Convergence and Synchronization of Piecewise-smooth Networks

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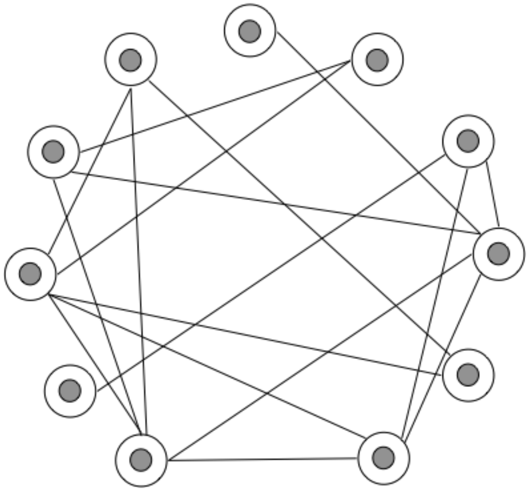
Outline

- Introduction and motivation
- From PWS systems to PWS networks
- Synchronization of networks of PWS systems
- Proving convergence
- A distributed multiplex approach
- Open Challenges and Future Work
- Conclusions

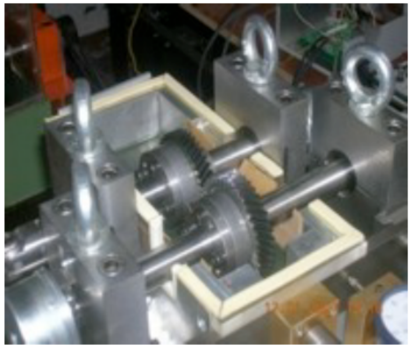


1. Intro & motivation

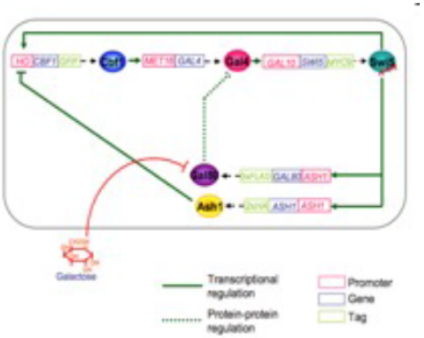
Introduction

- A lot of attention has been spent to study the dynamics of piecewise smooth systems in many application domains









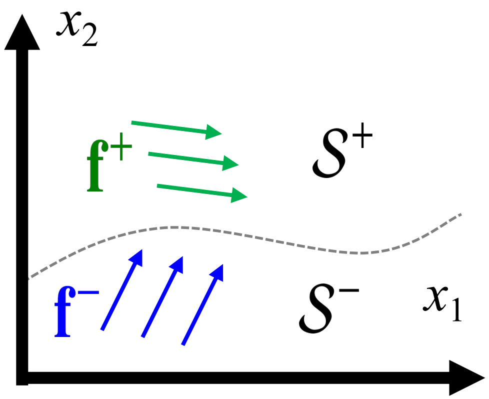
Bimodal PWS systems

- Results are available to study well-posedness, bifurcations, stability and other properties of PWSS
- Attention is often focussed on bimodal PWSS of the form

$$\dot{x} = \begin{cases} F_1(x,\mu), & H(x,\mu) > 0 \\ F_2(x,\mu), & H(x,\mu) < 0 \end{cases}$$

- ..with low-dimensional state vectors..
- ..and one discontinuity boundary

$$\Sigma := \{x \in \mathbb{R}^n : H(x,\mu) = 0\}$$



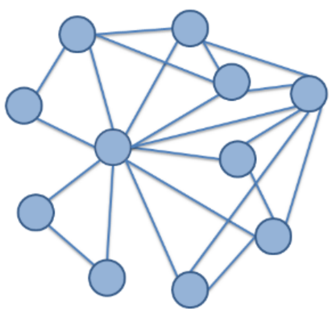
- ..no delays or noise etc etc

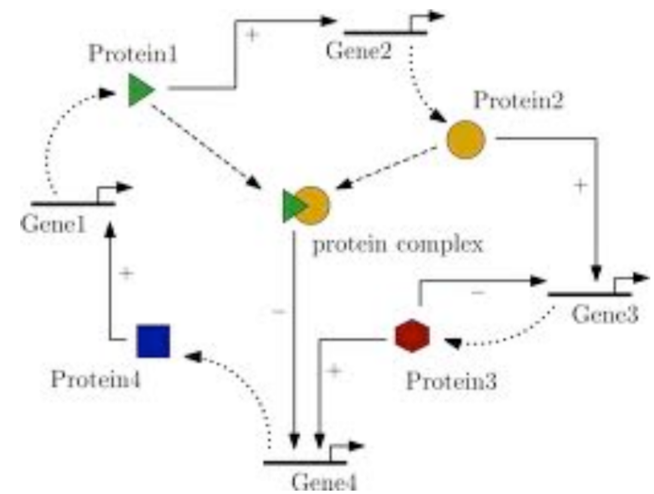
Common open challenges

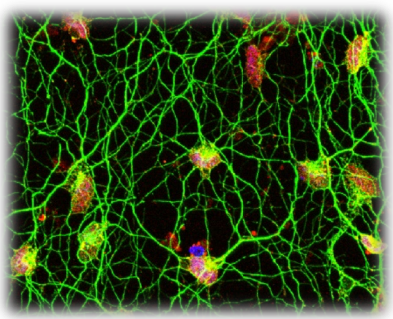
- Theoretical progress has been haunted by a number of key problems.
- Open challenges include how to deal both analytically and numerically with:
  - higher-dimensional flows
    [*the curse of dimensionality*]
  - intersections of multiple switching boundaries
  - bifurcations of higher-codimension
  - Sliding of sliding
  - etc

Crucial in many applications

- Solving these problems is crucial in many applications of the theory of piecewise smooth systems





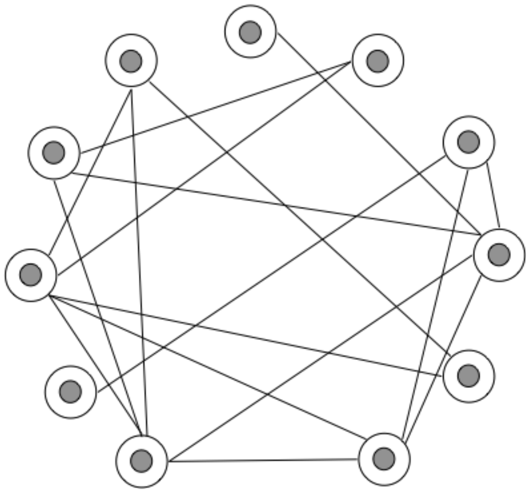


Complex networks..

- These problems are all instances of complex networks

Agent dynamics

$$\dot{x}_i = f_i(x_i) + g_i(x_i)u_i, \quad i = 1, \dots, N$$



Coupling protocol

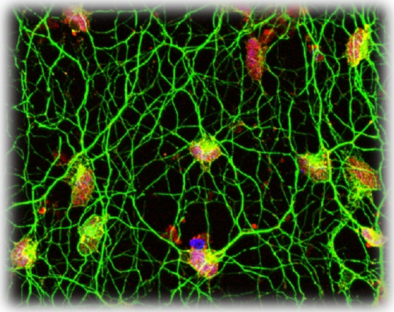
$$u_i = \sigma \sum_{j=1}^{N} a_{ij} \left[h(x_j) - h(x_i) \right]$$

Network structure

$$\mathcal{L} = \mathcal{D} - \mathcal{A}$$

..with a twist

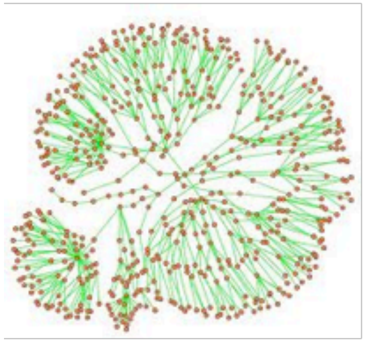
- There is a notable difference though…
- In complex networks theory it is often assumed that
  1. The dynamics of each agent is sufficiently smooth and differentiable
  2. Agents communicate via coupling functions that are smooth and differentiable
  3. The network structure is fixed in time..
- This is not the case in these applications

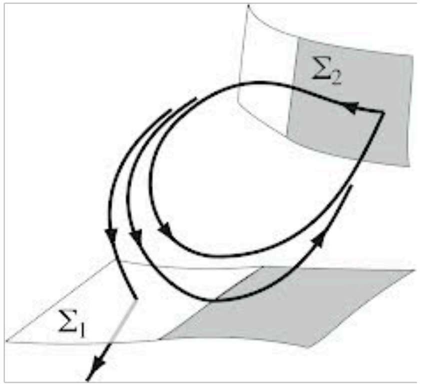


Bridging the gap

Complex Networks

Theory of PWSS



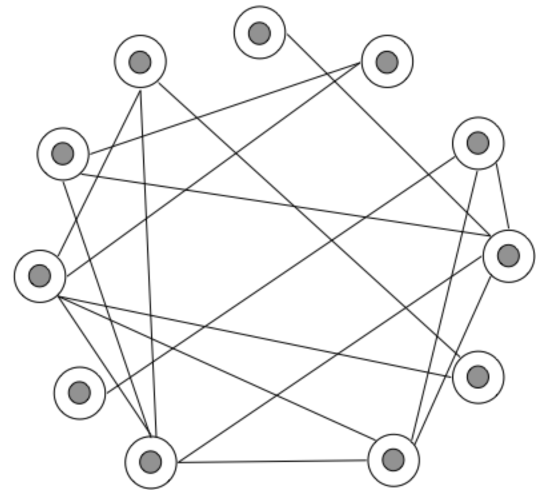


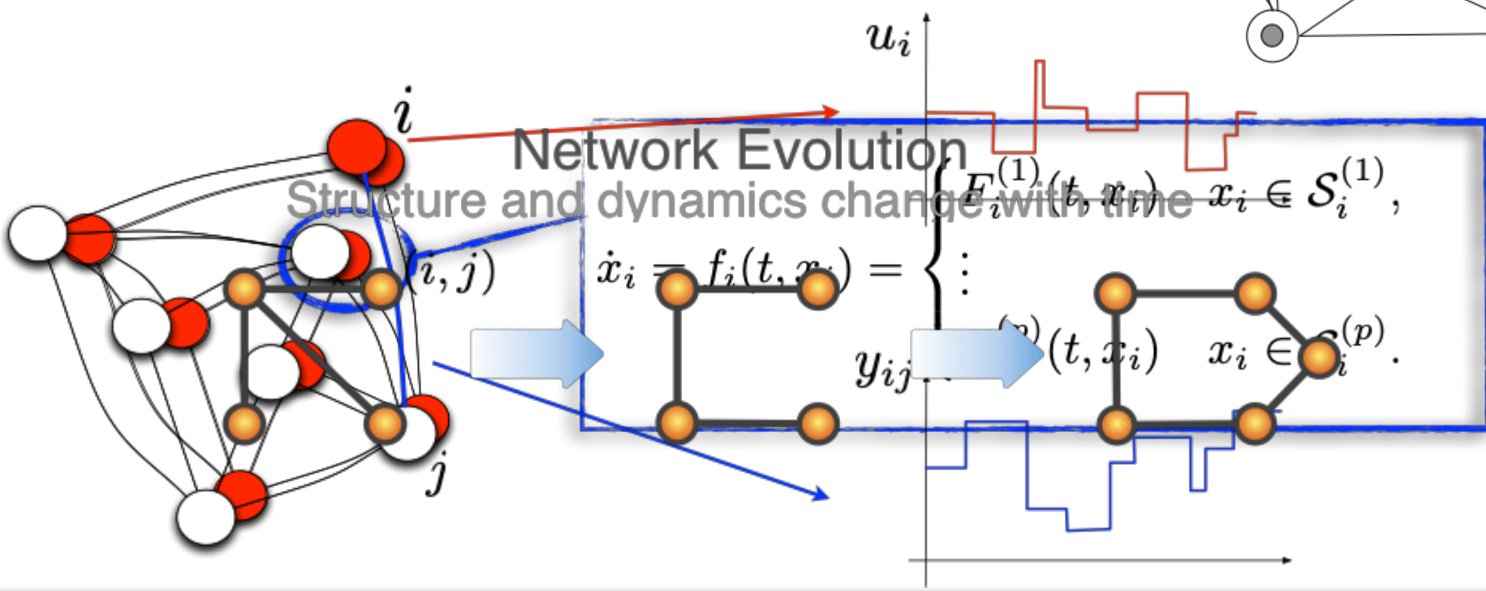
Complex Piecewise-smooth Networks

What is a piecewise-smooth network?

