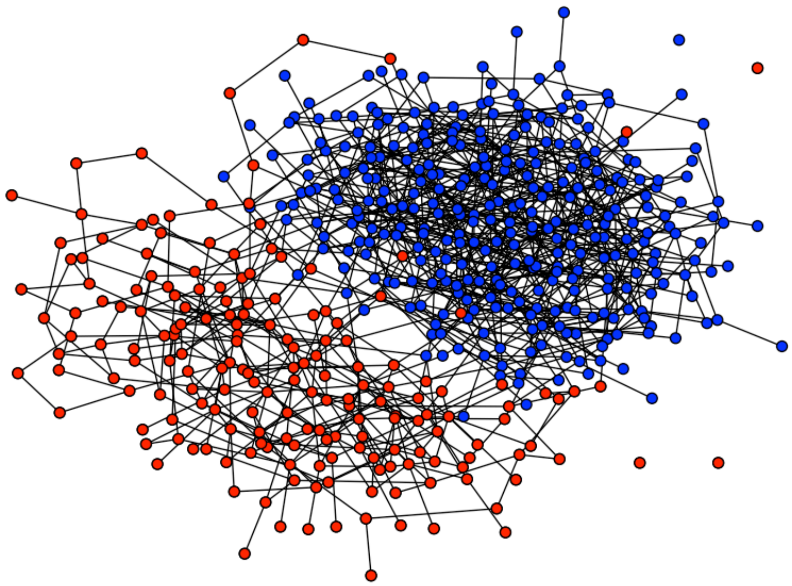
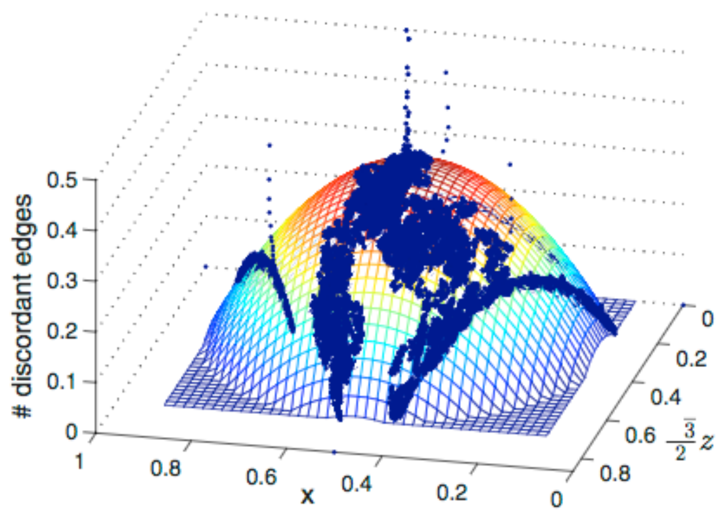




Coevolving voter models

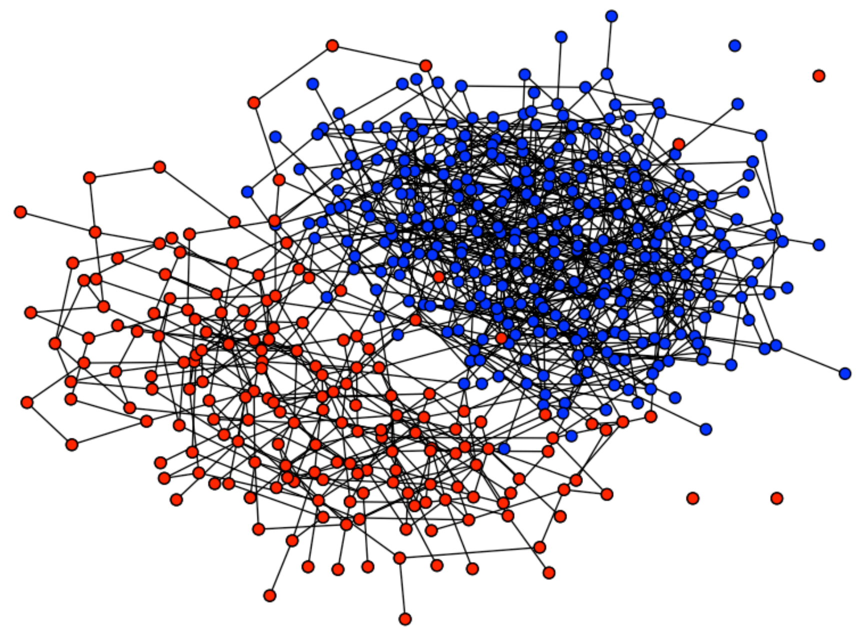
Peter J. Mucha
University of North Carolina at Chapel Hill

Durrett *et al.*, "Graph fission in an evolving voter model" (2012)
[variations on coevolving model of Holme & Newman (2006)]

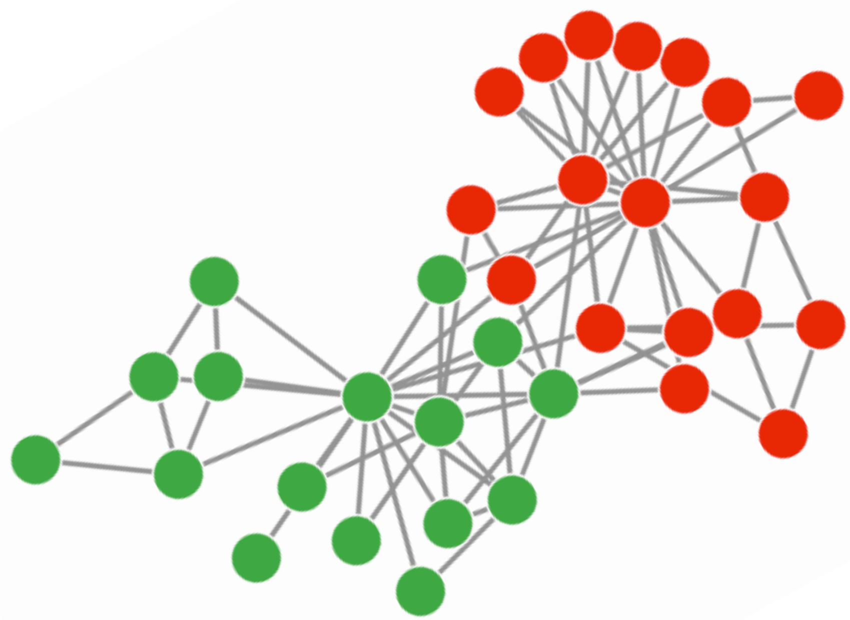
Two-opinion model ("0/1", "B/R", "S/I")

While there are discordant edges (i.e. 0-1)
    Select discordant edge (uniformly)
    w/probability (1-α)
        Voter step (change to 0-0 or 1-1)
    else (that is, w/probability α)
        Rewire*
End

*We consider two different rewiring rules. Both start by selecting one end to unilaterally drop the connection, then "Rewire-to-Random" v. "Rewire-to-Same"

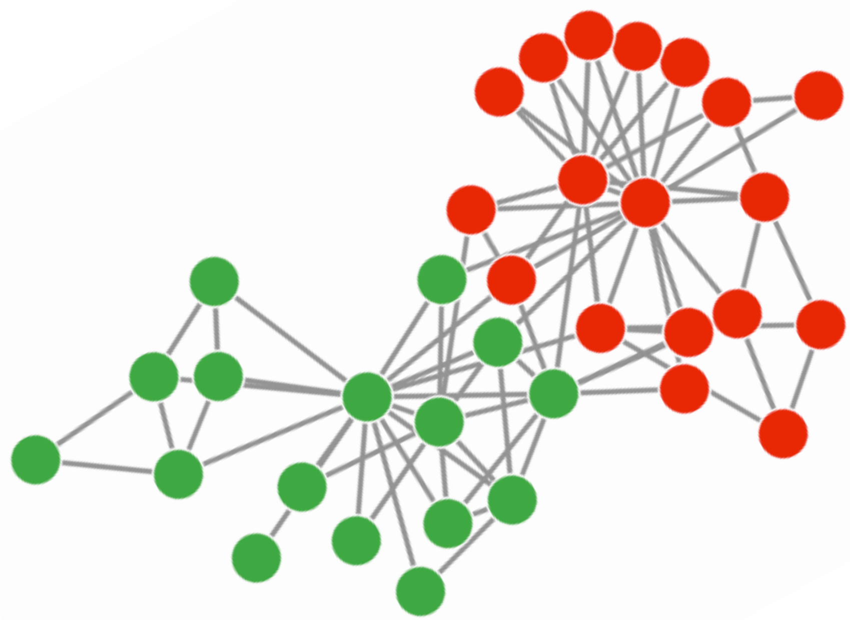


$\alpha = 0$: Classic Voter Model



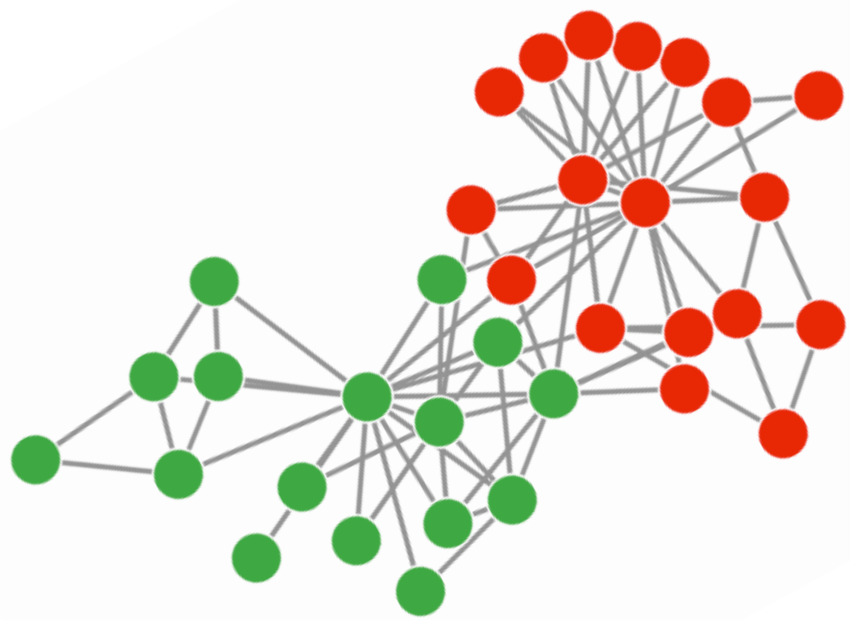
webweb, D. Larremore, M. Iuzzolino, H. Wapman
https://webwebpage.github.io/

$\alpha = 1$: Rewiring (simple and fast)



webweb, D. Larremore, M. Iuzzolino, H. Wapman
https://webwebpage.github.io/

$\alpha = 0.6$:

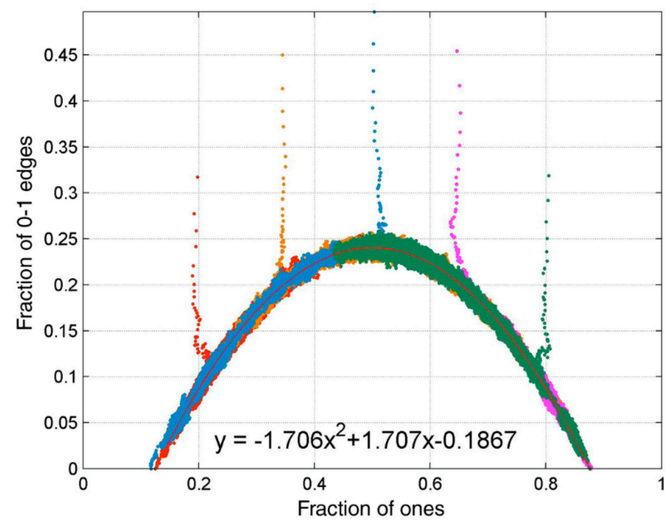


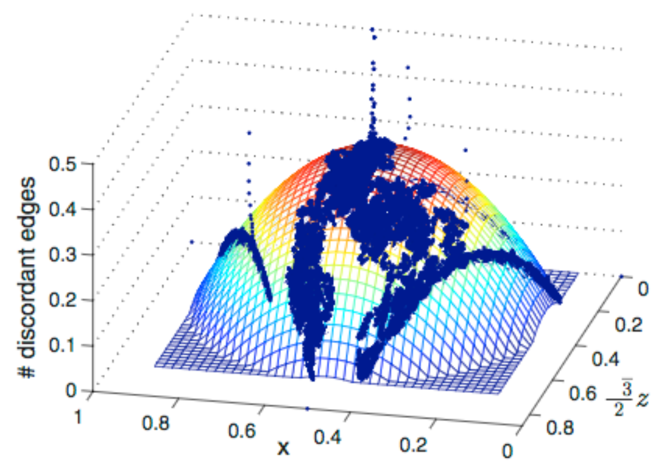
webweb, D. Larremore, M. Iuzzolino, H. Wapman
https://webwebpage.github.io/

