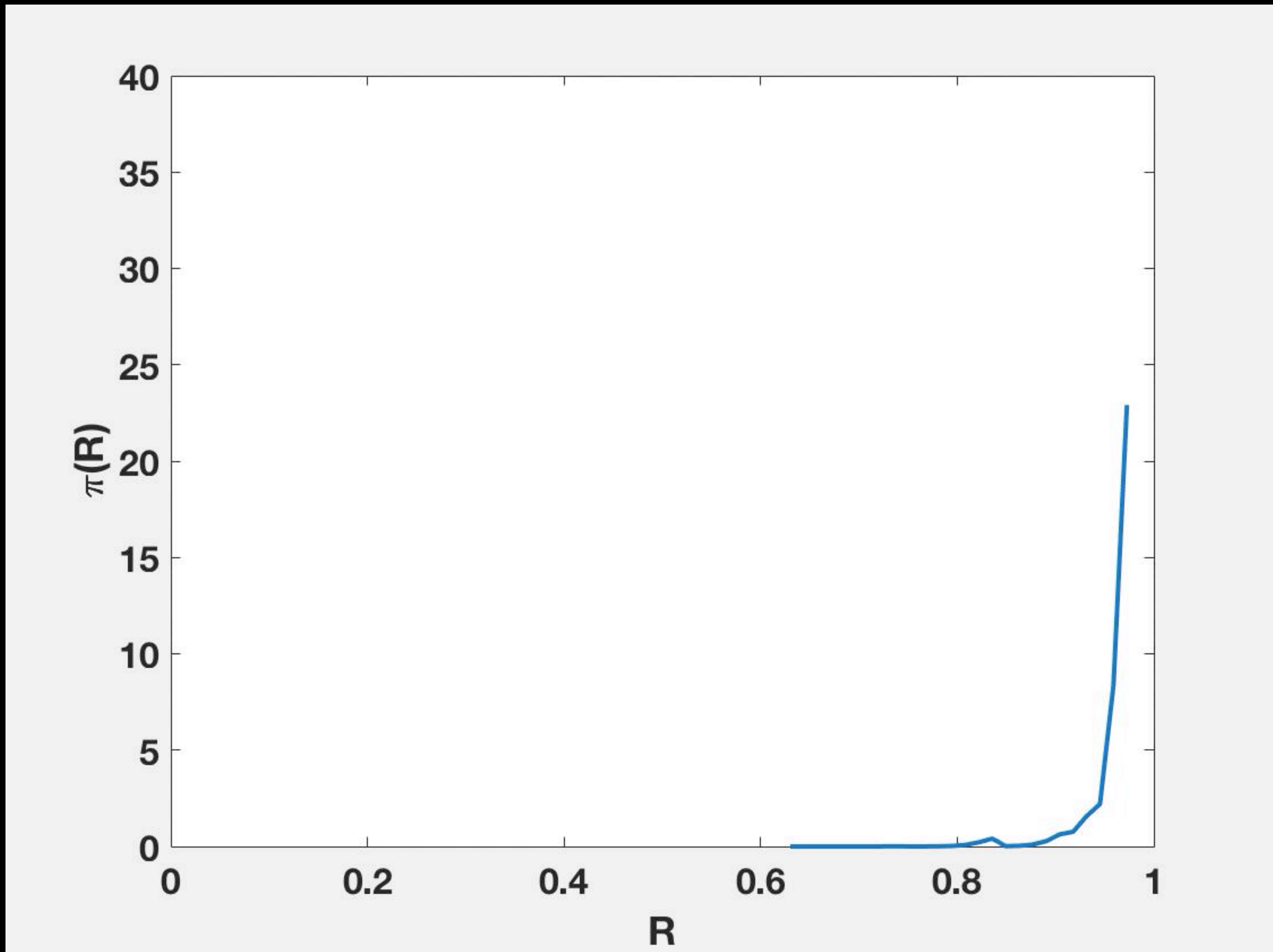
The average and the typical relative size of the MCGC are measured for every value of p