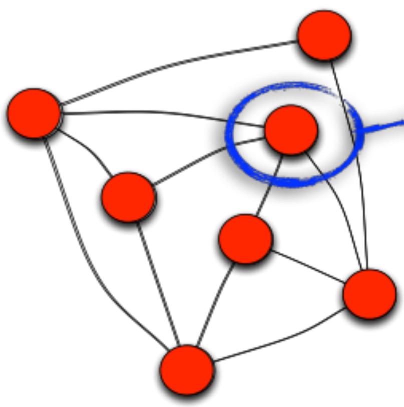
- A network can be piecewise smooth at different levels

1. At the agent level
2. At the coupling level
3. At the structural level
4. Any combination of the above



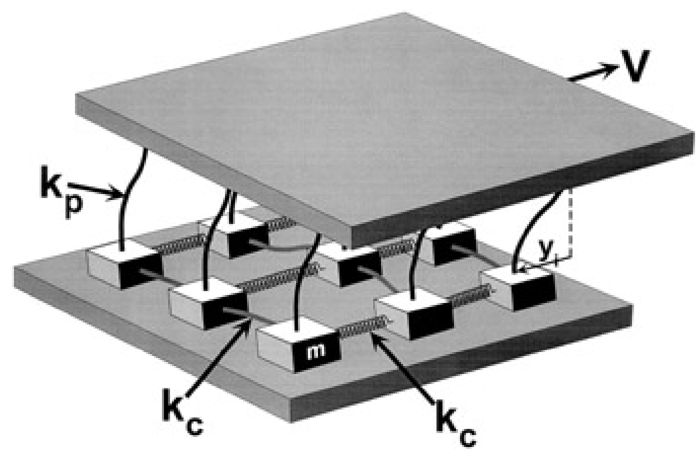


1. Networks of PWS systems



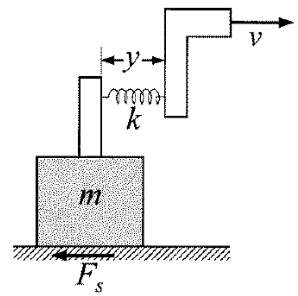
$$\dot{x}_i = f_i(t, x_i) = \begin{cases} F_i^{(1)}(t, x_i) & x_i \in \mathcal{S}_i^{(1)}, \\ \vdots & \\ F_i^{(p)}(t, x_i) & x_i \in \mathcal{S}_i^{(p)}. \end{cases}$$

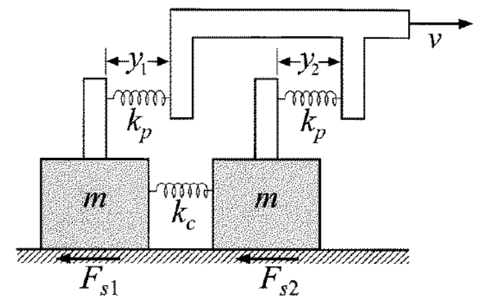
- Agents can
  - Have different degrees of discontinuity
  - Can be identical or heterogenous
  - Switching can be time-dependent or state-dependent

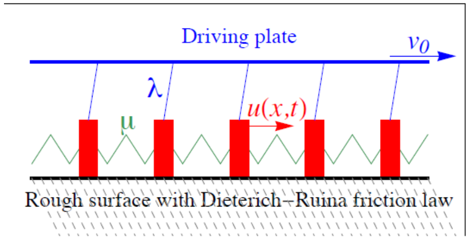


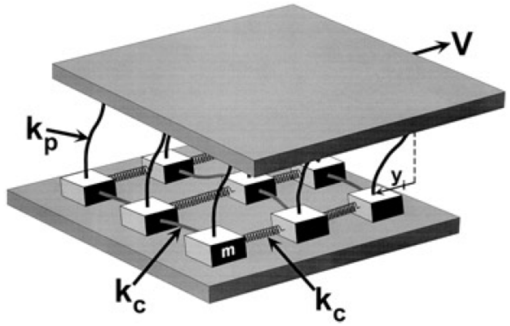
Examples

- Oscillations in Brake Systems [1]
- Burridge-Knopoff models in earthquake engineering [2]





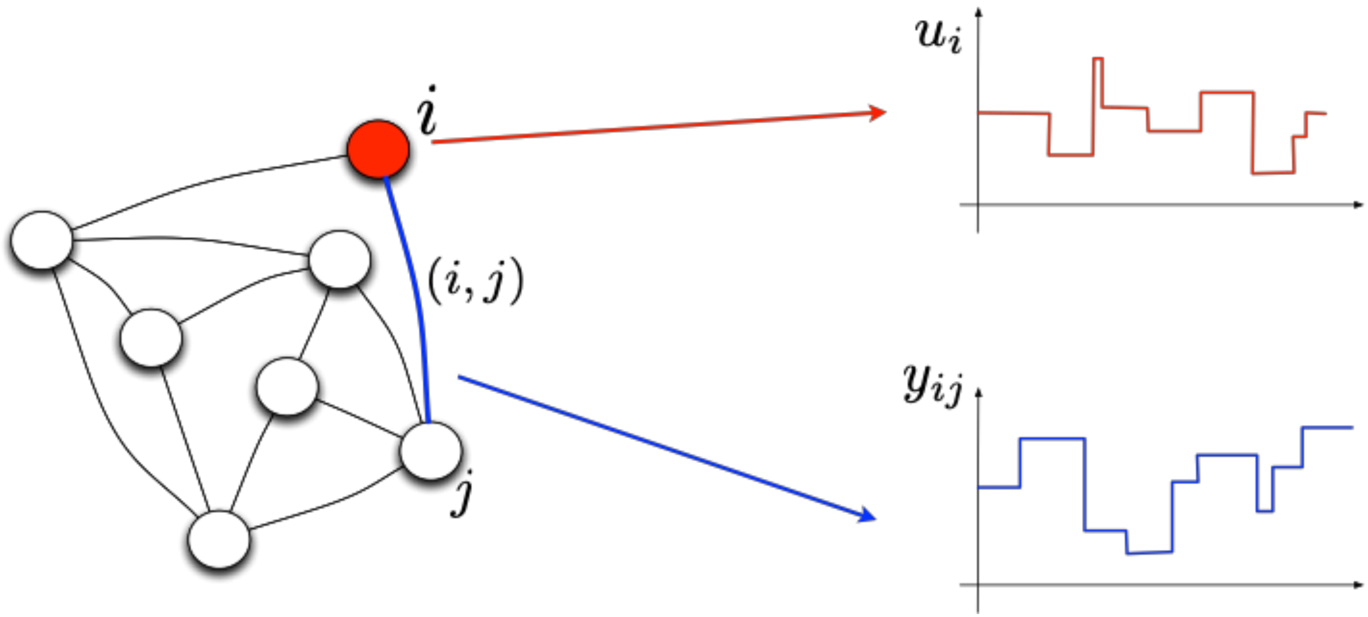




[1] G. Ostermeyer, "Dynamic Friction Laws and Their Impact on Friction Induced Vibrations", SAE technical paper, 2010.

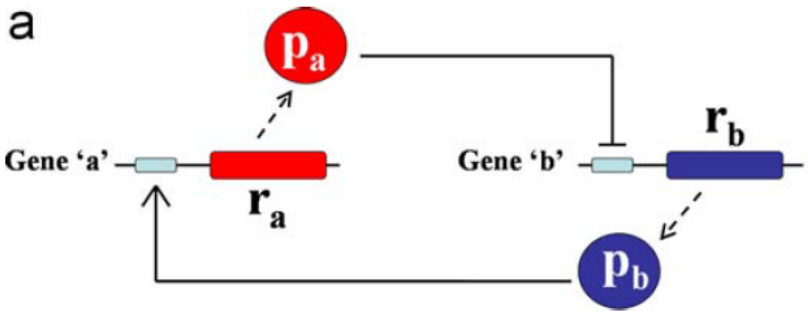
[2] D L. Turcotte, "Self-organized criticality." Reports on progress in physics 62.10, 1999.

2. Networks with PWS coupling



- Structure is fixed
- Agents communicate via PWS functions
  (e.g. switched functions, piecewise linear coupling)
- Delays, noise might be present