Outline

1. The predictable yet puzzling coevolving voter model
2. Approximate equations
   - Pair Approximation (PA)
   - Approximate Master Equations (AME)
   - Random Mutations
3. Reinforced transitivity

Funding: NSF, NIGMS, NICHD





Outline

1. The predictable yet puzzling coevolving voter model
2. Approximate equations
   – Pair Approximation (PA)
   – Approximate Master Equations (AME)
   – Random Mutations
3. Reinforced transitivity

Funding: NSF, NIGMS, NICHD

Graph fission in an evolving voter model

Richard Durrett[a,1], James P. Gleeson[b], Alun L. Lloyd[c,d], Peter J. Mucha[e], Feng Shi[e], David Sivakoff[a], Joshua E. S. Socolar[f], and Chris Varghese[f]

Multiopinion coevolving voter model with infinitely many phase transitions

Feng Shi,[1] Peter J. Mucha,[1] and Richard Durrett[2]

LOCAL SYMMETRY AND GLOBAL STRUCTURE IN ADAPTIVE VOTER MODELS*

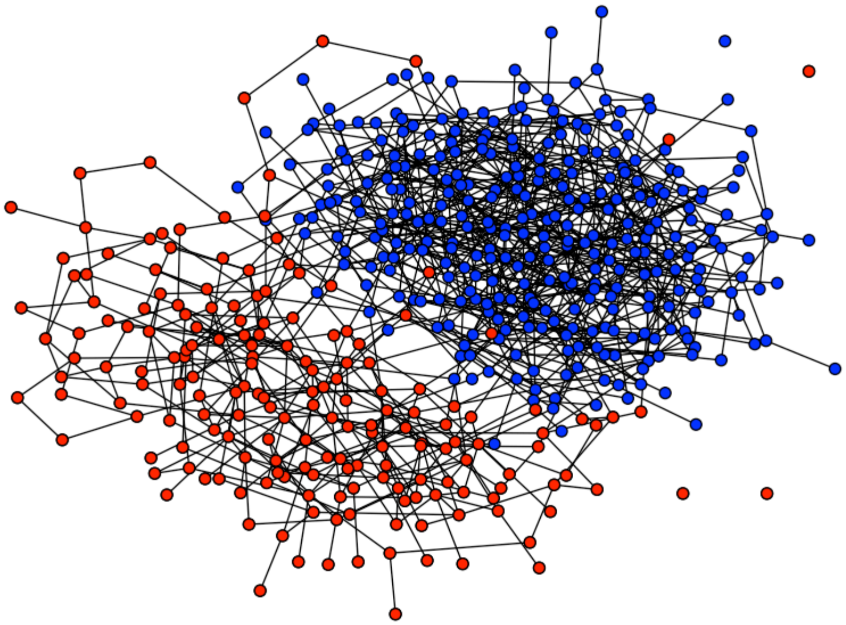
PHILIP S. CHODROW† AND PETER J. MUCHA‡

Transitivity reinforcement in the coevolving voter model

Nishant Malik, Feng Shi, Hsuan-Wei Lee, and Peter J. Mucha

Outline

1. **The predictable yet puzzling coevolving voter model**

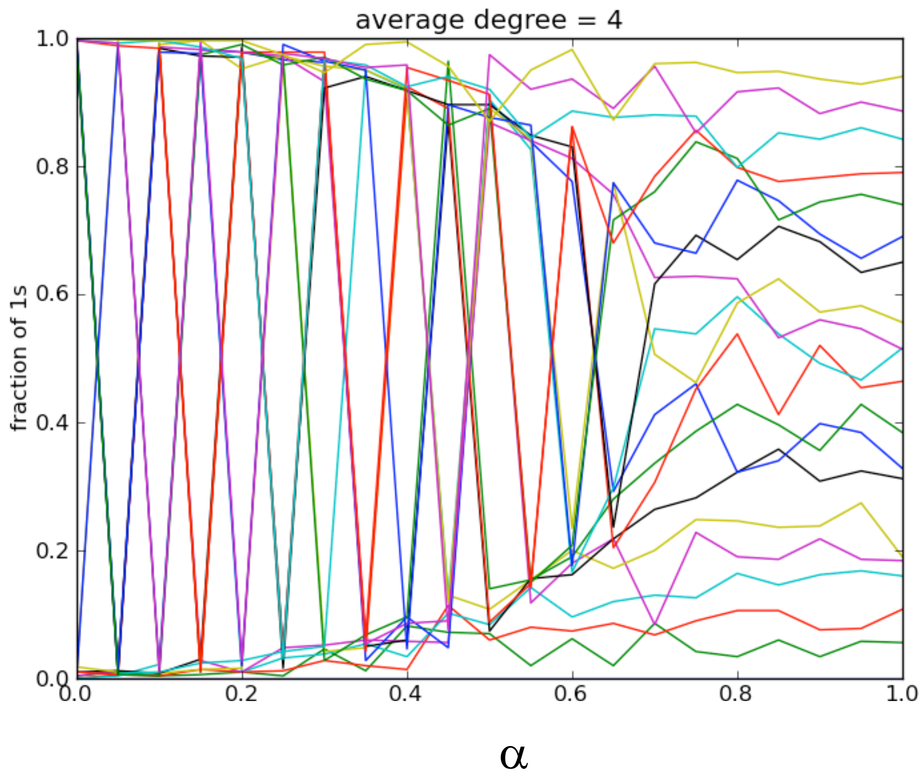


Durrett *et al.*, "Graph fission in an evolving voter model" (2012)

Two-opinion model ("0/1", "B/R", "S/I")

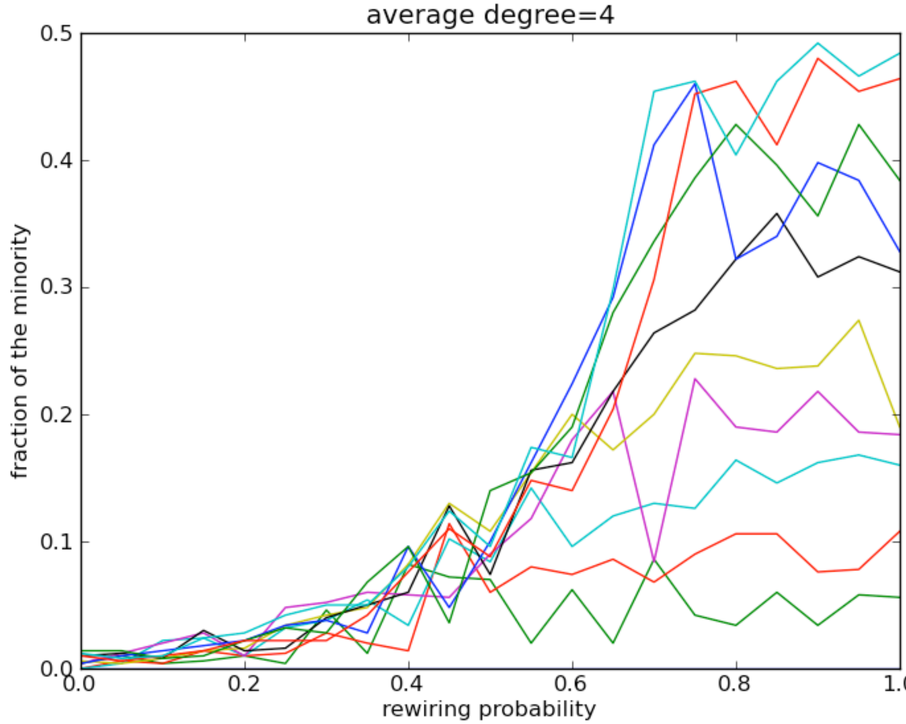
While there are discordant edges (i.e. 0-1)
    Select discordant edge (uniformly)
    w/probability (1-α)
        Voter step (change to 0-0 or 1-1)
    else (that is, w/probability α)
        Rewire*
End

*We consider two different rewiring rules. Both start by selecting one end to unilaterally drop the connection, then "Rewire-to-Random" v. "Rewire-to-Same"

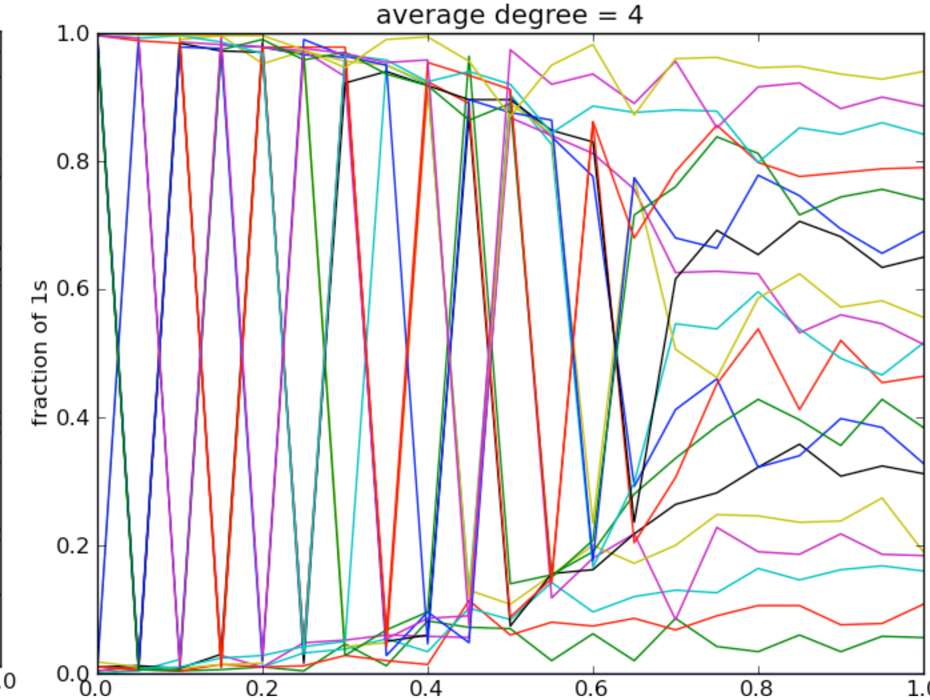


Durrett *et al.*, "Graph fission in an evolving voter model" (2012)

Two-opinion model. Select discordant edge. Voter step w/prob (1-α); otherwise rewire (α).
"Rewire-to-Random"



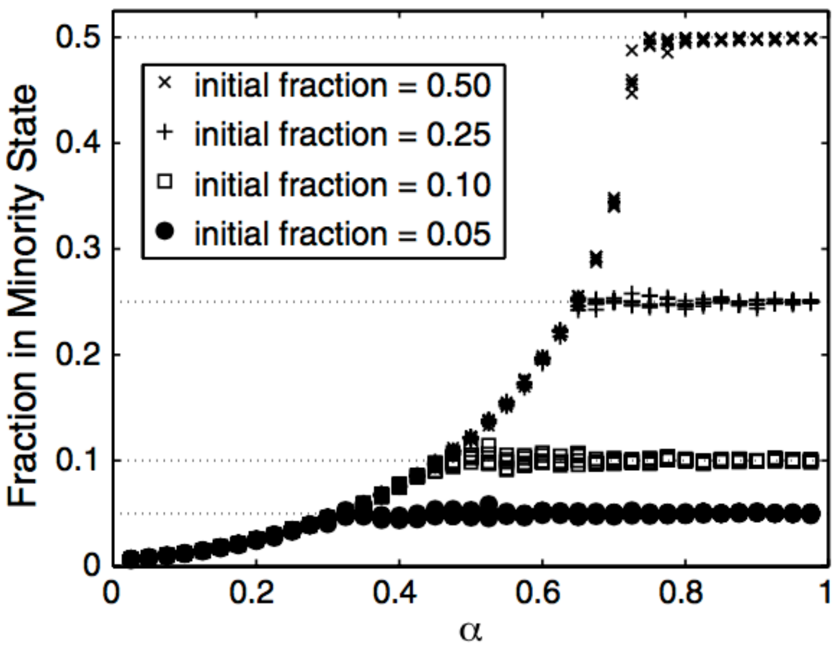
α

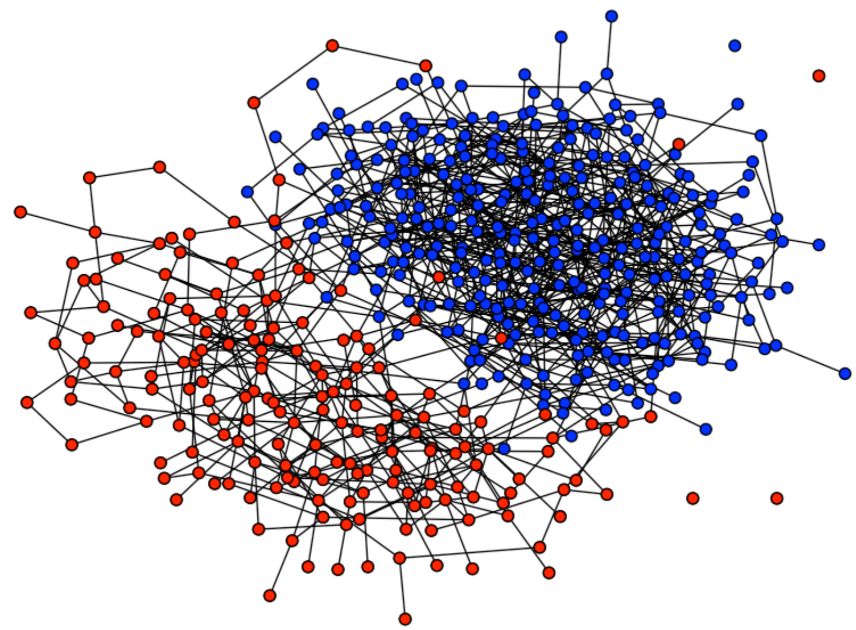


α

Durrett *et al.*, "Graph fission in an evolving voter model" (2012)