$$\bar{R} = \frac{1}{Q}\sum_{\mu=1}^{Q} R^{\mu}$$

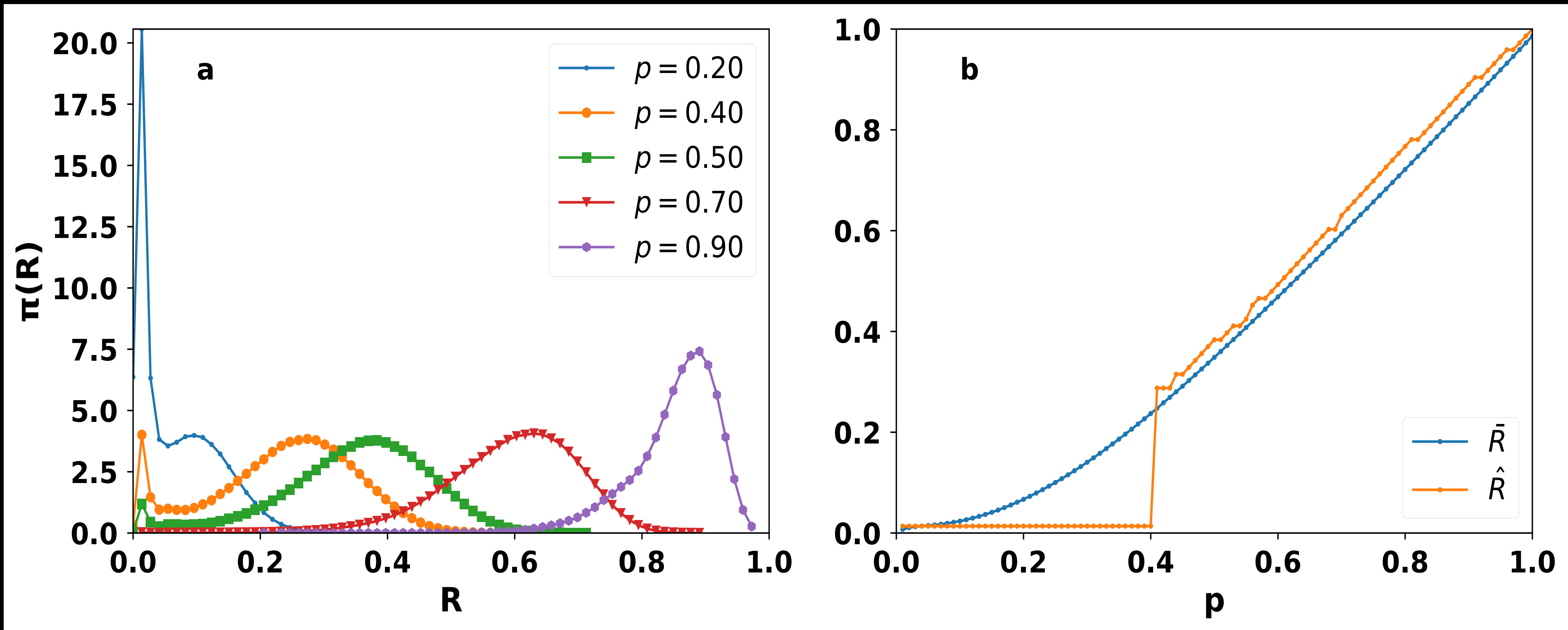
$$\hat{R} = \text{argmax}_R \pi(R)$$

Coghi Radicchi Bianconi (2018)