Example

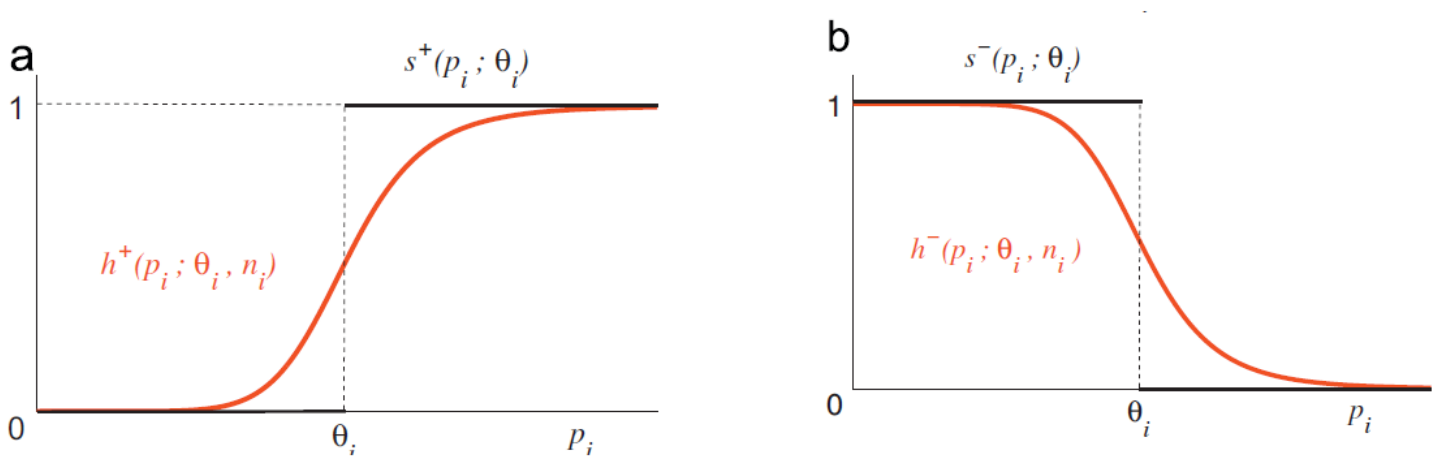


$$\dot{r}_a = m_a h^+(p_b; \theta_b, n_b) - \gamma_a r_a,$$

$$\dot{r}_b = m_b h^-(p_a; \theta_a, n_a) - \gamma_b r_b,$$

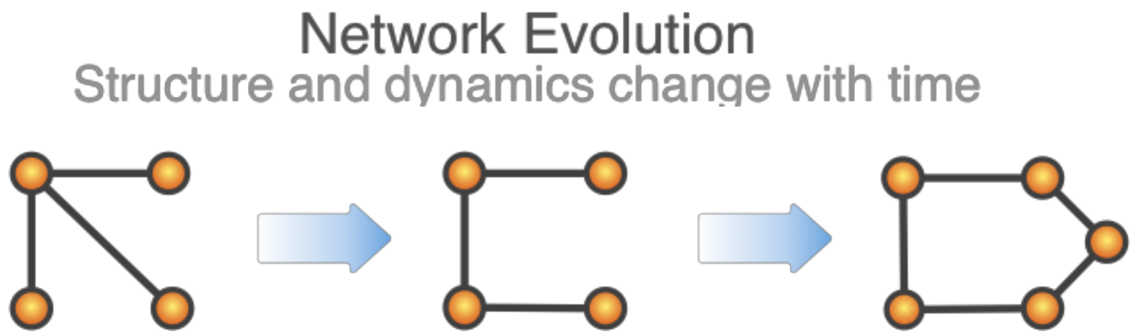
$$\dot{p}_a = k_a r_a - \delta_a p_a,$$

$$\dot{p}_b = k_b r_b - \delta_b p_b.$$



[1] L. Glass, S. A. Kauffman., "The logical analysis of continuous non-linear biochemical contro networks", J. Theor. Biol. 39, 103–129, 1973.

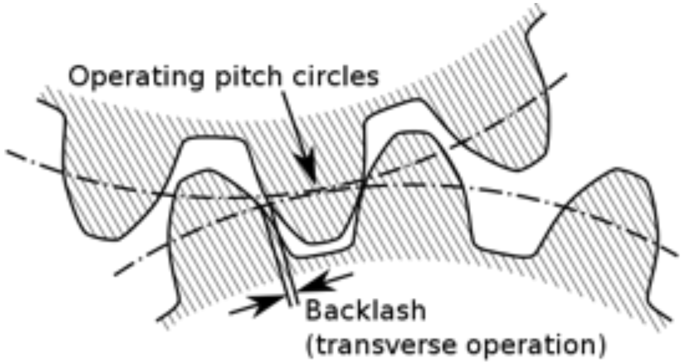
3. Networks with a PWS structure

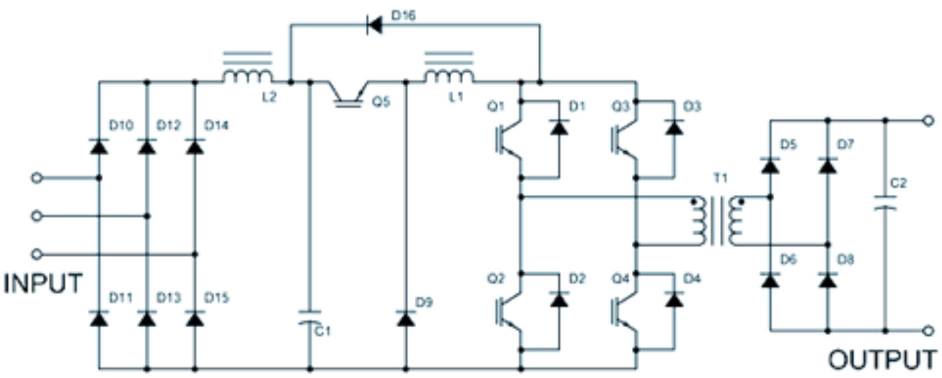


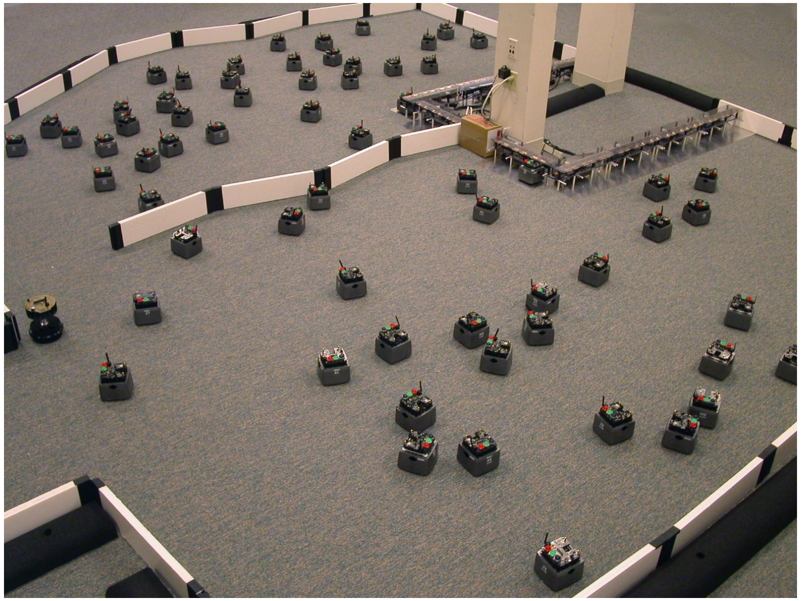
- In this case the structure of the graph evolves discontinously
- This is encoded in a time-varying network Laplacian
- ..that can change in a time-dependent or state-dependent manner
- e.g., event-based connections, proximity networks

Examples

- Temporal networks
- State-dependent connections (proximity networks)
- Power electronic networks

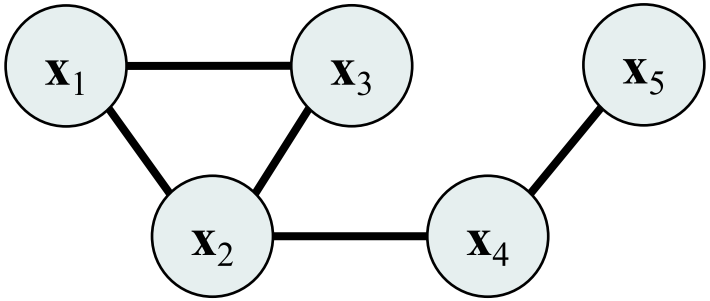






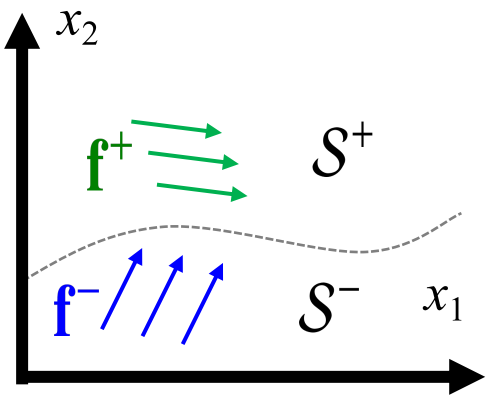
Our goal

- We want to study the collective behaviour of *networks of piecewise smooth systems*



$$\dot{\mathbf{x}}_i = \mathbf{f}_i(\mathbf{x}_i) - \sum_{j=1}^{N} L_{ij}\mathbf{g}(\mathbf{x}_i, \mathbf{x}_j), \quad i = 1, \dots, N.$$

$$\mathbf{f}(\mathbf{x}) = \begin{cases} \mathbf{f}^+(\mathbf{x}), & \mathbf{x} \in \mathcal{S}^+ \\ \mathbf{f}^-(\mathbf{x}), & \mathbf{x} \in \mathcal{S}^- \end{cases}$$



- We focus on synchronization as the simplest type of emerging behaviour

Key research questions

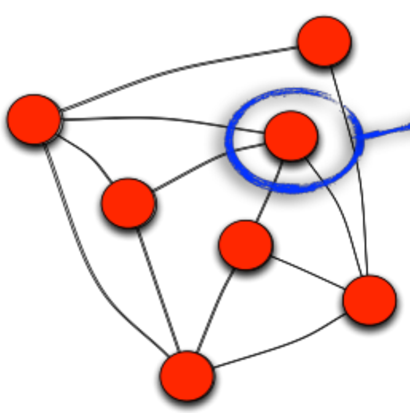
- Under what conditions can we guarantee convergence?
- Can we engineer a distributed coupling protocol able to induce synchronization?
- Can we achieve synchronization without controlling all of the system nodes?
- Can we *prove* local or global convergence to the synchronous behaviour? Bounded or asymptotic?

$$\lim_{t\to\infty} \sup \left\| \mathbf{x}_i(t) - \mathbf{x}_j(t) \right\| \leq \varepsilon, \quad \forall i,j \quad t > t_0.$$

$$\lim_{t\to\infty} \left\| \mathbf{x}_i(t) - \mathbf{x}_j(t) \right\| = 0, \quad \forall i,j.$$

Key hurdles

- A PWS network is a large high-dimensional PWS system
- Sliding motion can occur on several surfaces
- Asynchronous switching of different nodes can easily lead to instability
- Tools to prove global convergence of large PWS systems are few and very conservative



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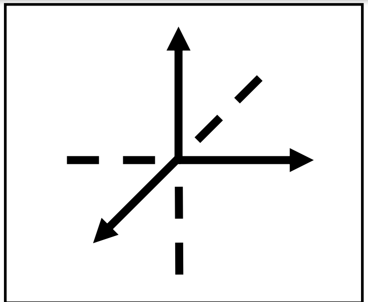
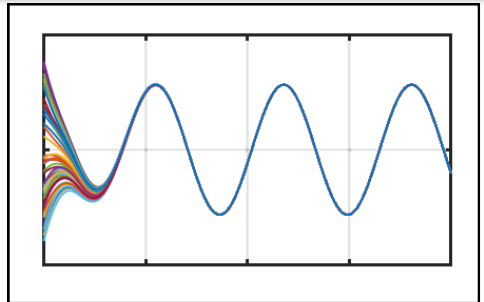
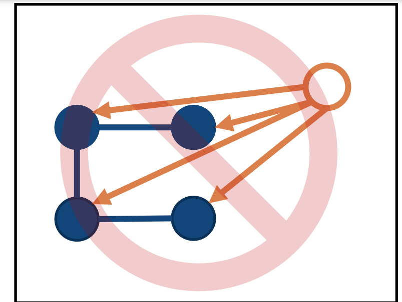
Relevant previous results







Paper	Global sync.	Asymptotic sync.	Control	Lack of centralised control
[1]			Yes	✗

[1] X. Yang, Z. Wu, J. Cao, “Finite-time synchronization of complex networks with nonidentical discontinuous nodes,” Nonlin. Dyn., 73(4), 2313–2327, 2013.

[2] P. De Lellis, M. di Bernardo, D. Liuzza, “Convergence and sync. in heterogeneous networks of smooth and piecewise smooth systems,” Automatica, 56, 1–11, 2015.

[3] S. Coombes, R. Thul, “Synchrony in networks of coupled non-smooth dynamical systems: Extending the master stability function”, Eur.J.Appl.Math.,27(6), 904–922, 2016.