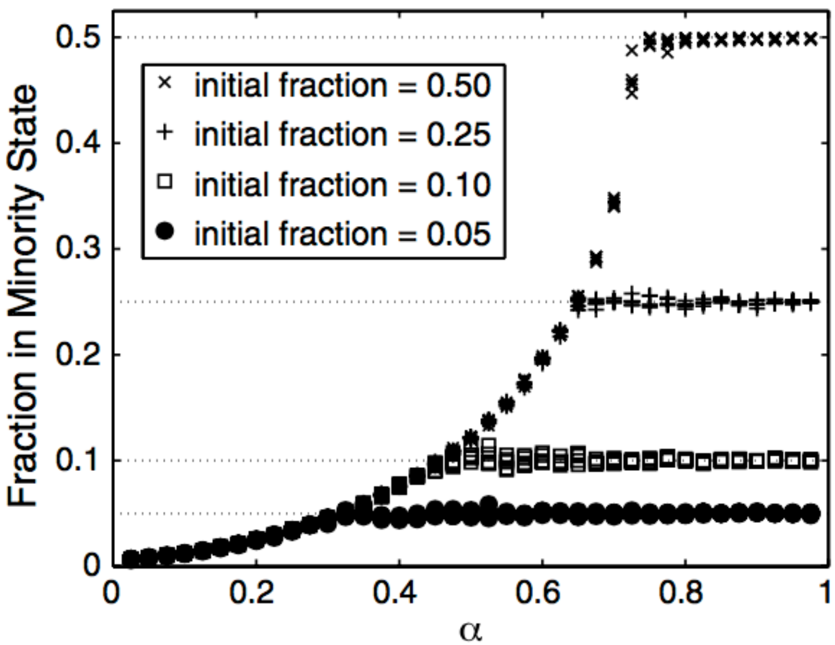
Two-opinion model. Select discordant edge. Voter step w/prob (1-α); otherwise rewire (α).
"Rewire-to-Random"

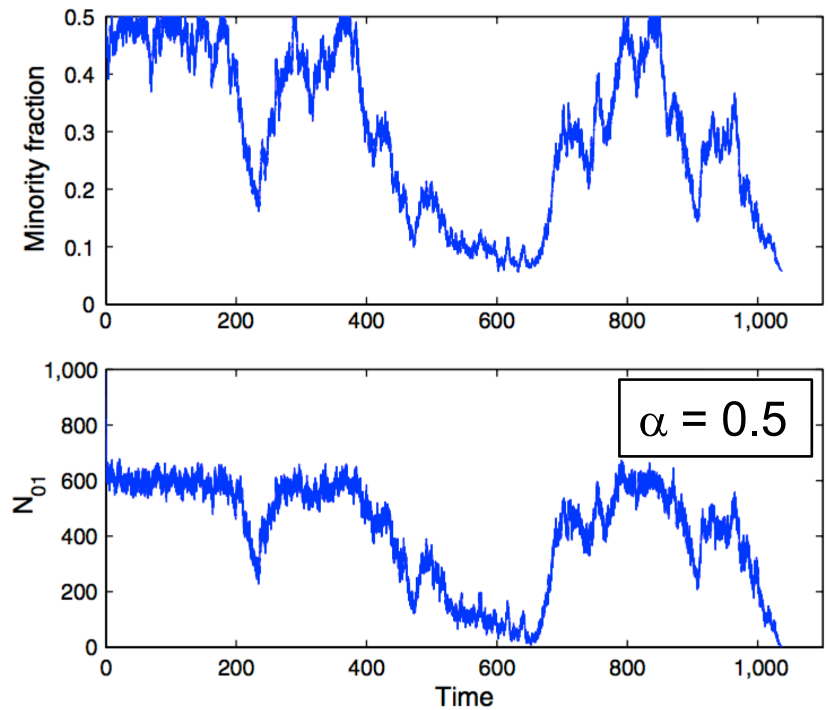




Durrett *et al.*, “Graph fission in an evolving voter model” (2012)

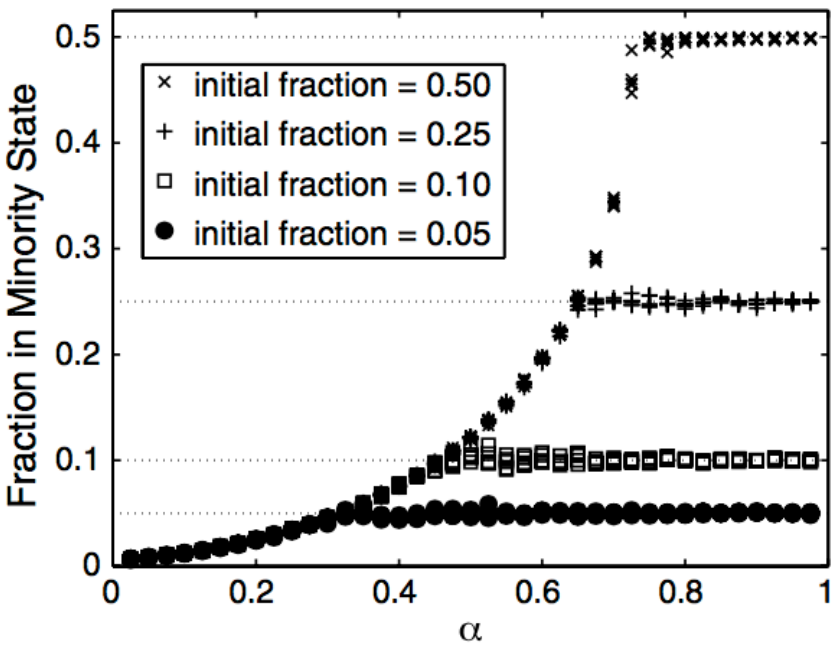
Two-opinion model. Select discordant edge. Voter step w/prob (1-α); otherwise rewire (α).
“Rewire-to-Random”

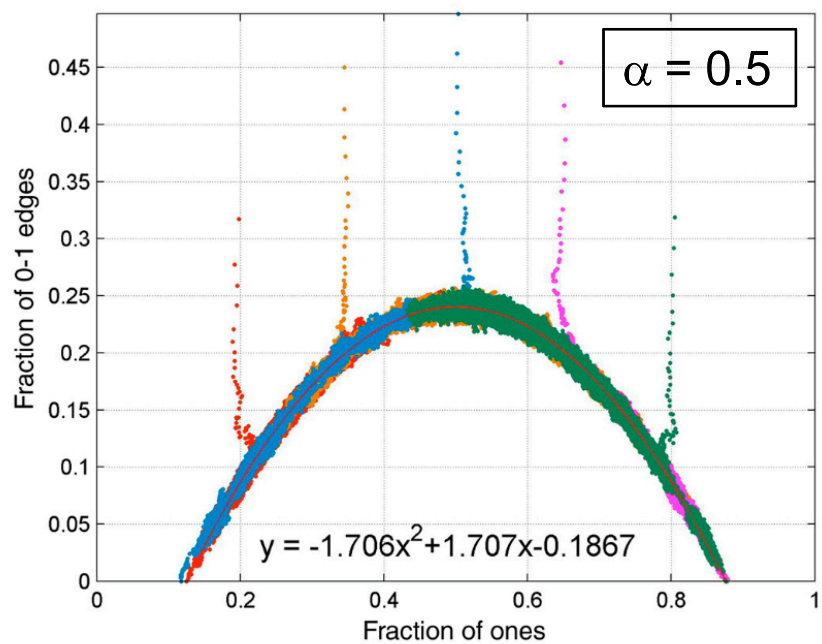




Durrett *et al.*, "Graph fission in an evolving voter model" (2012)

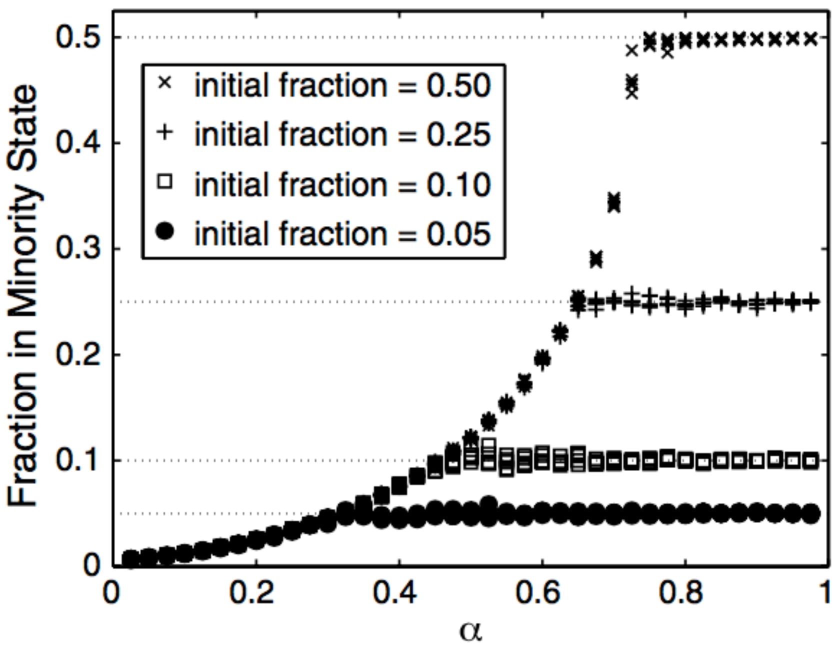
Two-opinion model. Select discordant edge. Voter step w/prob (1-α); otherwise rewire (α).
"Rewire-to-Random"

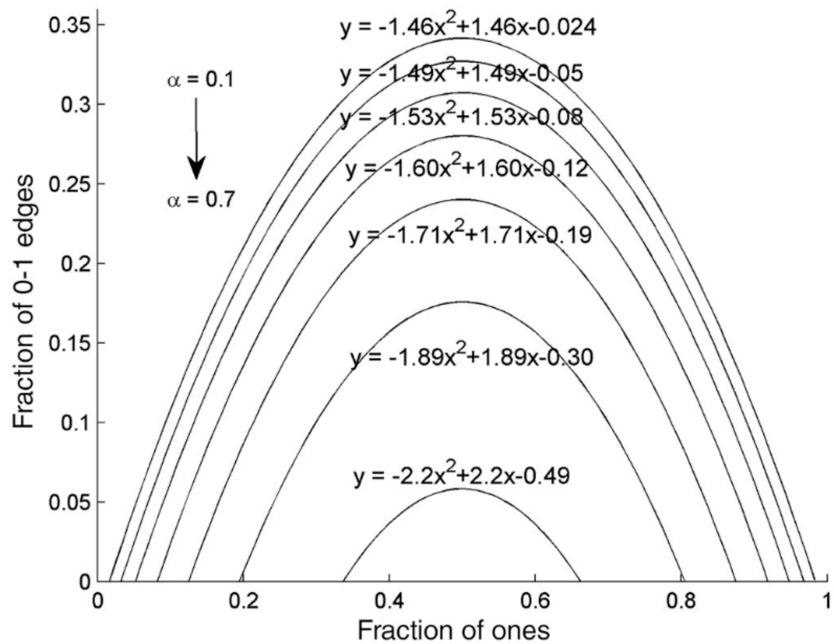




Durrett *et al.*, “Graph fission in an evolving voter model” (2012)

Two-opinion model. Select discordant edge. Voter step w/prob (1-α); otherwise rewire (α).
“Rewire-to-Random”