Distribution of the relative size of the MCGC



The typical size of the MCGC



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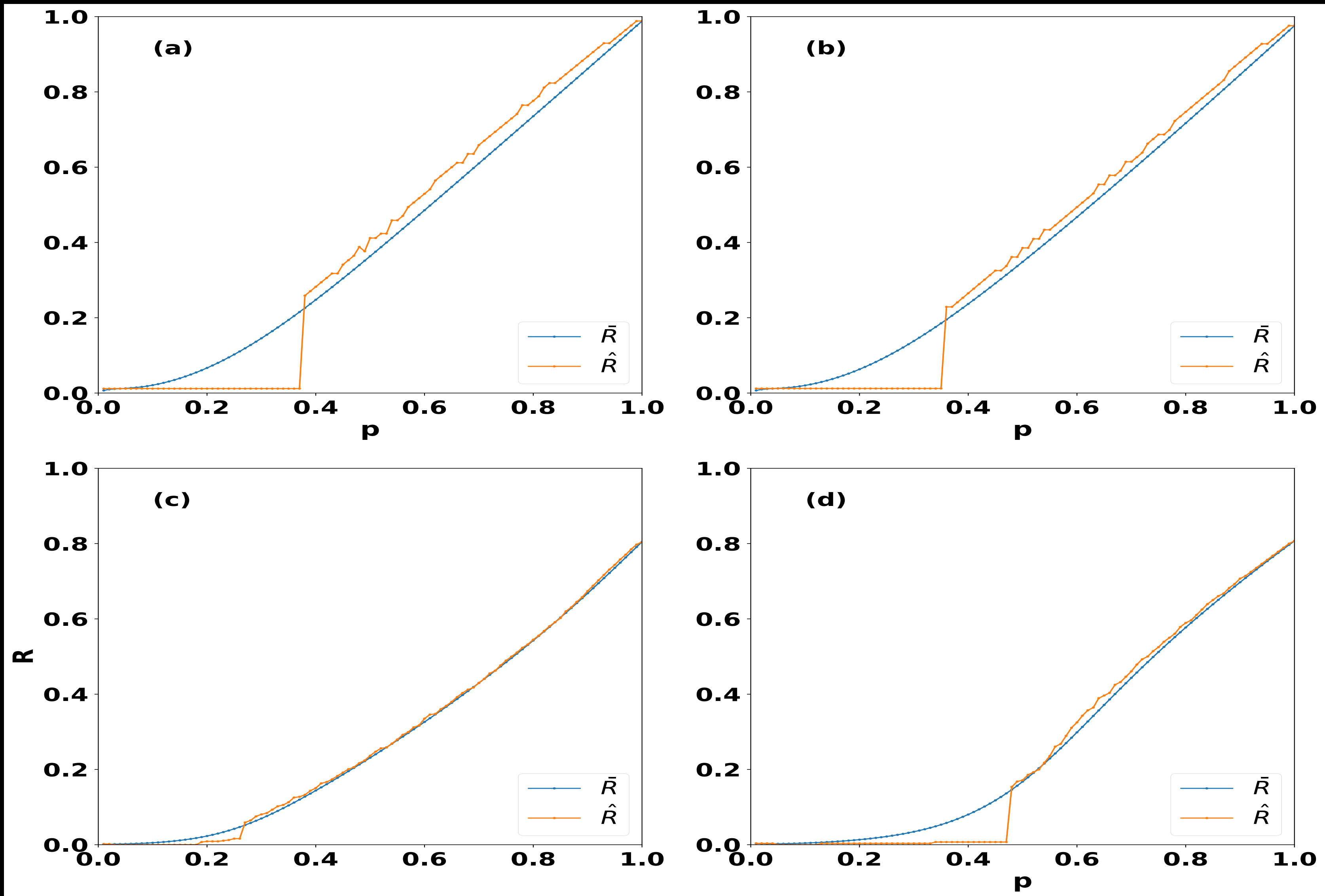
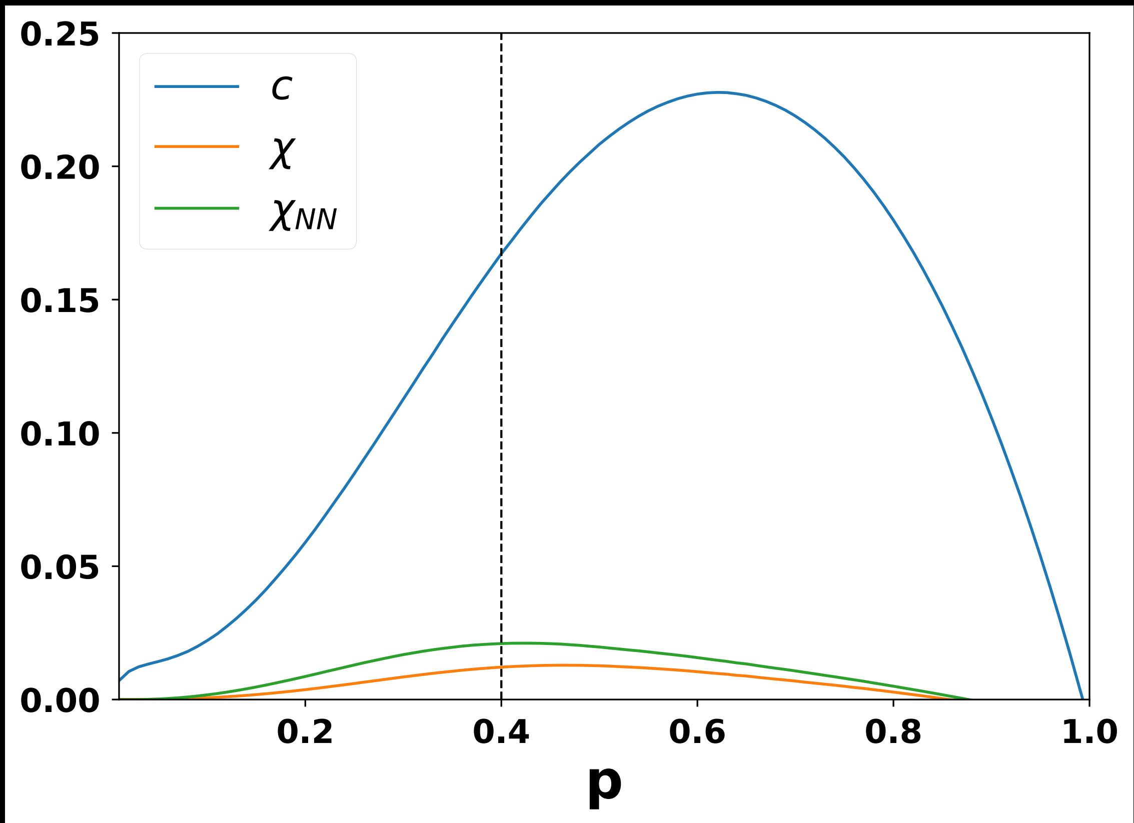
Average and Typical size of the MCGC