[4] M. Coraggio, P. De Lellis, M. di Bernardo, “Achieving convergence and synchronization in Networks of PWS Systems via distributed discontinuous coupling”, submitted to Automatica, 2019.

2. Proving convergence

Sufficient conditions for convergence

- Given a network of PWS systems

$$\dot{\mathbf{x}}_i = \mathbf{f}_i(\mathbf{x}_i) - \sum_{j=1}^{N} L_{ij}\mathbf{g}(\mathbf{x}_i, \mathbf{x}_j), \quad i = 1, \ldots, N.$$

- We want to find conditions on
  - the agent dynamics
  - the coupling function
  - the network structure
- in order to guarantee **global** convergence of all the agents towards synchronization

Agent dynamics: the QUAD condition

- A widely used condition for networks of smooth systems is the QUAD assumption

$$(\mathbf{v}_1 - \mathbf{v}_2)^{\mathrm{T}} \mathbf{P} [\mathbf{f}(\mathbf{v}_1) - \mathbf{f}(\mathbf{v}_2)] \leq (\mathbf{v}_1 - \mathbf{v}_2)^{\mathrm{T}} \underline{\mathbf{Q}} (\mathbf{v}_1 - \mathbf{v}_2) \quad \forall \mathbf{v}_1, \mathbf{v}_2.$$

- It is possible to show that some PWS systems fulfill this property
- In this case convergence results can immediately be extended to networks of PWS systems of the form

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) - c \sum_{j=1}^{N} L_{ij} \boldsymbol{\Gamma}(\mathbf{x}_j - \mathbf{x}_i), \quad i = 1, \ldots, N,$$

1.P. De Lellis, M. di Bernardo, G. Russo, "On QUAD, Lipschitz and contracting vector fields for consensus and synchronization of networks", IEEE Transactions on Circuits and Systems I, vol. 58, pp. 576–583, 2011

Theorem. Suppose

- $\mathbf{f}$ is QUAD($\mathbf{P}$, $\mathbf{Q}$), with $\mathbf{P} > 0$,
- $\boldsymbol{\Gamma} > 0$.

The network synchronizes if $c > c^*$, where

$$c^* \triangleq \frac{\|\mathbf{Q}\|_2}{\lambda_2(\mathbf{L}) \quad \lambda_{\min}(\mathrm{sym}(\mathbf{P}\boldsymbol{\Gamma}))}.$$

Agent dynamics

Topology

Coupling protocol

[1] M. Coraggio, P. De Lellis, S. J. Hogan, M. di Bernardo, “Synchronization of Networks of Piecewise-Smooth Systems,” IEEE Control Syst. Lett., 2(4), 653–658, 2018.

Example

- Consider a relay system $\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i)$, with

$$\mathbf{f}(\mathbf{x}_i) = \begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix} \mathbf{x}_i - \begin{bmatrix} 0 \\ 2\mathrm{sign}(x_{i,1} + x_{i,2}) \end{bmatrix}.$$

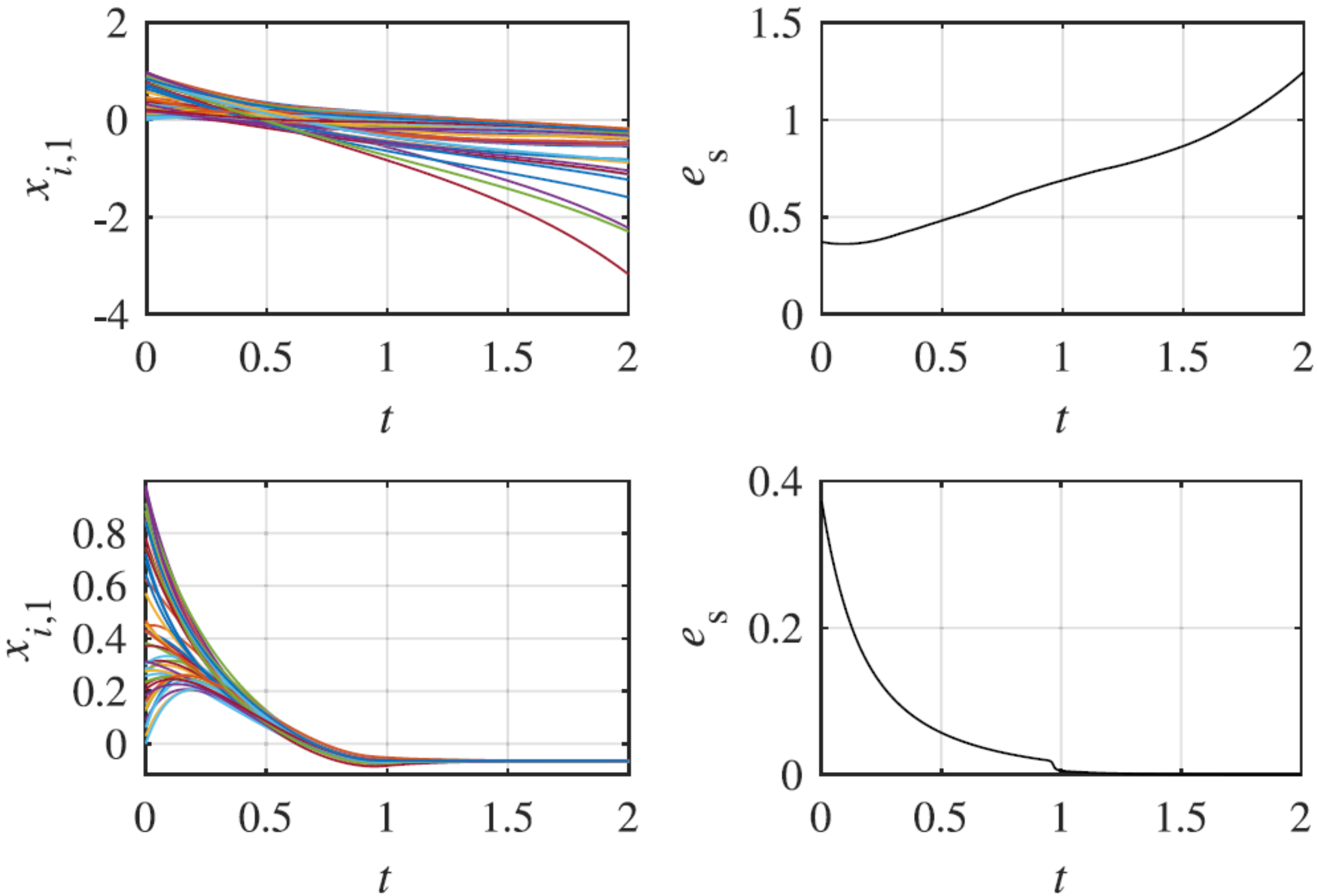
- The system is QUAD with

$$\mathbf{P} = \mathbf{I}, \qquad \mathbf{Q} = \begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix}, \quad \|\mathbf{Q}\|_2 = 3.06$$

- Take a network of $N = 50$ systems with $\lambda_2(\mathbf{L}) = 14.80$, $\mathbf{\Gamma} = \mathbf{I}$.
- Then we can estimate the critical coupling gain as

$$c^* = \|\mathbf{Q}\|_2 \,/\, \lambda_2(\mathbf{L}) = 0.21$$

Example

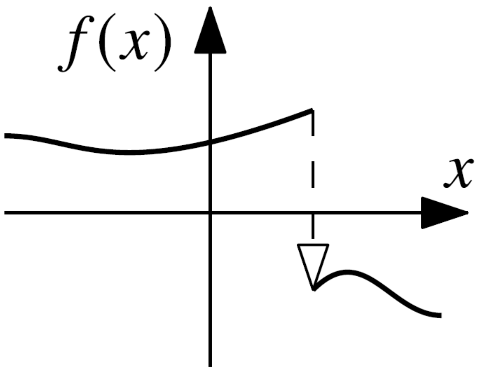


$c = 0.05 < c^*$

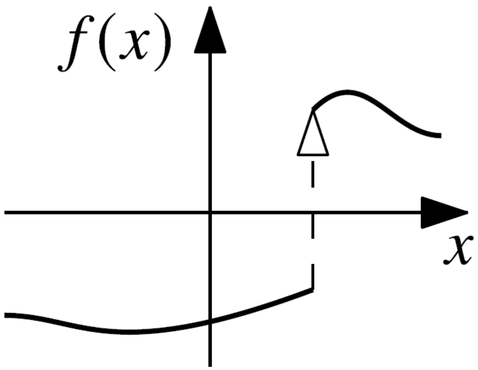
$c = 0.25 > c^*$

Problem

- For PWS systems the QUAD assumption is often too strong



QUAD function



Non-QUAD function

- So we need to relax that assumption in order to account for more realistic cases

A first relaxation of the QUAD property

- As a first attempt we introduced the concept of a **QUAD-affine** vector field such that

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}_1(\mathbf{x}) + \mathbf{f}_2(\mathbf{x})$$

$$\mathbf{f}_1(\mathbf{x}) \text{ is QUAD}, \quad \|\mathbf{f}_2(\mathbf{x})\|_2 < M \qquad\qquad 0 < M < +\infty$$

- Under this assumption we were able to find sufficient conditions for **bounded** synchronization

$$\lim_{t\to\infty} \sup \left\|\mathbf{x}_i(t) - \mathbf{x}_j(t)\right\| \leq \varepsilon, \quad \forall i, j \quad t > t_0.$$

P. De Lellis, M. di Bernardo, D. Liuzza, “Convergence and sync. in heterogeneous networks of smooth and piecewise smooth systems,” *Automatica*, 56, 1–11, 2015.

Bounded convergence result

- Here is a flavour of the conditions we were able to derive

Estimate of the critical coupling gain

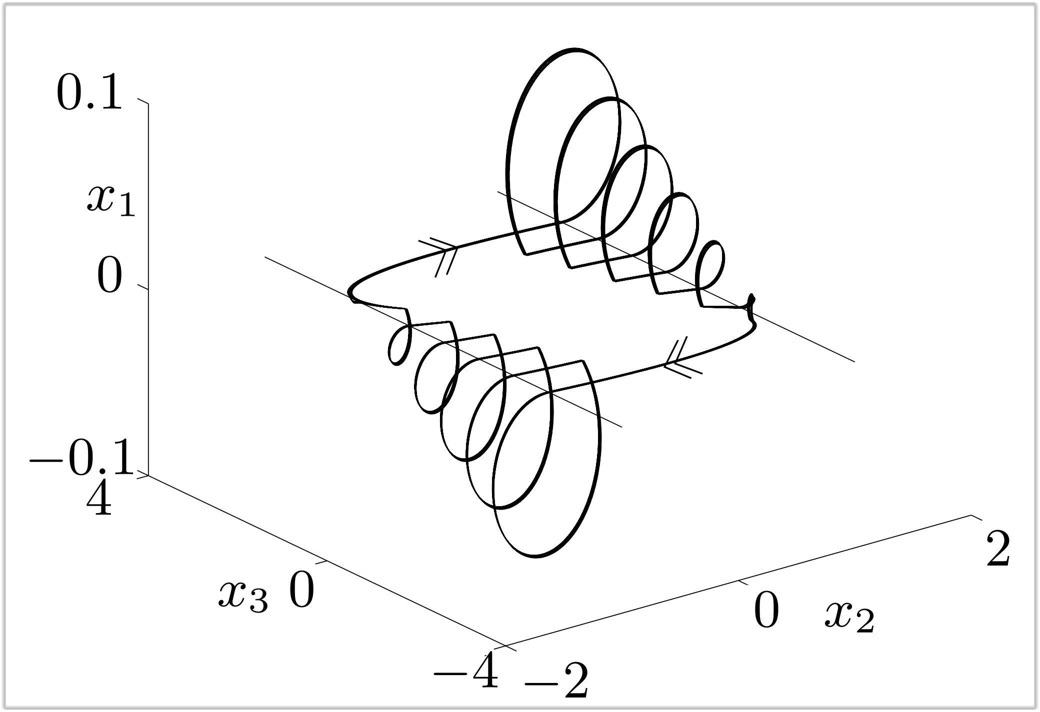
$$\tilde{c} = \max\left\{ \frac{1}{\lambda_2(L \otimes \Upsilon_r)} \left(\frac{2\sqrt{N}(\bar{M} + h_{\max})}{e_{\max}} + \lambda_{\max}(W_r^{\max}) \right), 0 \right\}$$