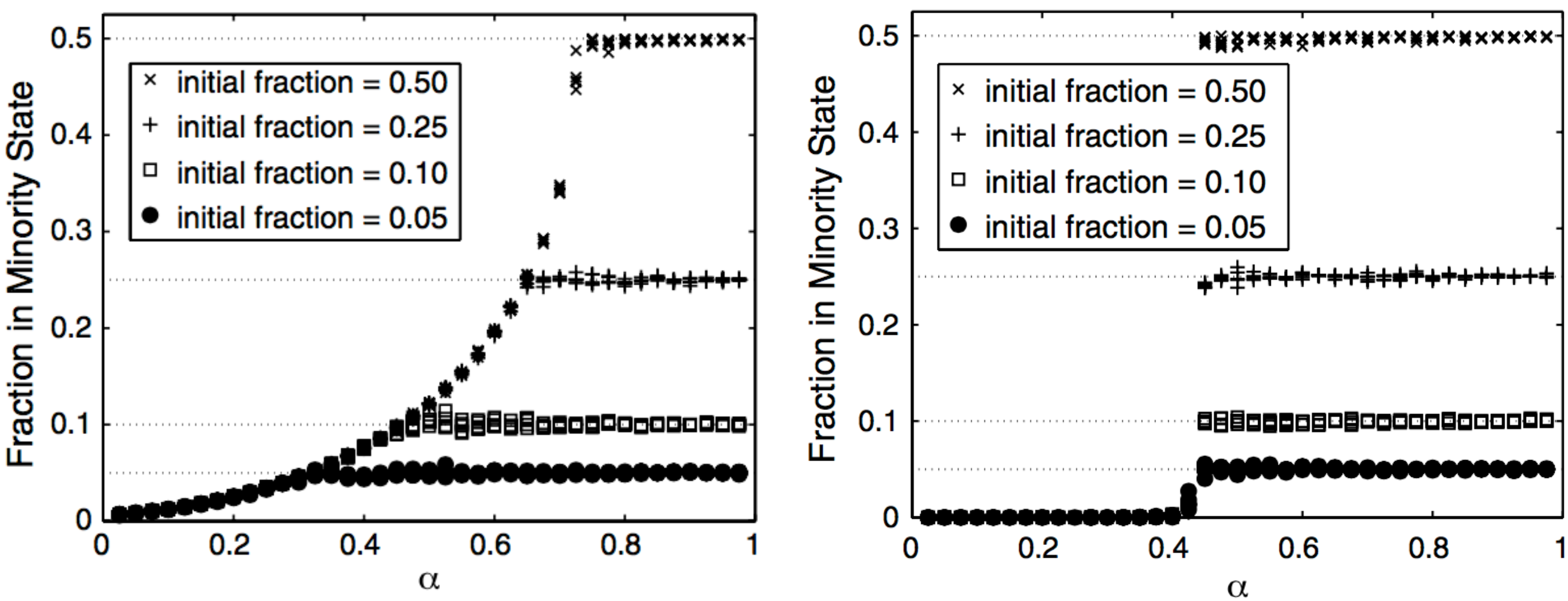




Durrett *et al.*, “Graph fission in an evolving voter model” (2012)

Two-opinion model. Select discordant edge. Voter step w/prob (1-α); otherwise rewire (α).

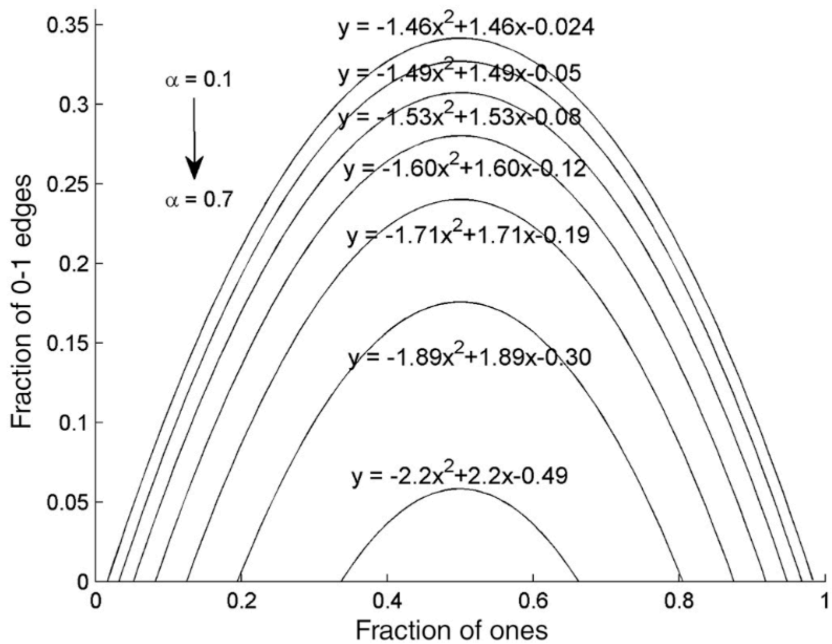
“Rewire-to-Random” v. “Rewire-to-Same”

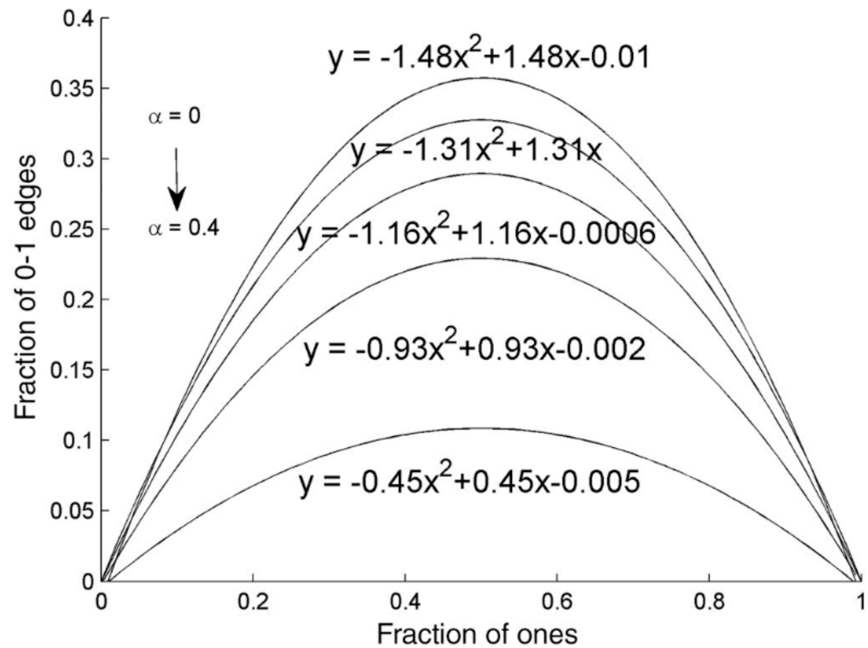


Durrett *et al.*, "Graph fission in an evolving voter model" (2012)

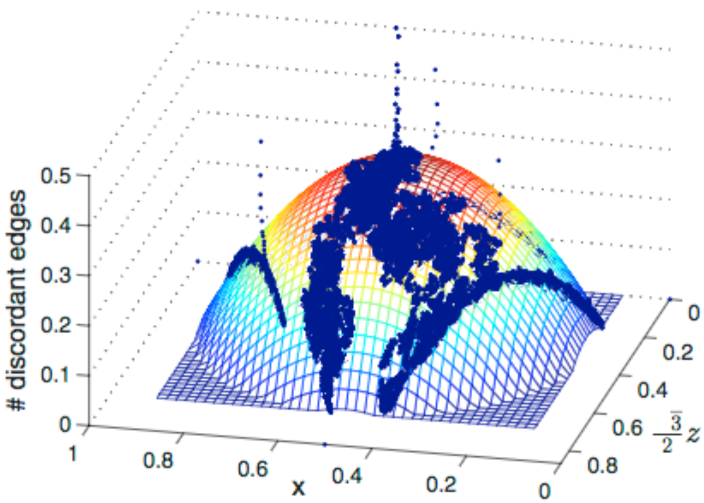
Two-opinion model. Select discordant edge. Voter step w/prob (1-α); otherwise rewire (α).

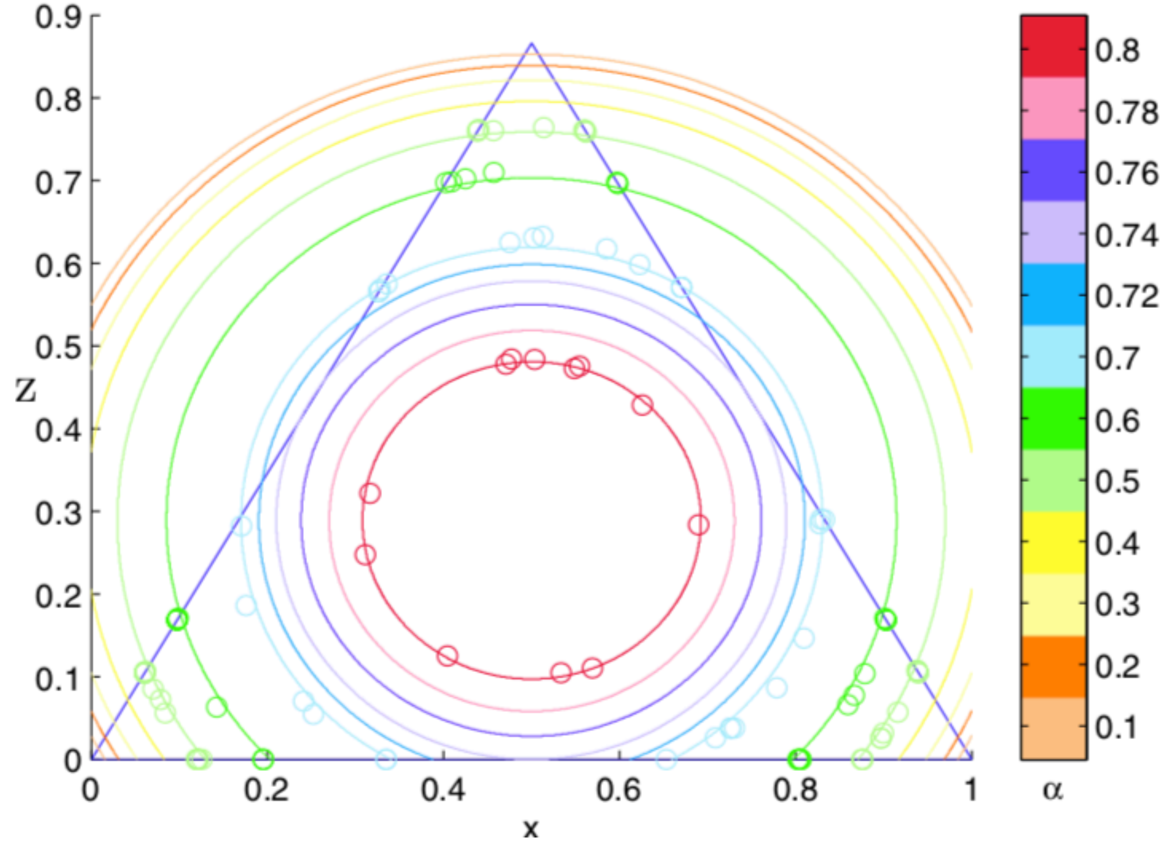
"Rewire-to-Random" v. "Rewire-to-Same"





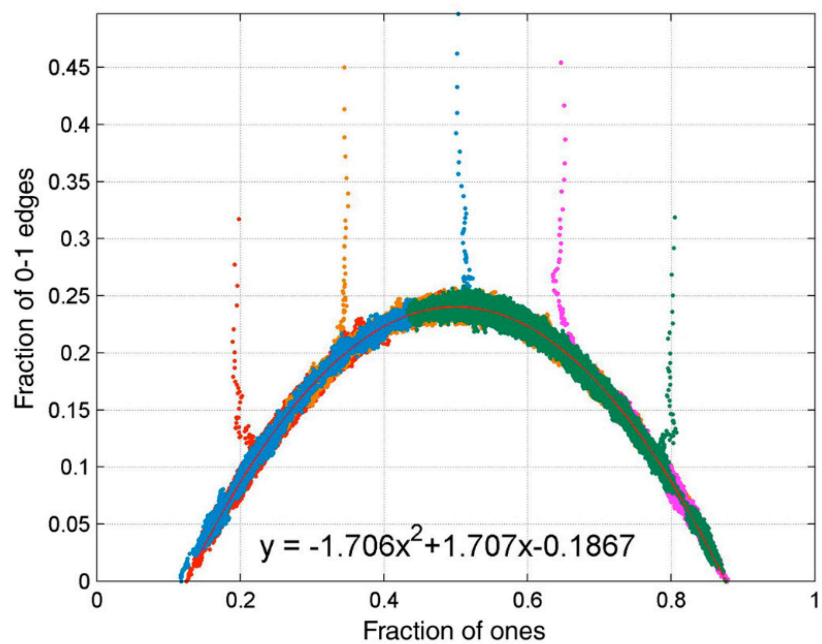
Shi, Mucha & Durrett: Multiple Opinions





Outline

1. The predictable yet puzzling coevolving voter model
2. **Approximate equations**
   - **Pair Approximation (PA)**
   - **Approximate Master Equations (AME)**
   - **Random Mutations**



Pair Approximation for Coevolving Voters

Two-opinion model. Select discordant edge. Voter step w/prob (1-α); otherwise rewire (α).