Specific heat, Correlations, Susceptibility



$$c = \frac{1}{N}\sum_i \langle\sigma_i\rangle(1 - \langle\sigma_i\rangle)$$

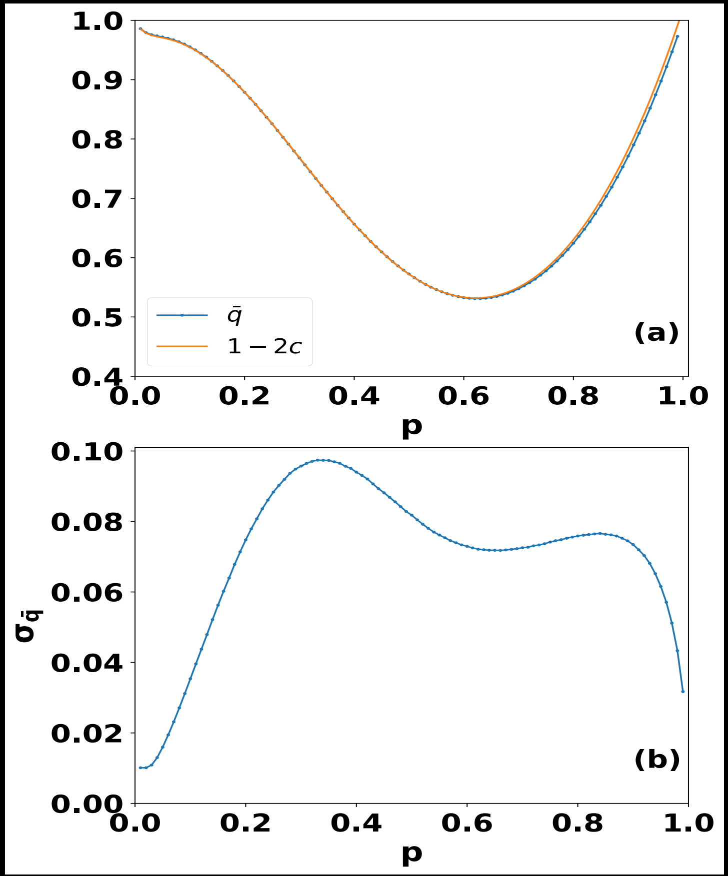
$$\chi_{NN} = \frac{1}{\langle k\rangle N}\sum_{<i,j>}\left(\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_i\rangle\right)$$