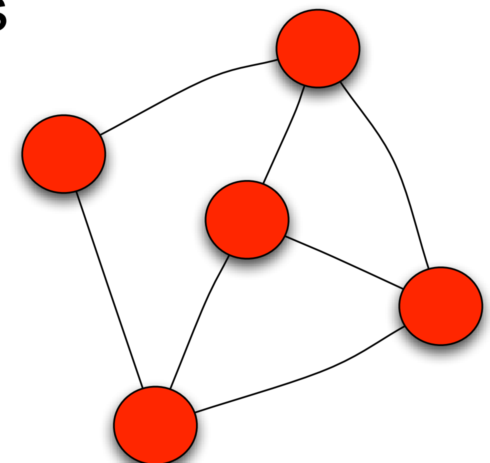
Upper bound of the synchronization error

$$\bar{\epsilon} = \min\left\{ -\frac{\sqrt{N}\,(\bar{M} + \bar{h}_0)}{w_{\max}}, \frac{\sqrt{N}(\bar{M} + h_{\max})}{-\max\left\{\lambda_{\max}(W_r^{\max}) - c\lambda_2(L \otimes \Upsilon_r), \lambda_{\max}(W_{n-r}^{\max})\right\}} \right\}$$

Example

- Take a network of five chaotic relay systems

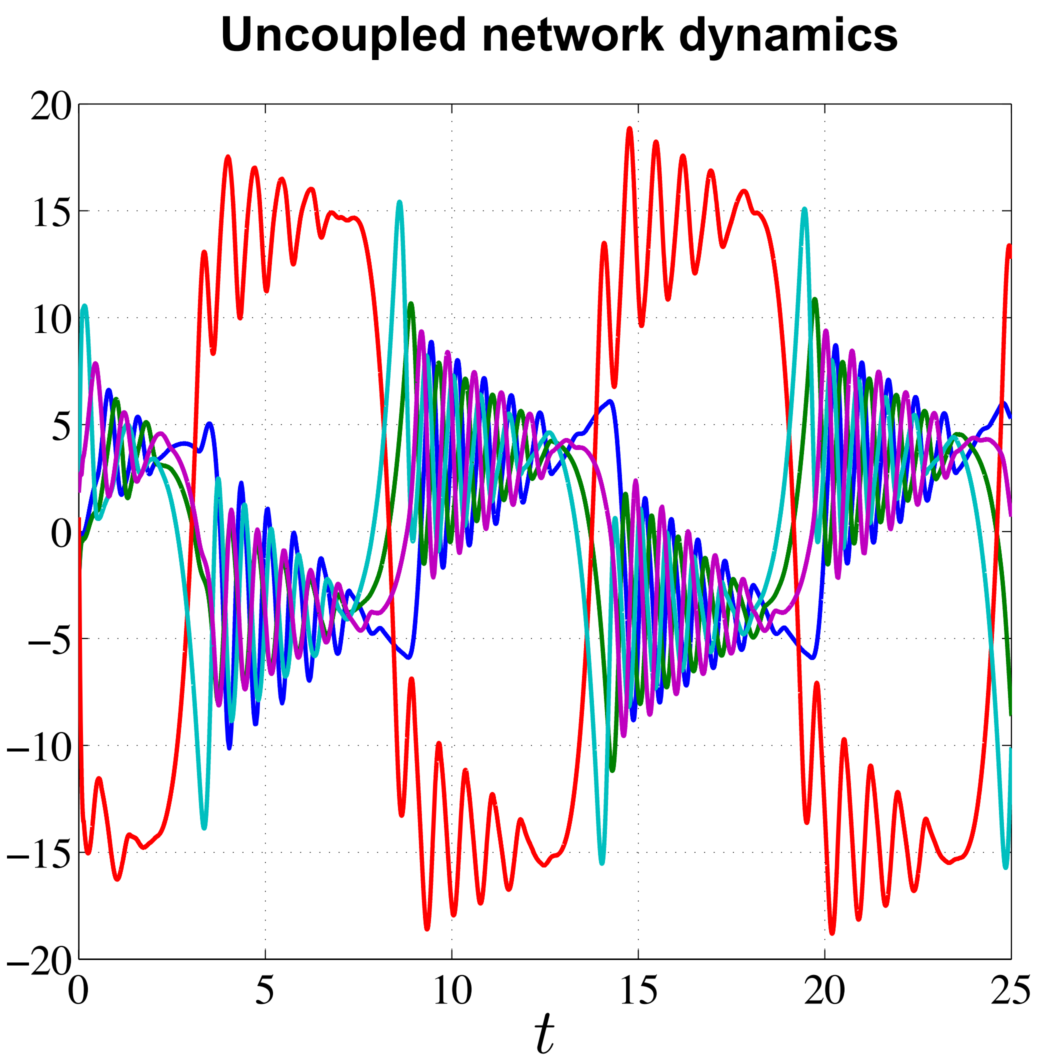




$$\dot{x}_i = Ax_i + Br_i + u,$$
$$y_i = Cx_i,$$
$$r_i = -\mathrm{sgn}(y_i).$$

M. di Bernardo, C. J. Budd, A. R. Champneys, and P. Kowalczyk, “Piecewise-smooth Dynamical Systems”, Springer-Verlag, 2008

Numerical results



Remarks

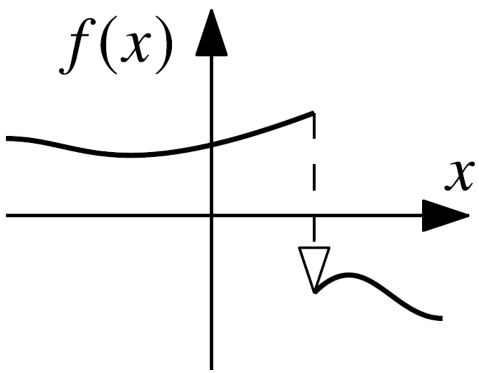
- The results are very general and also apply to networks of heterogeneous systems
- Conditions not always easy to apply (lots of algebraic manipulations)
- Most importantly analytical estimates are often too conservative
- We can only prove bounded convergence while often in simulations (e.g. relays' example) we observe asymptotic convergence
- What if we want to achieve asymptotic convergence?

3.

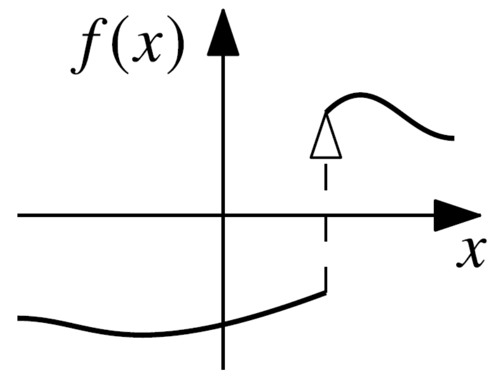
A multiplex discontinuous approach

From QUAD to σ-QUAD

- For those PWS functions that are not QUAD…



QUAD | σ-QUAD



not QUAD | σ-QUAD

- We define a vector field as **σ-QUAD** if

$$(\mathbf{v}_1 - \mathbf{v}_2)^{\mathrm{T}} \mathbf{P} [\mathbf{f}(\mathbf{v}_1) - \mathbf{f}(\mathbf{v}_2)] \leq (\mathbf{v}_1 - \mathbf{v}_2)^{\mathrm{T}} \mathbf{Q} (\mathbf{v}_1 - \mathbf{v}_2) \qquad \forall \mathbf{v}_1, \mathbf{v}_2$$

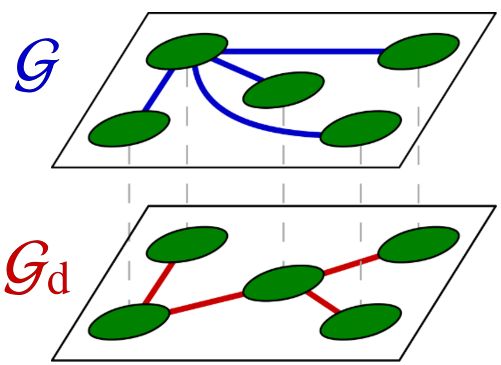
- Now the dynamics $\mathbf{f}$ can have any number of finite jumps.

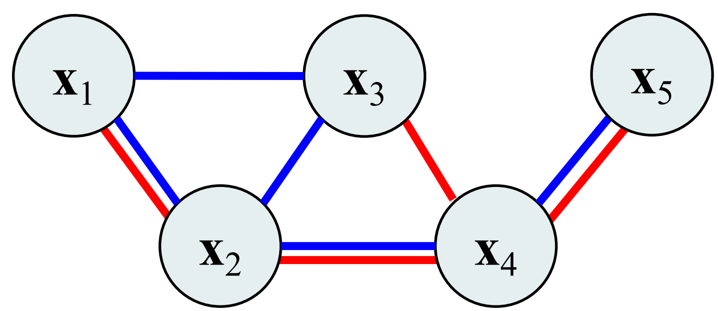
A distributed discontinuous approach

- In general diffusive coupling does not suffice on its own to achieve convergence in these cases

$$\dot{x}_i = f(x_i) - c\sum_{j=1}^{N} L_{ij}\Gamma(x_j - x_i) - c_d \sum_{j=1}^{N} L^d_{ij}\Gamma_d \,\mathrm{sign}(x_j - x_i)$$

- So we added an extra discontinuous coupling layer to guarantee convergence

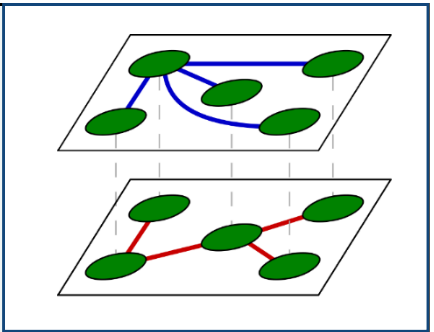




Synchronization of PWS QUAD systems

Theorem. Suppose

- f is σ-QUAD($\mathbf{P}$, $\mathbf{Q}$, $\mathbf{M}$), with $\mathbf{P} > 0$,
- $\mathbf{\Gamma}, \mathbf{\Gamma}_\mathrm{d} > 0$.