"Rewire-to-Same"

let $u = N_1/N$ be the initial fraction of vertices in state 1

$$N_{11} + 2N_{10} + N_{00} = M = \lambda N$$

$$\frac{1}{2}\frac{dN_{11}}{dt} = N_{10} + (1-\alpha)[N_{101} - N_{011}],$$

$$\frac{1}{2}\frac{dN_{00}}{dt} = N_{10} + (1-\alpha)[N_{010} - N_{100}].$$

Pair Approximation for Coevolving Voters

Two-opinion model. Select discordant edge. Voter step w/prob (1-α); otherwise rewire (α).
"Rewire-to-Same"

let $u = N_1/N$ be the initial fraction of vertices in state 1

$$N_{11} + 2N_{10} + N_{00} = M = \lambda N$$

$$\frac{1}{2}\frac{dN_{11}}{dt} = N_{10} + (1-\alpha)[N_{101} - N_{011}], \Rightarrow \frac{1}{1-\alpha}N_{10} = \frac{N_{01}N_{11}}{uN} - \frac{N_{10}N_{01}}{(1-u)N}$$

$$\frac{1}{2}\frac{dN_{00}}{dt} = N_{10} + (1-\alpha)[N_{010} - N_{100}]. \Rightarrow \frac{1}{1-\alpha}N_{10} = \frac{N_{10}N_{00}}{(1-u)N} - \frac{N_{01}N_{10}}{uN}$$

Pair Approximation for Coevolving Voters

Two-opinion model. Select discordant edge. Voter step w/prob (1-α); otherwise rewire (α).
"Rewire-to-Same"

let $u = N_1/N$ be the initial fraction of vertices in state 1

$$N_{11} + 2N_{10} + N_{00} = M = \lambda N$$

$$\frac{1}{2}\frac{dN_{11}}{dt} = N_{10} + (1-\alpha)[N_{101} - N_{011}], \Rightarrow \frac{1}{1-\alpha}N_{10} = \frac{N_{01}N_{11}}{uN} - \frac{N_{10}N_{01}}{(1-u)N}$$

$$\frac{1}{2}\frac{dN_{00}}{dt} = N_{10} + (1-\alpha)[N_{010} - N_{100}]. \Rightarrow \frac{1}{1-\alpha}N_{10} = \frac{N_{10}N_{00}}{(1-u)N} - \frac{N_{01}N_{10}}{uN}$$

Adding uN times the first to $(1-u)N$ times the second, we have

$$N_{11} + N_{00} - \left(\frac{u}{1-u} + \frac{1-u}{u}\right)N_{01} = \frac{N}{1-\alpha}.$$

$$N_{11} + N_{00} - \left(\frac{u}{1-u} + \frac{1-u}{u}\right) N_{01} = \frac{N}{1-\alpha}. \qquad \textbf{[S9]}$$

When $N_{01} = 0$, we have $N_{11} + N_{00} = \lambda N$, giving $1 - \alpha_c = 1/\lambda$ or

$$\alpha_c = 1 - \frac{1}{\lambda} = \frac{\lambda - 1}{\lambda}.$$

Using **[S9]** with $N_{11} + N_{00} = \lambda N - 2N_{01}$ and the algebra in **[S6]** yields

$$[\lambda - (1-\alpha)^{-1}]N = \frac{N_{01}}{u(1-u)}$$

and the following approximation for the arch

$$\frac{N_{01}}{N} = u(1-u)\left(\lambda - \frac{1}{1-\alpha}\right).$$

$$N_{11} + N_{00} - \left(\frac{u}{1-u} + \frac{1-u}{u}\right)N_{01} = \frac{N}{1-\alpha}. \qquad \textbf{[S9]}$$

When $N_{01} = 0$, we have $N_{11} + N_{00} = \lambda N$, giving $1 - \alpha_c = 1/\lambda$ or

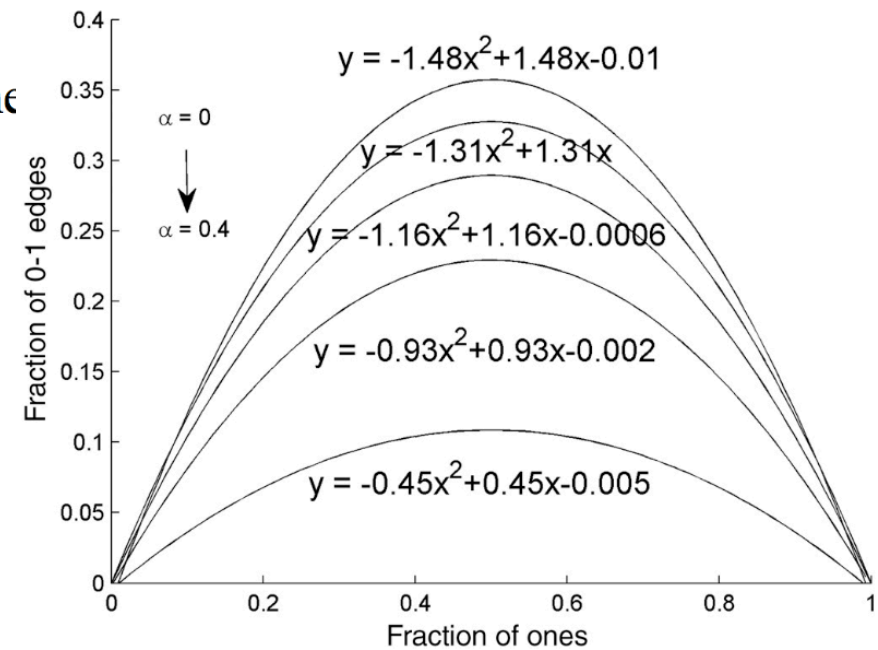
$$\alpha_c = 1 - \frac{1}{\lambda} = \frac{\lambda - 1}{\lambda}.$$

Using **[S9]** with $N_{11} + N_{00} = \lambda N - 2N_{01}$ and the
yields

$$[\lambda - (1-\alpha)^{-1}]N = \frac{N_{01}}{u(1-u)}$$

and the following approximation for the arch

$$\frac{N_{01}}{N} = u(1-u)\left(\lambda - \frac{1}{1-\alpha}\right).$$



$$N_{11} + N_{00} - \left(\frac{u}{1-u} + \frac{1-u}{u} \right) N_{01} = \frac{N}{1-\alpha}. \qquad \textbf{[S9]}$$

When $N_{01} = 0$, we have $N_{11} + N_{00} = \lambda N$, giving $1 - \alpha_c = 1/\lambda$ or

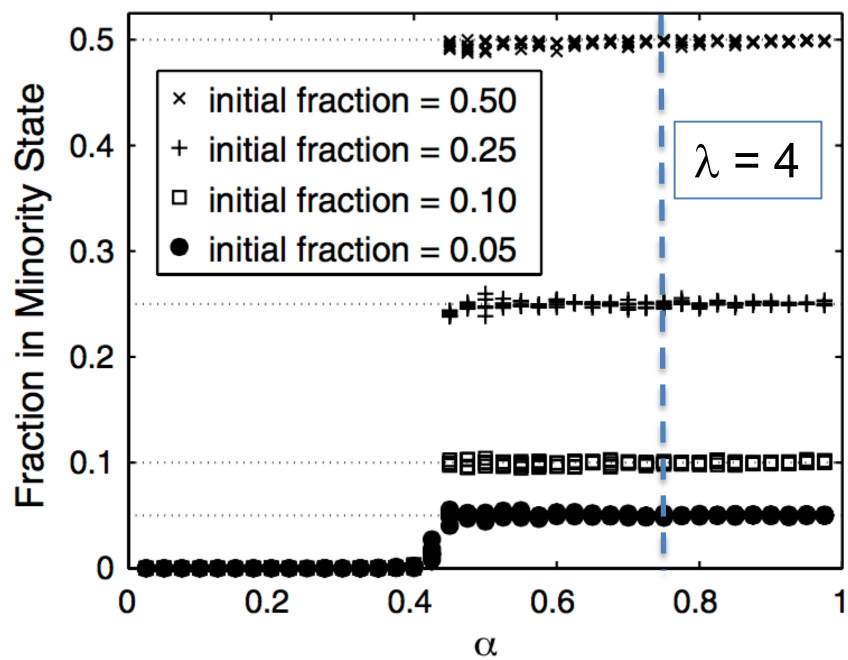
$$\alpha_c = 1 - \frac{1}{\lambda} = \frac{\lambda - 1}{\lambda}.$$

Using **[S9]** with $N_{11} + N_{00} = \lambda N - 2N_{01}$ and th
yields

$$[\lambda - (1-\alpha)^{-1}]N = \frac{N_{01}}{u(1-u)}$$

and the following approximation for the arch

$$\frac{N_{01}}{N} = u(1-u)\left(\lambda - \frac{1}{1-\alpha}\right).$$



Pair Approximation for Coevolving Voters

Two-opinion model. Select discordant edge. Voter step w/prob (1-α); otherwise rewire (α).
"Rewire-to-Random"

let $u = N_1/N$ be the initial fraction of vertices in state 1

$$N_{11} + 2N_{10} + N_{00} = M = \lambda N$$

$$\frac{1}{2}\frac{dN_{11}}{dt} = [1 - \alpha(1-u)]N_{10} + (1-\alpha)[N_{101} - N_{011}],$$

$$\frac{1}{2}\frac{dN_{00}}{dt} = (1 - \alpha u)N_{10} + (1-\alpha)[N_{010} - N_{100}].$$

Pair Approximation for Coevolving Voters

Two-opinion model. Select discordant edge. Voter step w/prob (1-α); otherwise rewire (α).

"Rewire-to-Random"

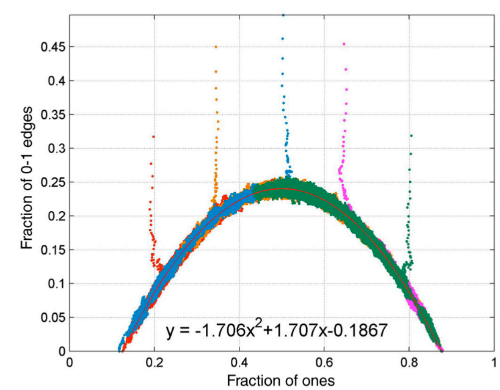
let $u = N_1/N$ be the initial fraction of vertices in state 1

$$N_{11} + 2N_{10} + N_{00} = M = \lambda N$$

$$\frac{1}{2}\frac{dN_{11}}{dt} = [1-\alpha(1-u)]N_{10} + (1-\alpha)[N_{101} - N_{011}],$$

$$\frac{1}{2}\frac{dN_{00}}{dt} = (1-\alpha u)N_{10} + (1-\alpha)[N_{010} - N_{100}].$$

$$\Rightarrow \quad \frac{N_{01}}{N} = u(1-u)\left(\lambda - 1 - \frac{[u^2+(1-u)^2]\alpha}{1-\alpha}\right).$$