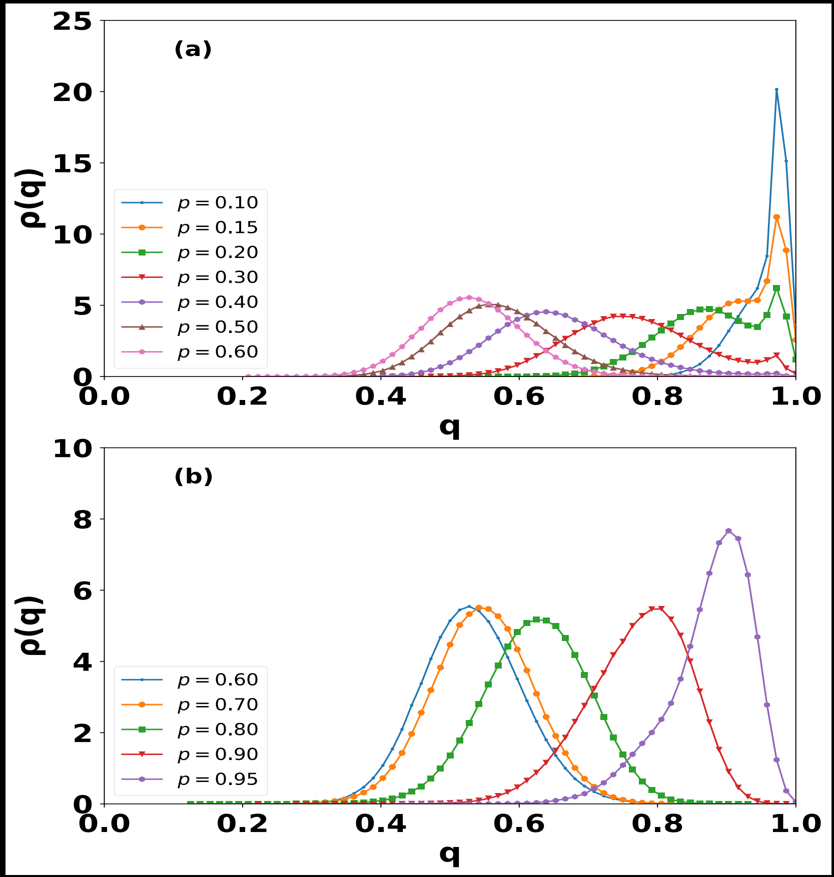
$$\chi = \frac{1}{N(N-1)}\sum_{i,j}\left(\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_i\rangle\right)$$

Coghi Radicchi Bianconi (2018)

Overlap of configurations

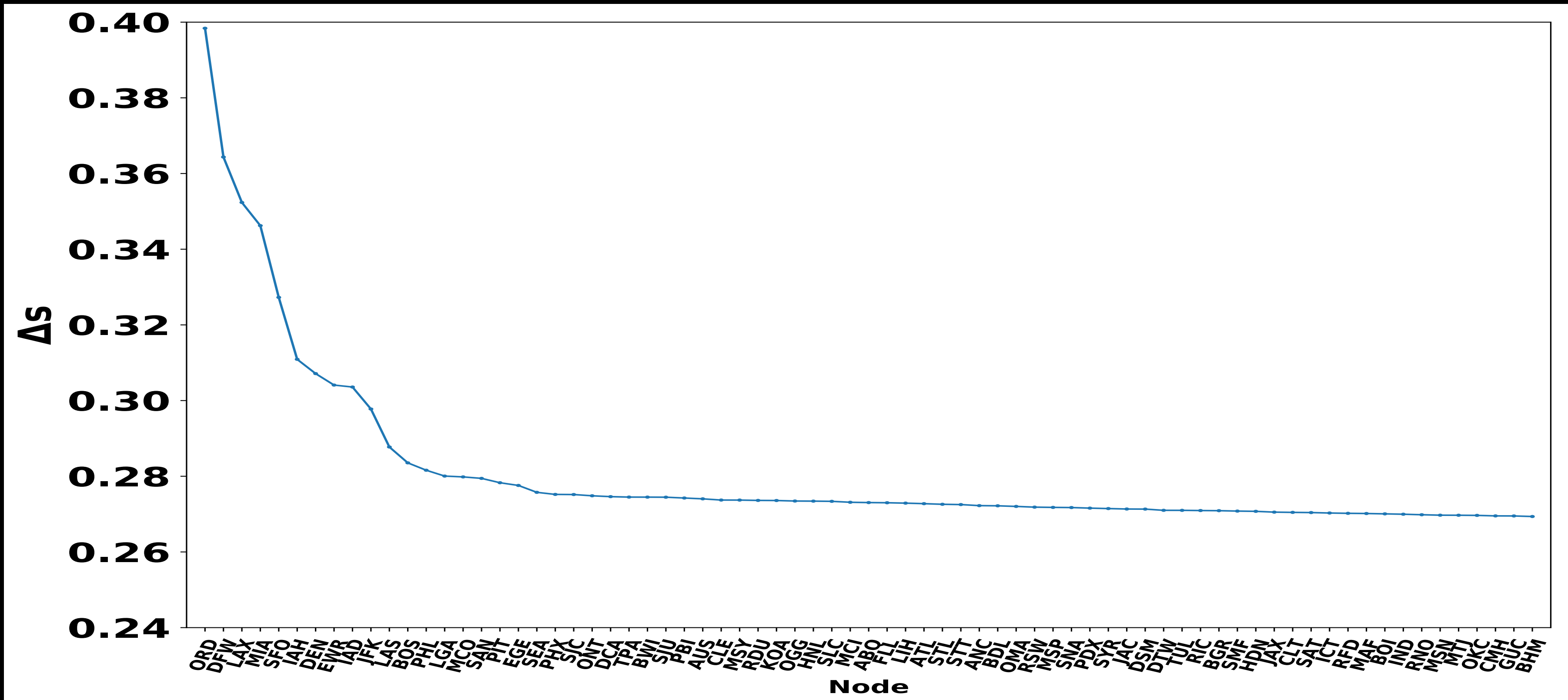
$$q^{\mu,\nu} = \frac{1}{N}\sum_{i=1}^{N}\left[\sigma_i^{\mu}\sigma_i^{\nu} + (1-\sigma_i^{\mu})(1-\sigma_i^{\nu})\right]$$