The network synchronizes if $c > c^*$ and $c_\mathrm{d} \geq c_\mathrm{d}^*$, where

$$c^* = \frac{\|\mathbf{Q}\|_2}{\lambda_2(\mathbf{L}) \quad \lambda_{\min}(\mathrm{sym}\ \mathbf{P\Gamma})}, \qquad c_\mathrm{d}^* = \frac{\|\mathbf{M}\|_\infty}{\delta_{\mathcal{G}_\mathrm{d}} \quad \mu_\infty^-(\mathbf{P\Gamma}_\mathrm{d})}.$$

Agent dynamics

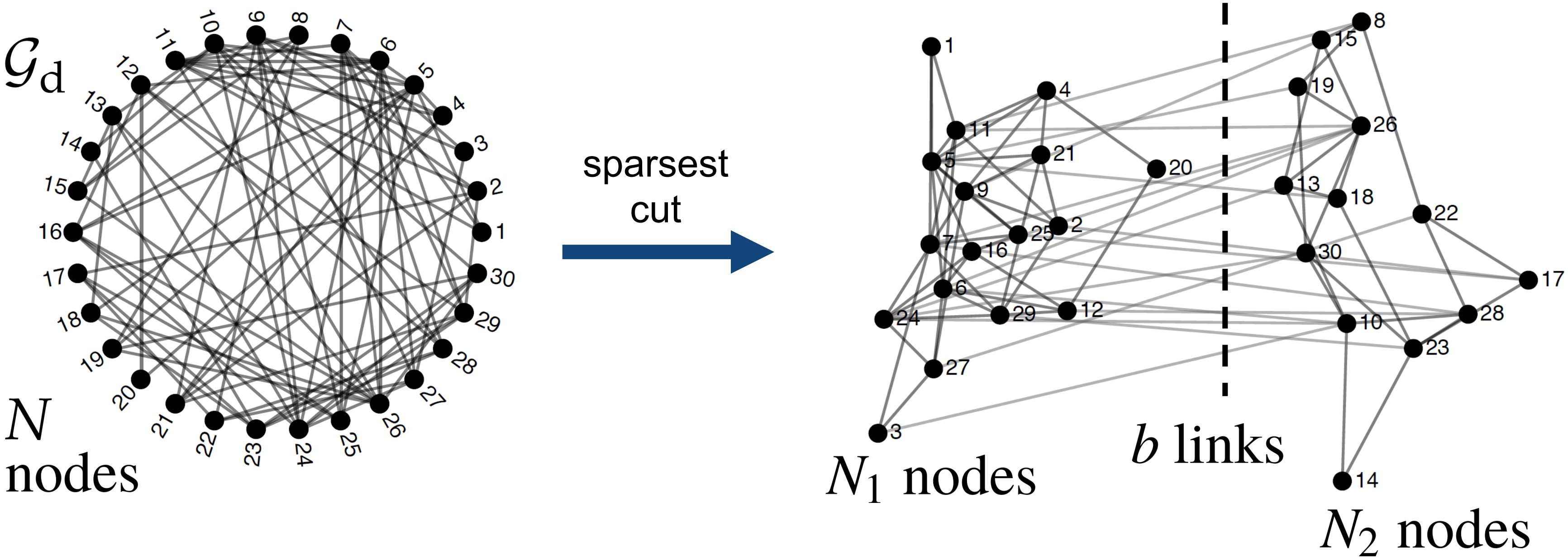
Topology

Coupling protocol

M. Coraggio, P. De Lellis, M. di Bernardo, “Achieving Convergence and Synchronization in Networks of Piecewise-Smooth Systems via Distributed Discontinuous Coupling”, submitted to Automatica (see arXiv), 2019.

Minimum density

- $\delta_{\mathcal{G}_\mathrm{d}}$: *minimum density* of $\mathcal{G}_\mathrm{d}$; computed from *sparsest cut*.

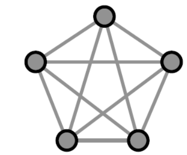


$$\delta_{\mathcal{G}_\mathrm{d}} \triangleq \frac{N}{2} \min \frac{b}{N_1 N_2}$$

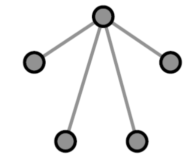
- We derived an algorithm and formulae for selected structures

Minimum density for selected topologies

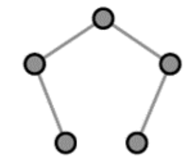
- Complete: $\delta_{\mathcal{G}} = \dfrac{N}{2}$



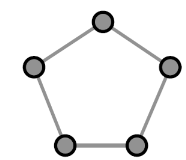
- Star: $\delta_{\mathcal{G}} = \dfrac{N}{2(N-1)}$



- Path: $\delta_{\mathcal{G}} = \begin{cases} 2/N, & N \text{ even} \\ 2N/(N^2-1), & N \text{ odd} \end{cases}$



- Ring: $\delta_{\mathcal{G}} = \begin{cases} 4/N, & N \text{ even} \\ 4N/(N^2-1), & N \text{ odd} \end{cases}$



- *l*-nearest neighbours: $\delta_{\mathcal{G}} = \begin{cases} \frac{4\sum_{k=0}^{l-1}(l-k)}{N}, & N \text{ even} \\ \frac{4N\sum_{k=0}^{l-1}(l-k)}{N^2-1}, & N \text{ odd} \end{cases}$

Example: back to relay systems

- Consider $N = 30$ chaotic relay systems $\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i)$, with

$$\mathbf{f}(\mathbf{x}_i) = \begin{bmatrix} 1.51 & 1 & 0 \\ -99.922 & 0 & 1 \\ -5 & 0 & 0 \end{bmatrix} \mathbf{x}_i - \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \operatorname{sign}(x_{i,1}).$$

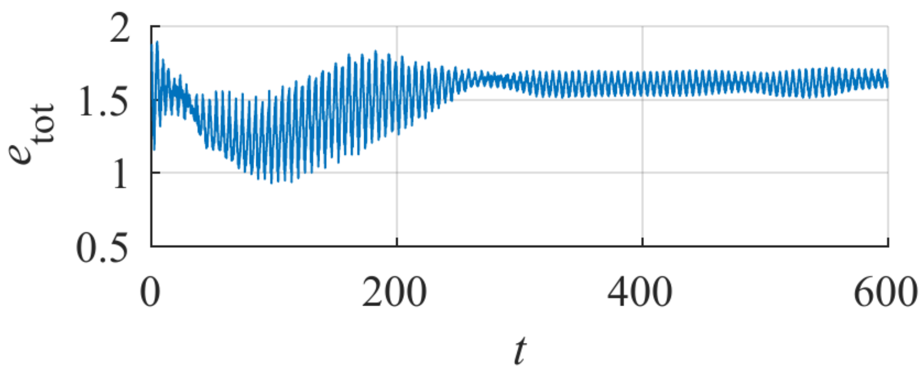
- These systems are σ-QUAD with

$$\mathbf{P} = \mathbf{I}, \quad \mathbf{Q} = \begin{bmatrix} 1.51 & 1 & 0 \\ -99.922 & 0 & 1 \\ -5 & 0 & 0 \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

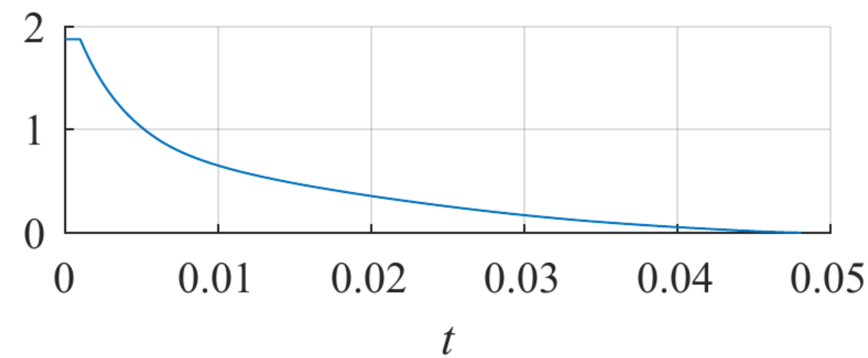
- $||\mathbf{Q}||_2 = 100.063, \quad ||\mathbf{M}||_\infty = 4.$

Numerical validation

- Coupling protocol: $\mathbf{\Gamma} = \mathbf{\Gamma}_{\mathrm{d}} = \mathbf{I}$;
  Diffusive layer: $\lambda_2(\mathbf{L}) = 1$;
  Discontinuous layer: $\delta_{\mathcal{G}\mathrm{d}} = 1.290$.
- $c^* = \|\mathbf{Q}\|_2 \,/\, \lambda_2(\mathbf{L}) = 100.063$, $\quad c_{\mathrm{d}}^* = \|\mathbf{M}\|_\infty \,/\, \delta_{\mathcal{G}\mathrm{d}} = 3.102$.



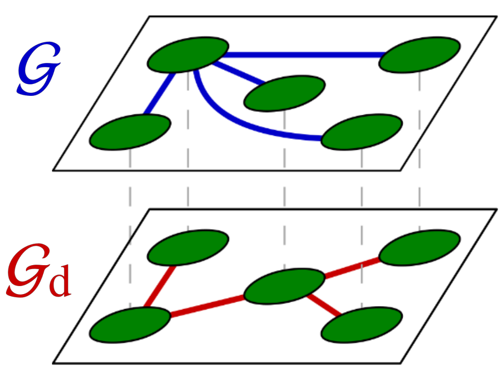
$c = 0.1 < c^*$, $\quad c_{\mathrm{d}} = 0.001 < c_{\mathrm{d}}^*$

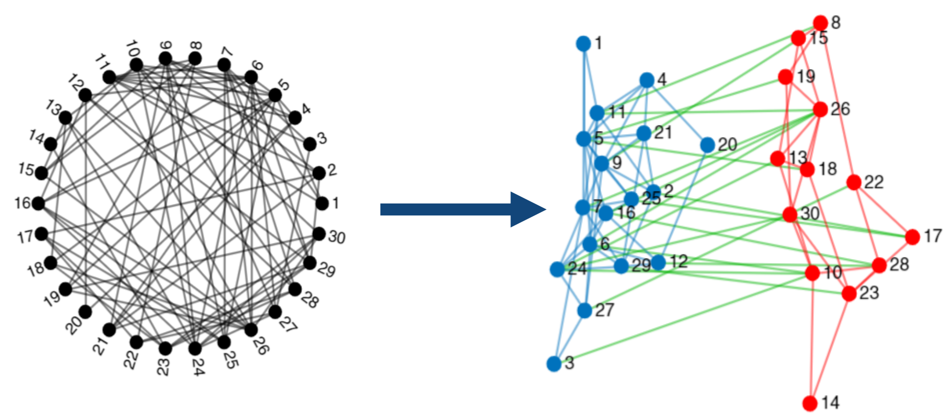


$c = 101 > c^*$, $\quad c_{\mathrm{d}} = 3.2 > c_{\mathrm{d}}^*$

Remarks

- Still a conservative result but..
- ..the structural nature of the sufficient conditions we obtained can be exploited to evaluate the effect on synchronization of structural changes in the network
- ..and its multiplex structure gives us additional degrees of freedom to enhance convergence