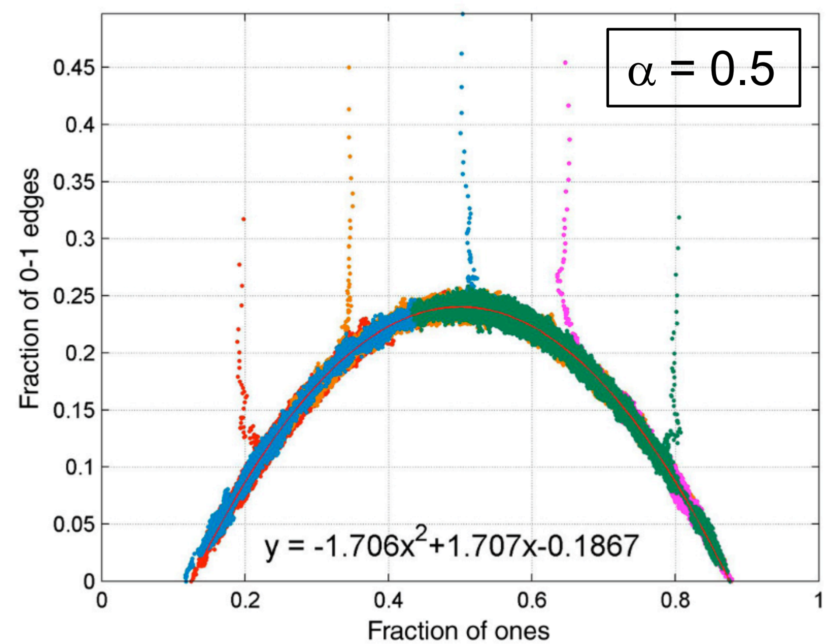
Approximate Master Equations

Two-opinion model. Select discordant edge. Voter step w/prob (1-α); otherwise rewire (α).
"Rewire-to-Random"
Approximate Master Equations (AME):

$$\frac{d}{dt}\bar{S}_{k,m} = \alpha\{-(2-u)m\bar{S}_{k,m} + (1-u)(m+1)\bar{S}_{k,m+1} + (m+1)\bar{S}_{k+1,m+1}\} + \alpha N_{01}[-2\bar{S}_{k,m} + \bar{S}_{k-1,m-1} + \bar{S}_{k-1,m}]/N + (1-\alpha)[-m\bar{S}_{k,m} + (k-m)\bar{I}_{k,m}] + (1-\alpha)[-\beta^S(k-m)\bar{S}_{k,m} + \beta^S(k-m+1)\bar{S}_{k,m-1} - \gamma^S m\bar{S}_{k,m} + \gamma^S(m+1)\bar{S}_{k,m+1}],$$

$$\beta^S = \frac{\sum_{k,m}(k-m)m\bar{S}_{k,m}}{\sum_{k,m}(k-m)\bar{S}_{k,m}} = \frac{N_{001}}{N_{00}}$$

$$\gamma^S = \frac{\sum_{k,m}(k-m)^2\bar{I}_{k,m}}{\sum_{k,m}(k-m)\bar{I}_{k,m}} = \frac{N_{010}}{N_{01}} + 1.$$



Durrett *et al.*, "Graph fission in an evolving voter model" (2012)

Two-opinion model. Select discordant edge. Voter step w/prob (1-α); otherwise rewire (α).
"Rewire-to-Random"
Approximate Master Equations (AME):

$$\begin{aligned}\frac{d}{dt}\bar{S}_{k,m} &= \alpha\{-(2-u)m\bar{S}_{k,m} + (1-u)(m+1)\bar{S}_{k,m+1} \\ &\quad + (m+1)\bar{S}_{k+1,m+1}\} \\ &\quad + \alpha N_{01}[-2\bar{S}_{k,m} + \bar{S}_{k-1,m-1} + \bar{S}_{k-1,m}]/N \\ &\quad + (1-\alpha)[-m\bar{S}_{k,m} + (k-m)\bar{I}_{k,m}] \\ &\quad + (1-\alpha)[-\beta^S(k-m)\bar{S}_{k,m} + \beta^S(k-m+1)\bar{S}_{k,m-1} \\ &\quad - \gamma^S m\bar{S}_{k,m} + \gamma^S(m+1)\bar{S}_{k,m+1}],\end{aligned}$$

$$\beta^S = \frac{\sum_{k,m}(k-m)m\bar{S}_{k,m}}{\sum_{k,m}(k-m)\bar{S}_{k,m}} = \frac{N_{001}}{N_{00}}$$

$$\gamma^S = \frac{\sum_{k,m}(k-m)^2\bar{I}_{k,m}}{\sum_{k,m}(k-m)\bar{I}_{k,m}} = \frac{N_{010}}{N_{01}} + 1.$$

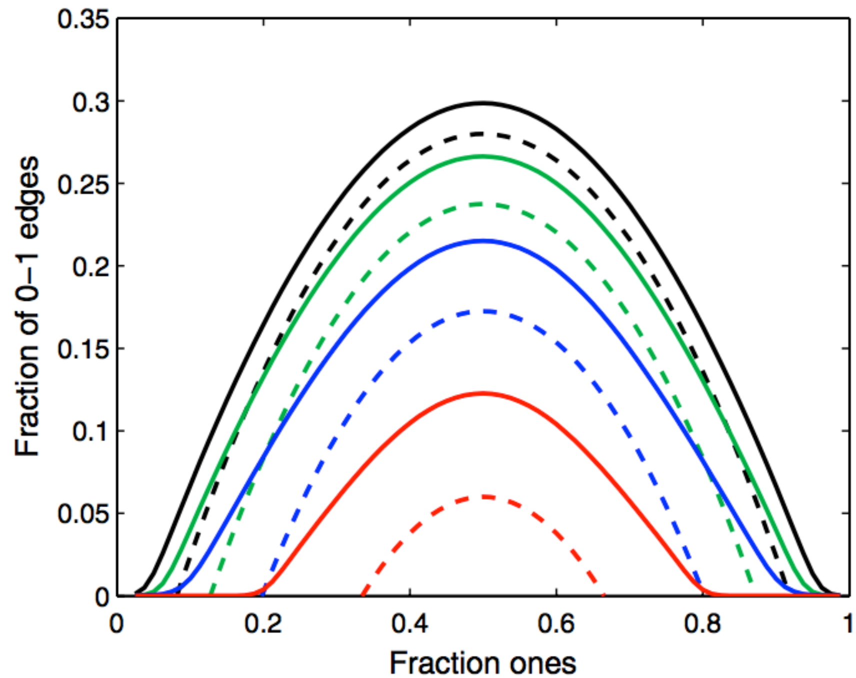


Fig. 10. Arches computed by approximate master equation (solid lines) versus simulation (dashes) for rewire-to-random model with $\alpha = 0.4, 0.5, 0.6, 0.7$. The curves decrease as α increases.

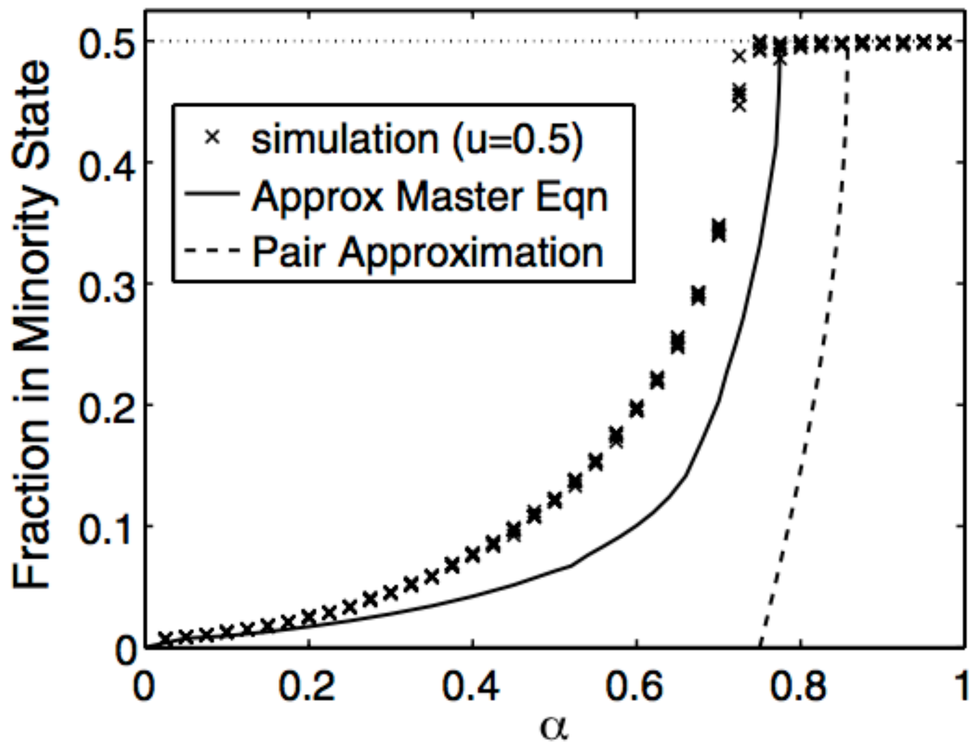
Durrett *et al.*, "Graph fission in an evolving voter model" (2012)

Two-opinion model. Select discordant edge. Voter step w/prob (1-α); otherwise rewire (α).
"Rewire-to-Random"
Approximate Master Equations (AME):

$$\frac{d}{dt}\bar{S}_{k,m} = \alpha\{-(2-u)m\bar{S}_{k,m} + (1-u)(m+1)\bar{S}_{k,m+1} + (m+1)\bar{S}_{k+1,m+1}\} + \alpha N_{01}[-2\bar{S}_{k,m} + \bar{S}_{k-1,m-1} + \bar{S}_{k-1,m}]/N + (1-\alpha)[-m\bar{S}_{k,m} + (k-m)\bar{I}_{k,m}] + (1-\alpha)[-\beta^S(k-m)\bar{S}_{k,m} + \beta^S(k-m+1)\bar{S}_{k,m-1} - \gamma^S m\bar{S}_{k,m} + \gamma^S(m+1)\bar{S}_{k,m+1}],$$

$$\beta^S = \frac{\sum_{k,m}(k-m)m\bar{S}_{k,m}}{\sum_{k,m}(k-m)\bar{S}_{k,m}} = \frac{N_{001}}{N_{00}}$$

$$\gamma^S = \frac{\sum_{k,m}(k-m)^2\bar{I}_{k,m}}{\sum_{k,m}(k-m)\bar{I}_{k,m}} = \frac{N_{010}}{N_{01}} + 1.$$



Analyze with random mutations

Chodrow & Mucha (arXiv:1812.05464)

Mutation step: with probability $\lambda << 1$, change opinion of uniformly random node else proceed as usual per above (see also Ji et al. NJP, 2013)

Ergodic for $\lambda > 0$, study the equilibrium measure

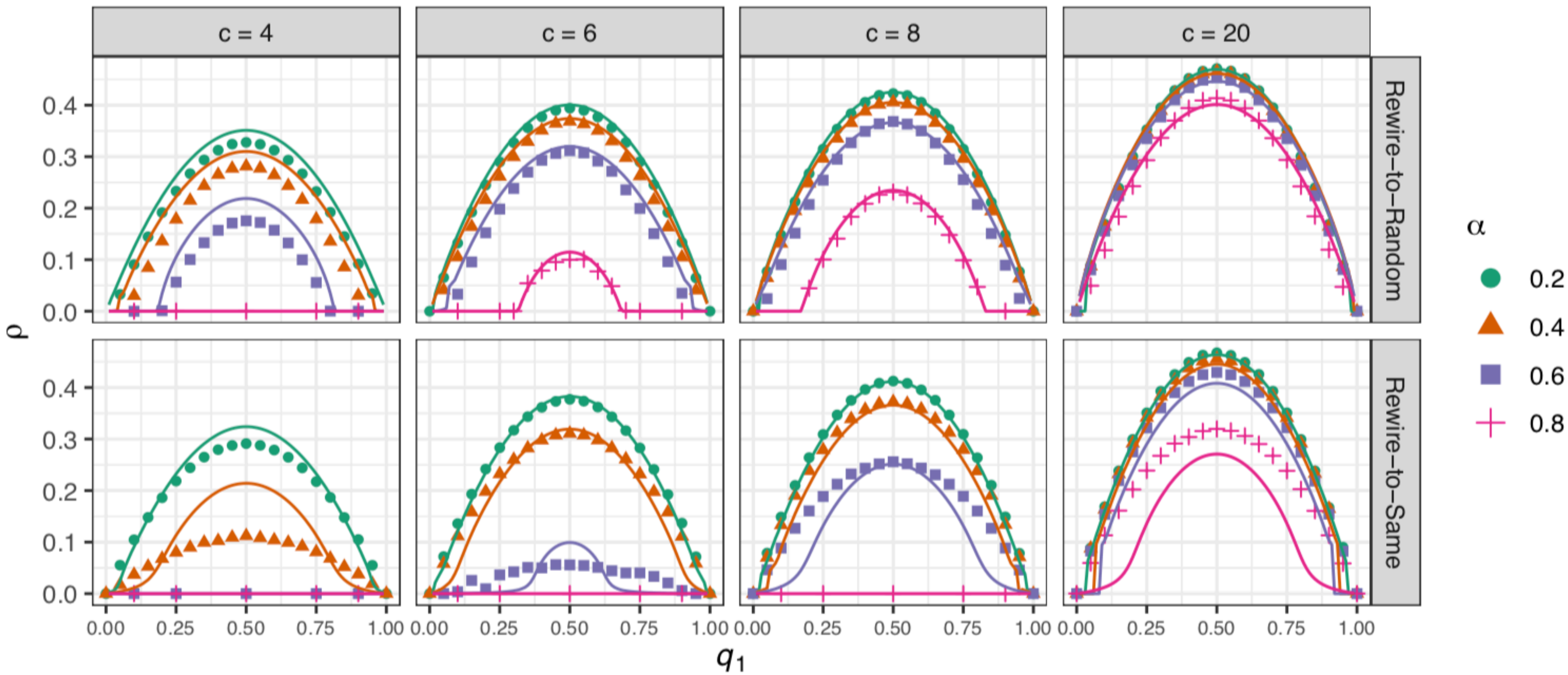
Assume number of active edges is small.
Consider a node u immediately after its opinion mutates, so that it is a local minority.
Solve for steady state of mean field approximation for changes in the edge counts:

$$\mathbf{m}(t+1) - \mathbf{m}(t) = \lambda \mathbf{w}(\mathbf{x}) + (1-\lambda)\alpha \mathbf{r}(\mathbf{q}) + (1-\lambda)(1-\alpha)\hat{\mathbf{v}}(\mathbf{q}, \mathbf{x})$$