Safeguard centrality

$$\Delta s_i = \frac{1}{Q} \sum_{\mu=1}^{Q} s_i \left[\theta(R^{\mu} - R^{\star}) - \theta(R^{\star} - R^{\mu})\right]$$



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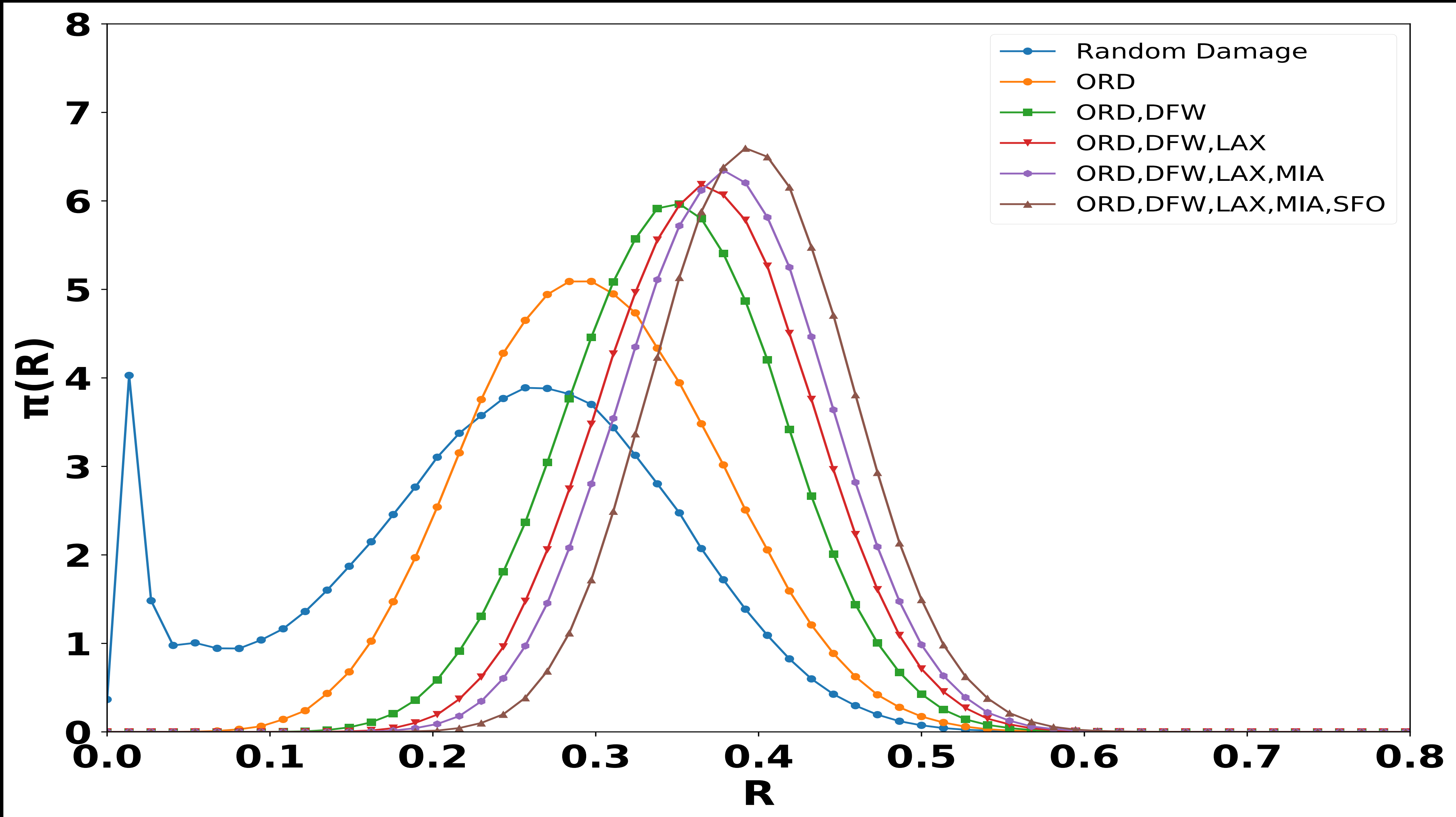
Safeguard central nodes