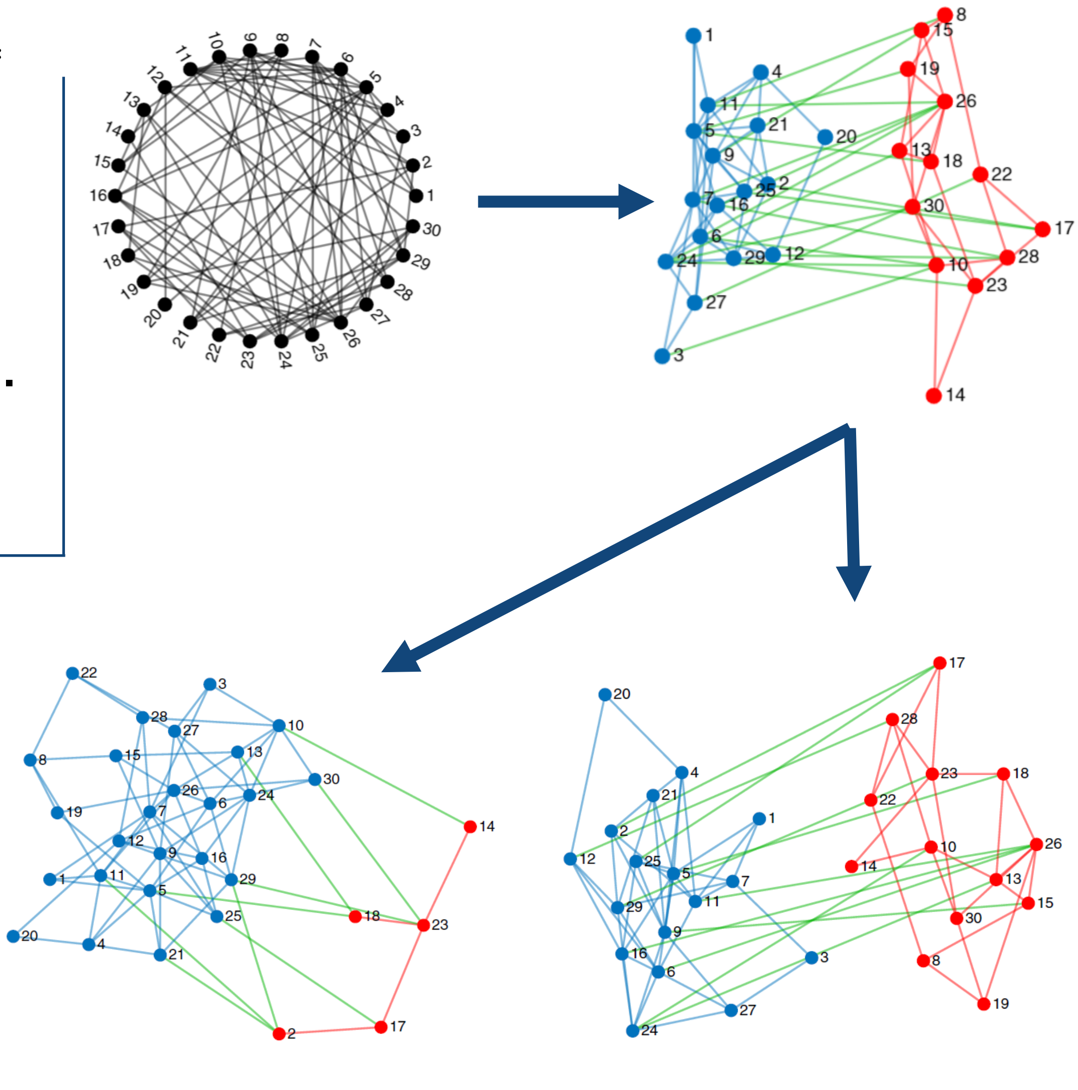
Resilience to faults

- Resilience: how does c_d^* change when links are removed?

a) Remove 4 blue & 4 red.

b) Remove 8 green.

- $\delta_{\mathcal{G}\,\mathrm{d,a}} > \delta_{\mathcal{G}\,\mathrm{d,b}}$ (a better than b). So *here* inter-cluster links are more important to have a low threshold c_d^*

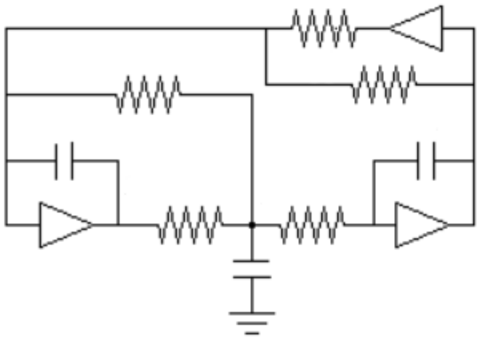


(a) (b)

Varying the multiplex layer structure

- Consider $N = 10$ chaotic Sprott circuits, with

$$\mathbf{f}(\mathbf{x}_i) = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -0.5 \end{bmatrix}}_{\mathbf{Q}} \mathbf{x}_i + \begin{bmatrix} 0 \\ 0 \\ \mathrm{sign}(x_{i,1}) \end{bmatrix}.$$

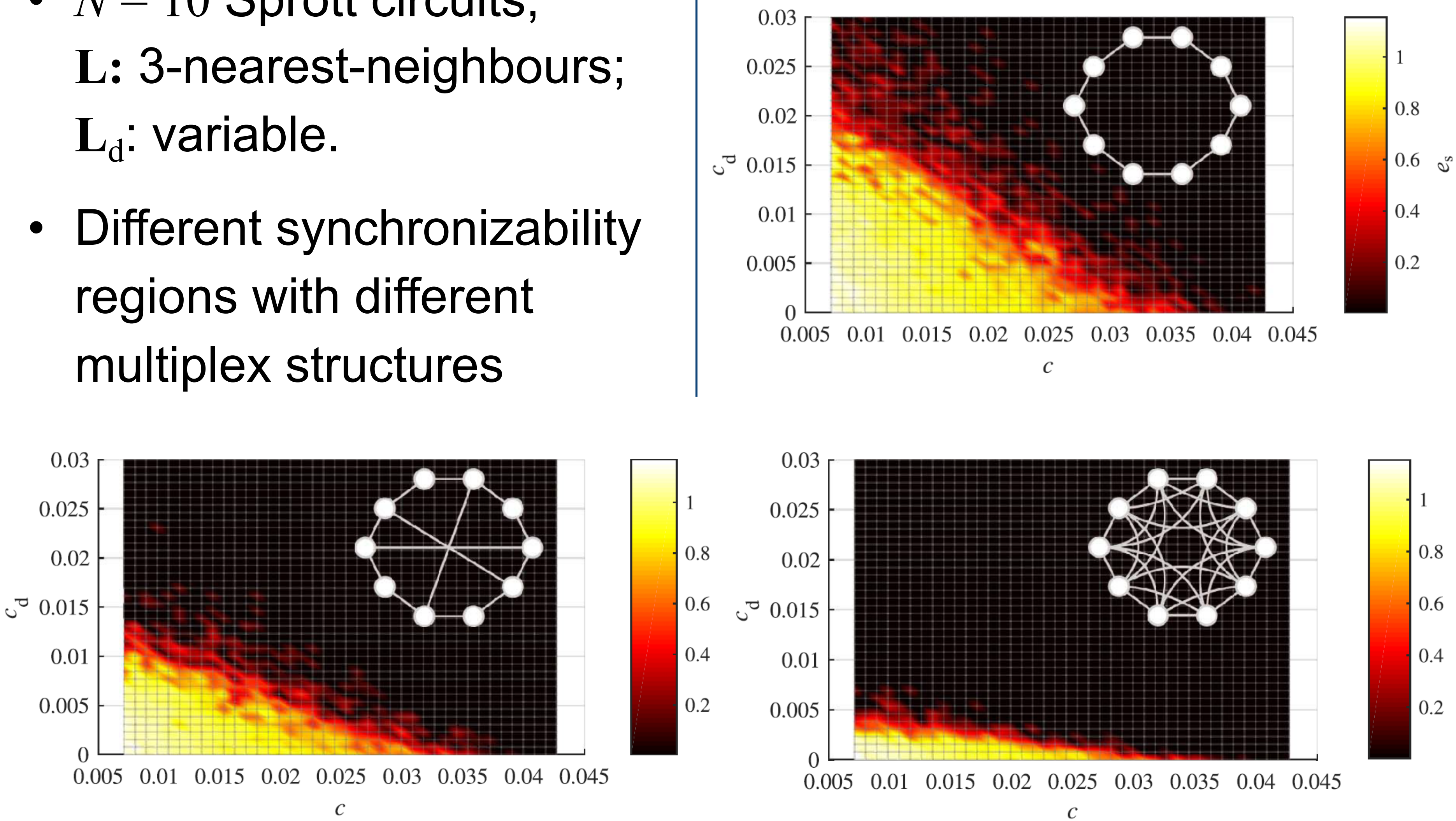


- **f** is σ-QUAD with

$$\mathbf{P} = \mathbf{I}, \quad \|\mathbf{Q}\|_2 = 1.7, \quad \mathbf{M} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

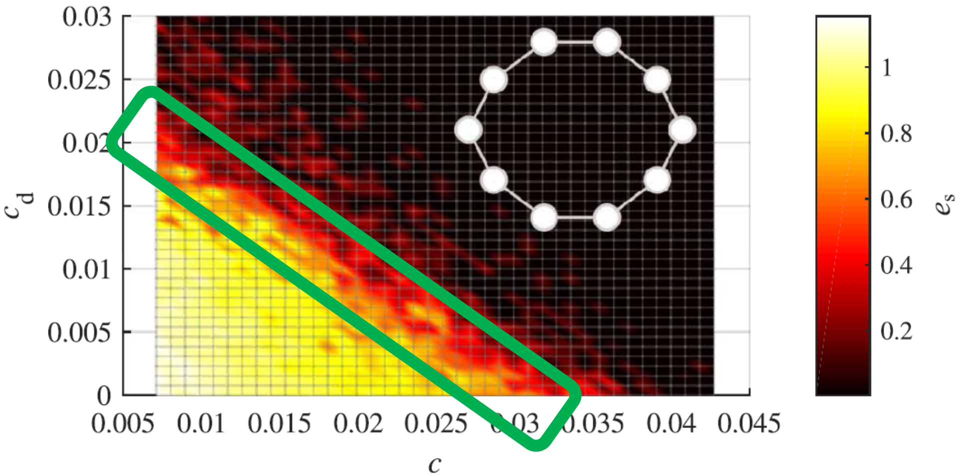
Varying the upper layer structure

- $N = 10$ Sprott circuits;
  L: 3-nearest-neighbours;
  $\mathbf{L}_\mathrm{d}$: variable.
- Different synchronizability regions with different multiplex structures



A pressing open problem

- Our convergence results give very conservative estimates of the region where synchronization is attained



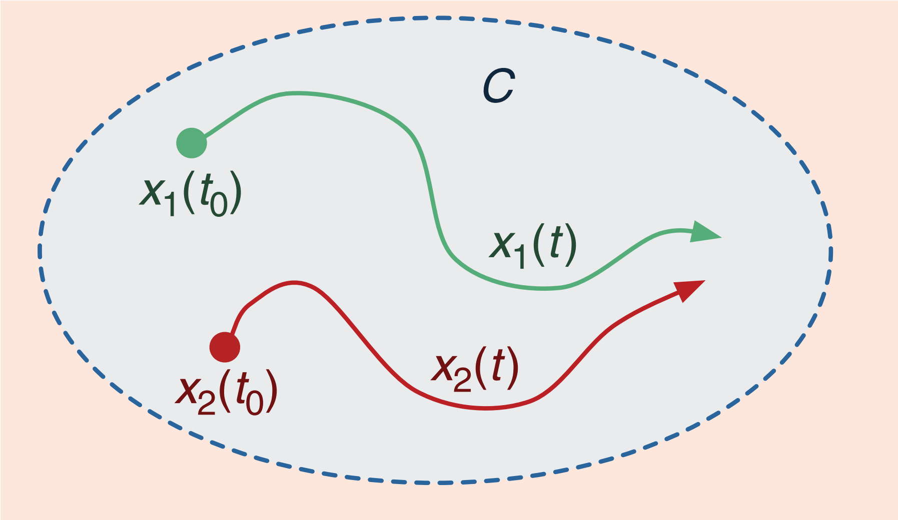
- We need better conditions to capture the sophisticated link between the multiplex nature of the coupling layers and the synchronizability of the network