(Bookkeeping in the voter term for multiple different kinds of events)

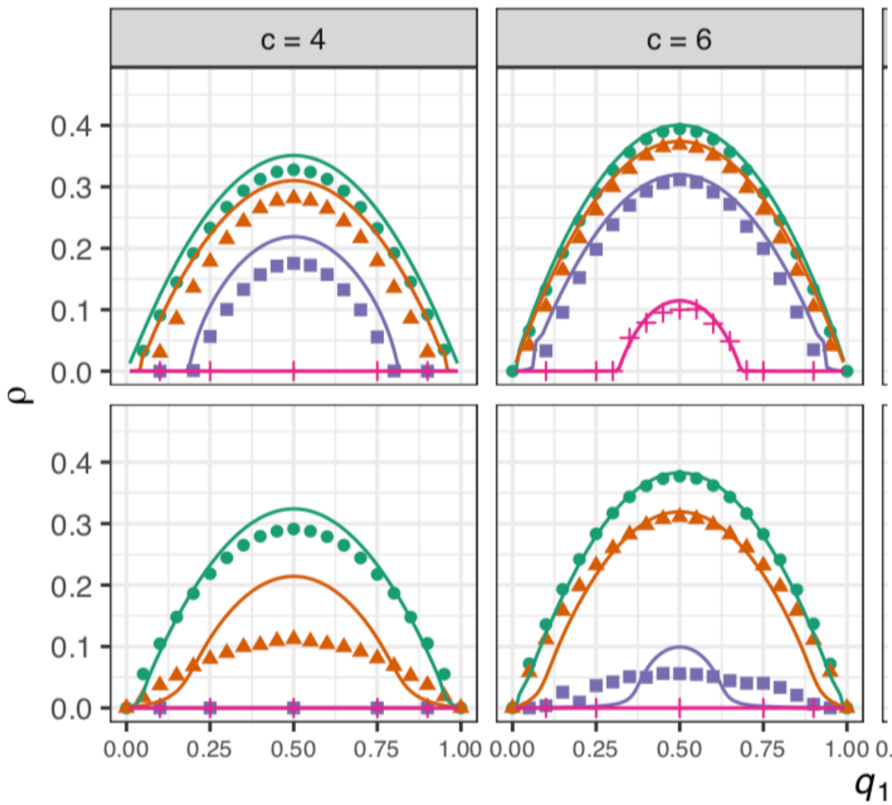
Analyze with random mutations

Chodrow & Mucha (arXiv:1812.05464)



Analyze with random mutations

Chodrow & Mucha (arXiv:1812.05464)



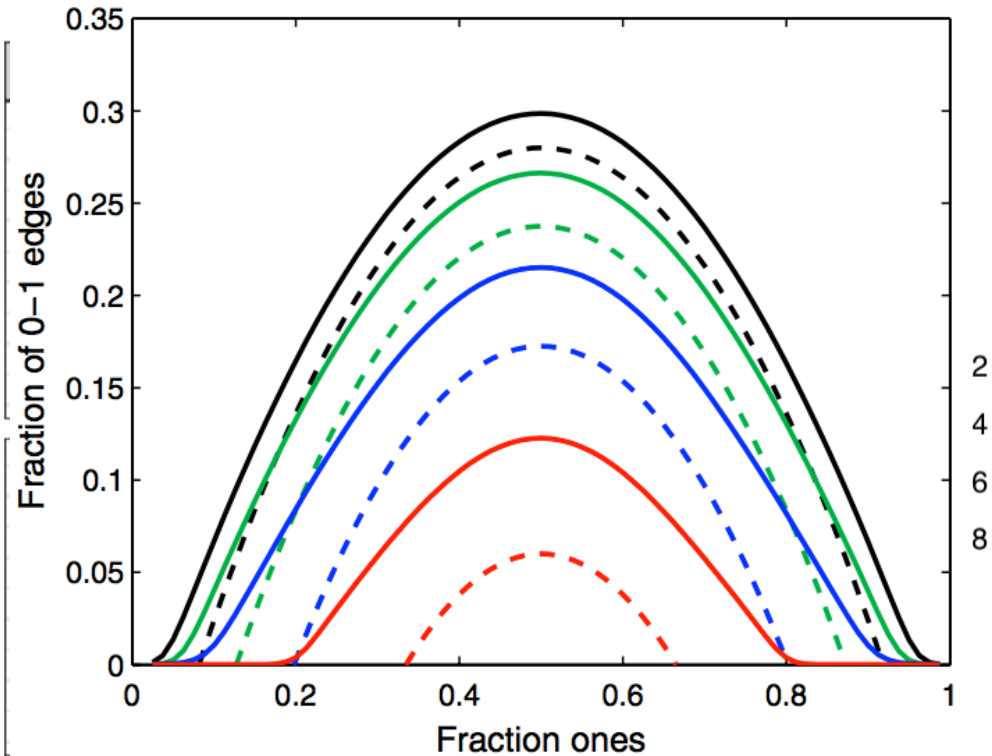
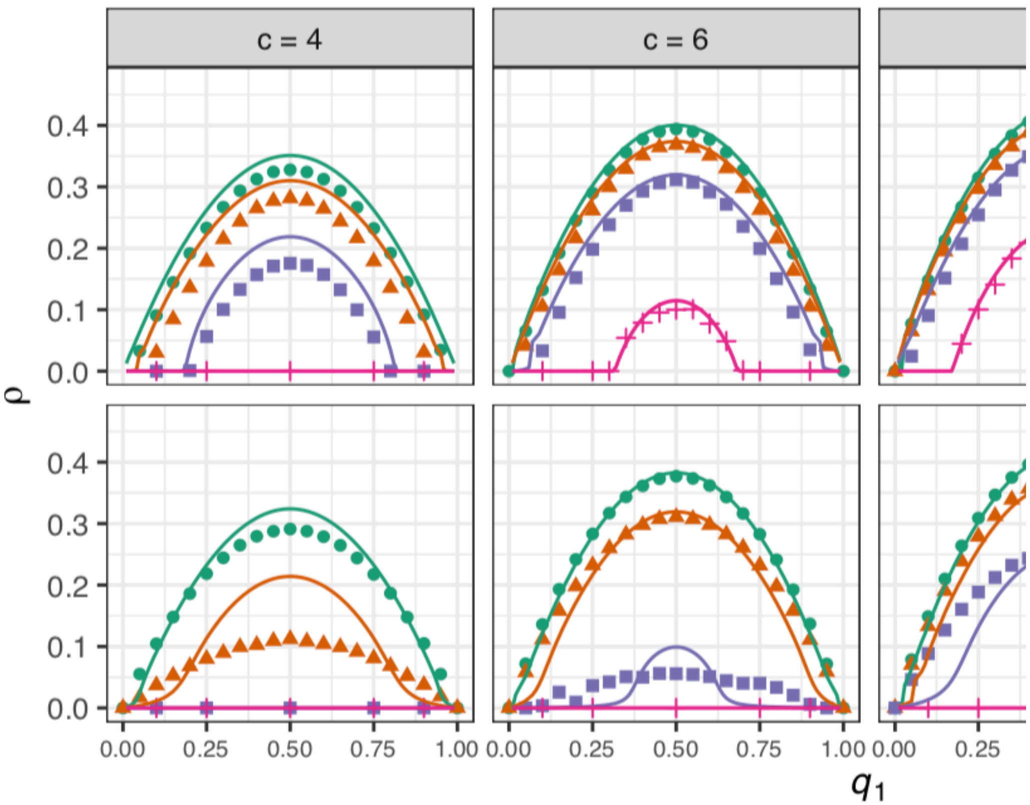
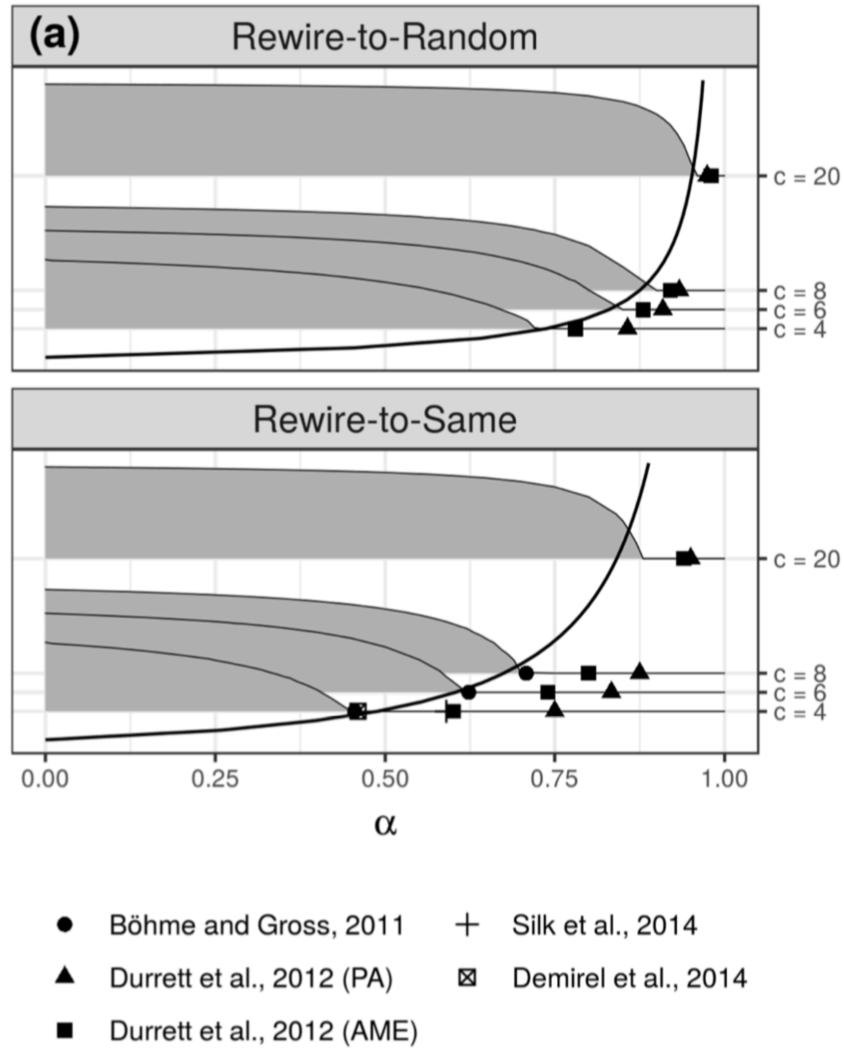


Fig. 10. Arches computed by approximate master equation (solid lines) versus simulation (dashes) for rewire-to-random model with $\alpha = 0.4, 0.5, 0.6, 0.7$. The curves decrease as α increases.