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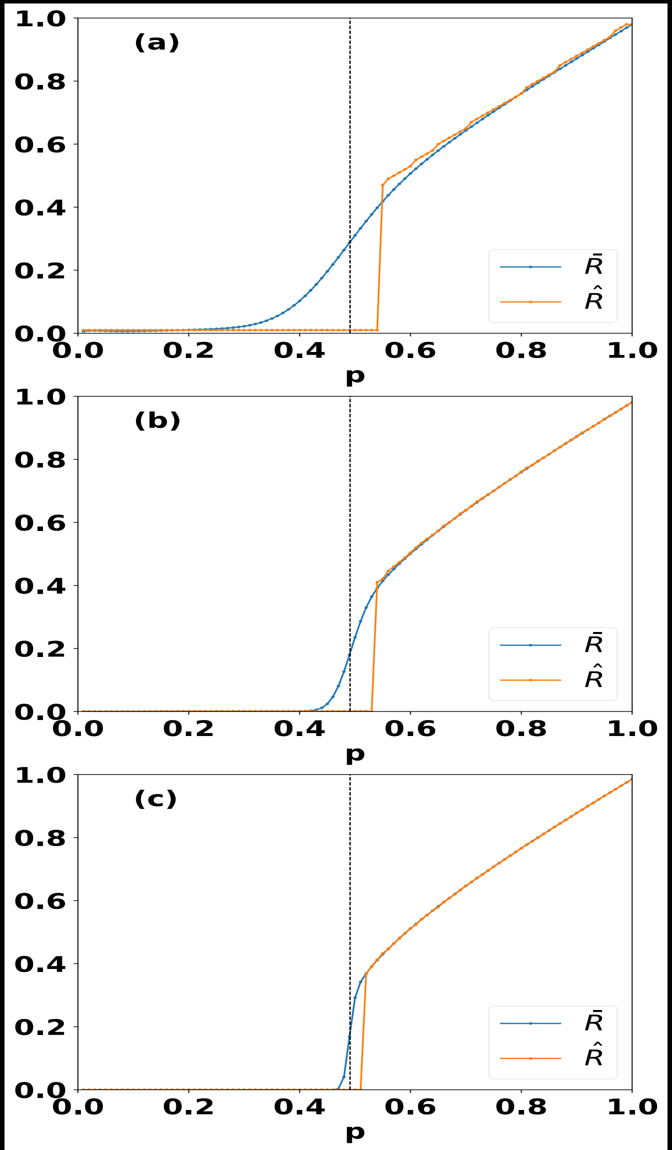
Finite size effects on Poisson duplex networks



N=10^2

N=10^3

N=10^4

Conclusions

Multilayer networks are able to capture the complexity of systems where links have different connotations

Multilayer networks allow encode more information than its single layers taken in isolation

Multlilinks detect important information and they can be correlated with the weights of the links.

Multilinks are central to determine the centrality of the nodes in the Functional Multiplex PageRank

Redundant interdependencies are fundamental design principles to book the robustness of multiplex networks

New large-deviation theory of percolation can capture the fragility of real multiplex networks and the safeguard centrality can be used to suppress the role of dramatic cascades of failures

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Codes and collaboration

Codes are available from my GitHub page
https://github.com/ginestrab

G.Bianconi

J. Iacovacci

A.Arenas

C. Rahmede

F. Radicchi