Analyze with random mutations

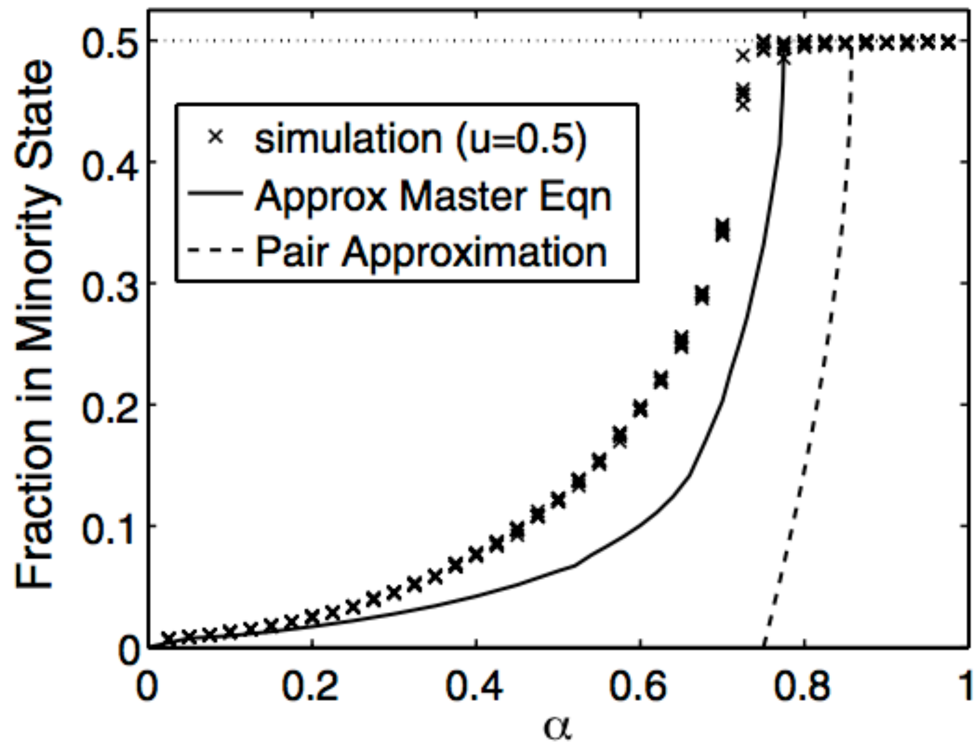
Chodrow & Mucha (arXiv:1812.05464)





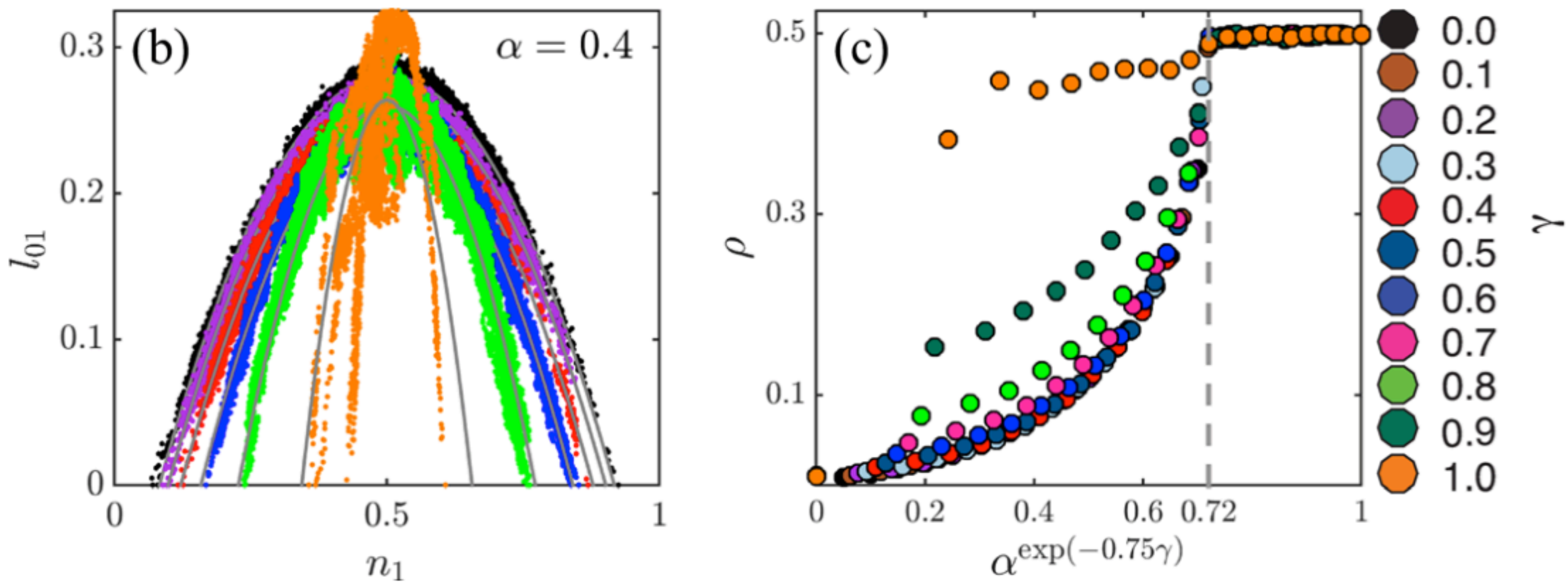
Outline

1. The predictable yet puzzling coevolving voter model
2. Approximate equations
   - Pair Approximation (PA)
   - Approximate Master Equations (AME)
   - Random Mutations
3. **Reinforced transitivity**



Malik *et al.*, “Transitivity reinforcement in the coevolving voter model” (2016)

Voter step w/prob (1-α); otherwise rewire (α). Then rewire to friend-of-friend w/prob γ.



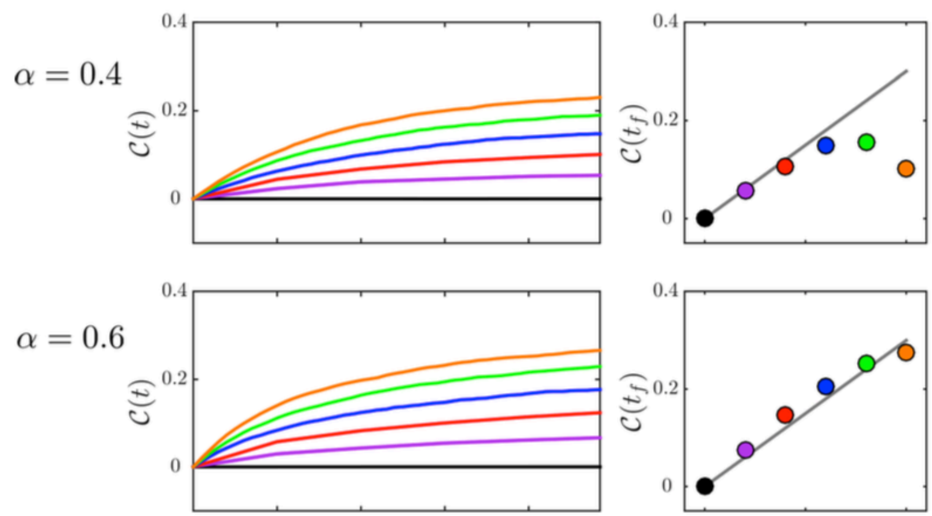
Malik *et al.*, "Transitivity reinforcement in the coevolving voter model" (2016)

Voter step w/prob (1-α); otherwise rewire (α). Then rewire to friend-of-friend w/prob γ.

Mean field theory for rate of change in triangle count and final level of transitivity:

$$\dot{T} = -\alpha\mathcal{C}(\langle k \rangle - 1) + \alpha\gamma$$

$$\mathcal{C} = \frac{3\gamma}{3\langle k \rangle - 2}$$

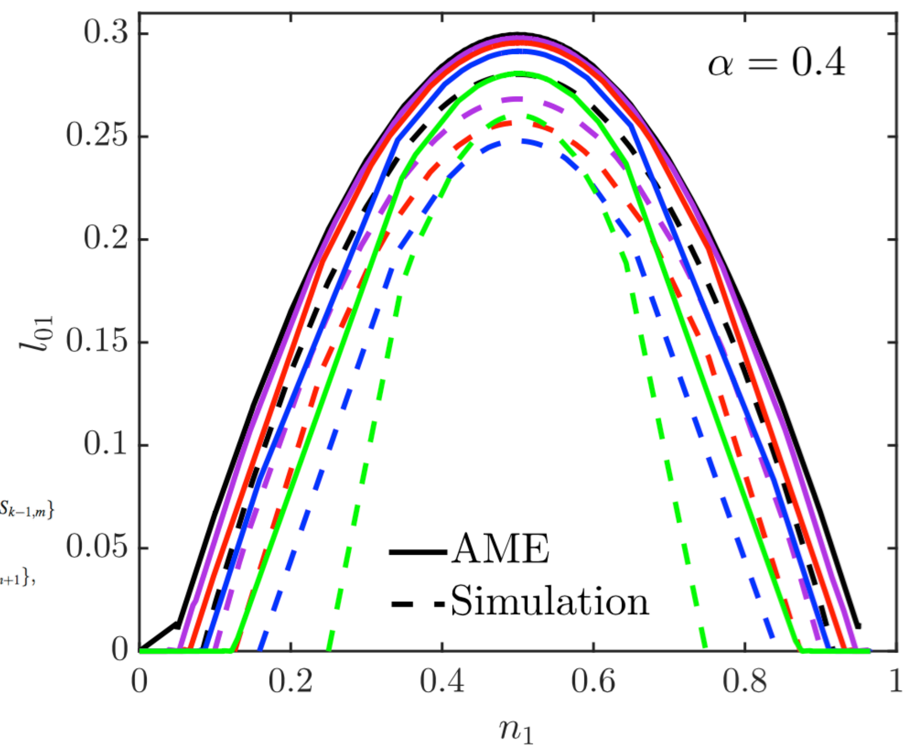


Malik *et al.*, “Transitivity reinforcement in the coevolving voter model” (2016)

Voter step w/prob (1-α); otherwise rewire (α). Then rewire to friend-of-friend w/prob γ.

Approximate Master Equations (AME):

$$
\begin{aligned}
\frac{d}{dt}S_{k,m} = {} & \alpha\gamma\{-[1+P(nn0|S_{k,m})]mS_{k,m} + P(nn0|S_{k,m+1})(m+1)S_{k,m+1} + (m+1)S_{k+1,m+1}\} \\
& + \alpha(1-\gamma)\{-(2-u)mS_{k,m} + (1-u)(m+1)S_{k,m+1} + (m+1)S_{k+1,m+1}\} \\
& + \alpha\gamma\Bigg\{-\left[\left(\frac{m}{k}\cdot\frac{l_{10}}{\frac{1}{2}l_{11}+l_{10}}\right)\cdot\frac{l_{01}}{N_0} + \left(\frac{k-m}{k}\cdot\frac{\frac{1}{2}l_{00}}{\frac{1}{2}l_{00}+l_{01}}\right)\cdot\frac{l_{01}}{N_0}\right]\cdot S_{k,m} \\
& - \left[\left(\frac{m}{k}\cdot\frac{\frac{1}{2}l_{11}}{\frac{1}{2}l_{11}+l_{10}}\right)\cdot\frac{l_{10}}{N_1} + \left(\frac{k-m}{k}\cdot\frac{l_{01}}{\frac{1}{2}l_{00}+l_{01}}\right)\cdot\frac{l_{10}}{N_1}\right]\cdot S_{k,m} \\
& + \left[\left(\frac{m-1}{k-1}\cdot\frac{\frac{1}{2}l_{11}}{\frac{1}{2}l_{11}+l_{10}}\right)\cdot\frac{l_{10}}{N_1} + \left(\frac{k-m}{k-1}\cdot\frac{l_{01}}{\frac{1}{2}l_{00}+l_{01}}\right)\cdot\frac{l_{10}}{N_1}\right]\cdot S_{k-1,m-1} \\
& + \left[\left(\frac{m}{k-1}\cdot\frac{l_{10}}{\frac{1}{2}l_{11}+l_{10}}\right)\cdot\frac{l_{01}}{N_0} + \left(\frac{k-m-1}{k-1}\cdot\frac{\frac{1}{2}l_{00}}{\frac{1}{2}l_{00}+l_{01}}\right)\cdot\frac{l_{01}}{N_0}\right]\cdot S_{k-1,m}\Bigg\} + \alpha(1-\gamma)\frac{l_{01}}{N}\{-2S_{k,m} + S_{k-1,m-1} + S_{k-1,m}\} \\
& + (1-\alpha)\{-mS_{k,m} + (k-m)I_{k,m}\} + (1-\alpha)\{-\beta^s(k-m)S_{k,m} + \beta^s(k-m+1)S_{k,m-1} - \gamma^s mS_{k,m} + \gamma^s(m+1)S_{k,m+1}\},
\end{aligned}
$$



Outline

1. The predictable yet puzzling coevolving voter model
2. Approximate equations
   - Pair Approximation (PA)
   - Approximate Master Equations (AME)
   - Random Mutations
3. Reinforced transitivity

Funding: NSF, NIGMS, NICHD

Graph fission in an evolving voter model

Richard Durrett[a,1], James P. Gleeson[b], Alun L. Lloyd[c,d], Peter J. Mucha[e], Feng Shi[e], David Sivakoff[a], Joshua E. S. Socolar[f], and Chris Varghese[f]

Multiopinion coevolving voter model with infinitely many phase transitions

Feng Shi,[1] Peter J. Mucha,[1] and Richard Durrett[2]

LOCAL SYMMETRY AND GLOBAL STRUCTURE IN ADAPTIVE VOTER MODELS*

PHILIP S. CHODROW† AND PETER J. MUCHA‡

Transitivity reinforcement in the coevolving voter model

Nishant Malik, Feng Shi, Hsuan-Wei Lee, and Peter J. Mucha