4. Concluding remarks

Conclusions

- We focussed on the problem of finding sufficient conditions for **global convergence** in networks of dynamical systems with discontinuous vector fields
- We saw a set of **different conditions** on the agent dynamics, the coupling protocol and the network structure
- Common to all was the assumption that the agent vector field was **QUAD, QUAD-affine or σ-QUAD**
- We saw that a ***multiplex* coupling strategy** can be effective to achieve global asymptotic convergence..
- ..with conditions that can be nicely related to **structural properties** of the coupling layers

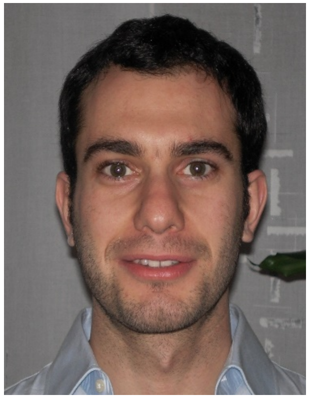
- Still the conditions are not always easy to apply
- They are very conservative and fail to capture the intricate link between the structure of the coupling layers and regions of stability
- As for networks of smooth systems maybe the goal is not to prove asymptotic stability of the network towards the synchronization manifold..
- ..but to prove **incremental stability** of the agent trajectories



Acknowledgements

- Marco Coraggio (PWS networks)
- Piero De Lellis (PWS networks)
- Davide Liuzza (QUAD-affine systems)
- John Hogan (PWS networks)
- For more info check

  https://sites.google.com/site/dibernardogroup/



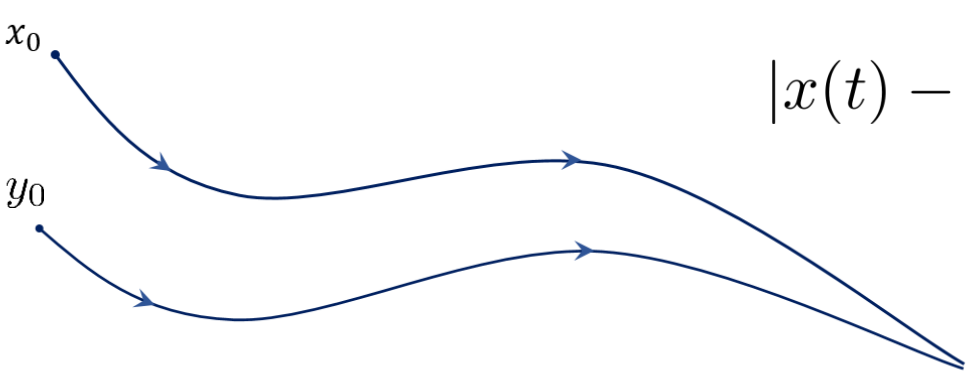
Extending contraction theory to PWSS

- A separate route we took was to extend contraction theory to PWS systems (this is another talk ☺)

Definition

A continuously differentiable system $\dot{x} = f(x)$ is said to be *contracting* in a subset $\mathcal{C} \subseteq \mathbb{R}^n$ if there exists some norm, with associated matrix measure μ, such that

$$\mu\left(\frac{\partial f}{\partial x}(x)\right) \leq -c, \quad c > 0 \qquad \forall x \in \mathcal{C}, \forall t \geq t_0$$



$$|x(t) - y(t)| \leq K\, e^{-c(t-t_0)}\, |x_0 - y_0|,$$

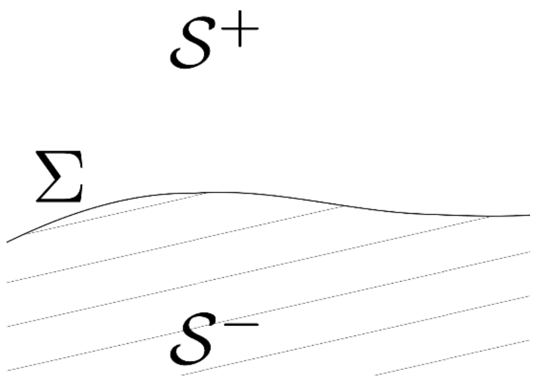
Contracting bimodal PWS systems

- <u>Theorem</u>: *A bimodal nonlinear switched system is incrementally exponentially stable if there exists a norm with associated matrix measure such that*

$$\mu\left(\frac{\partial F^+}{\partial x}(x)\right) \leq -c_1, \qquad \forall x \in \bar{\mathcal{S}}^+$$

$$\mu\left(\frac{\partial F^-}{\partial x}(x)\right) \leq -c_2, \qquad \forall x \in \bar{\mathcal{S}}^-$$

$$\mu\left(\left[F^+(x) - F^-(x)\right]\nabla H\right) = 0, \ \forall x \in \Sigma$$



D. Fiore, S. J. Hogan, M. di Bernardo, "Contraction analysis of switched systems via regularisation", *Automatica*, vol. 73, pp. 279–288, 2016

Application to PWA systems

- PWA systems

$$\dot{x} = \begin{cases} A_1 x + b_1 + Bu & \text{if } h^T x > 0 \\ A_2 x + b_2 + Bu & \text{if } h^T x < 0 \end{cases}$$

  - Our conditions become

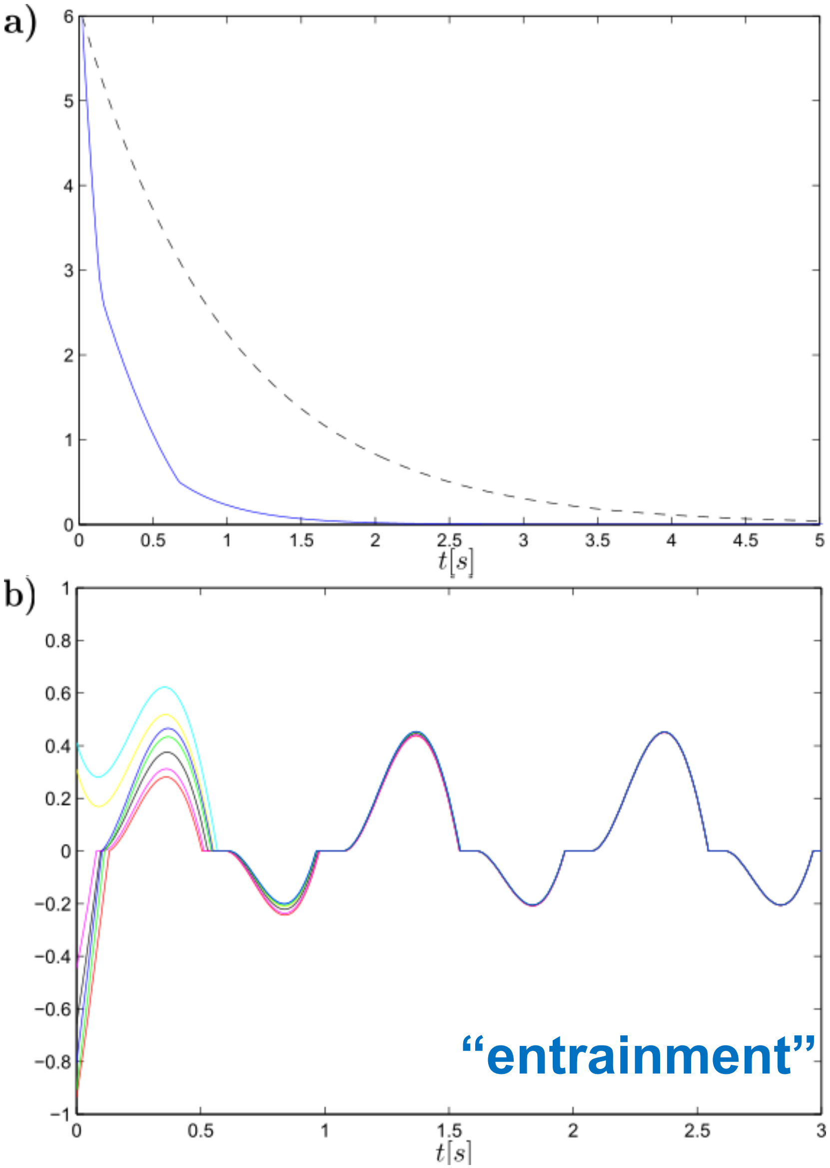
$$\begin{aligned} \mu\left(A_1\right) &\le -c_1 \\ \mu\left(A_2\right) &\le -c_2 \\ \mu\left(\Delta A x h^T\right) &= 0 \\ \mu\left(\Delta b h^T\right) &= 0 \end{aligned}$$

  - Using Euclidian norm

$$PA_i + A_i^T P < 0, \quad i = 1, 2,$$

$$\Delta A = g h^T$$

$$P \Delta b = -\gamma h$$



[1] Pavlov et al. – "On convergence properties of piecewise affine systems". *International Journal of Control* (2007)

Application to Relay fb systems

- Relay feedback systems

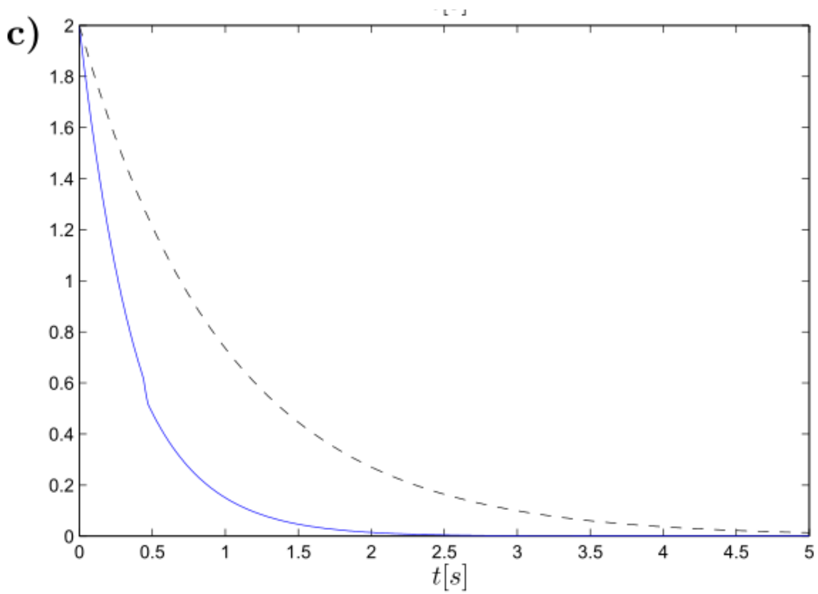
$$\dot{x} = Ax - b\,\mathrm{sgn}(y)$$
$$y = c^T x$$

- Our conditions become

$$\mu(A) \leq -c_1$$
$$\mu\left(-bc^T\right) = 0$$

- Using Euclidean norm they become the same as those from the well-known Kalman-Yakubovich Lemma

$$PA + A^T P < 0$$
$$Pb = c$$



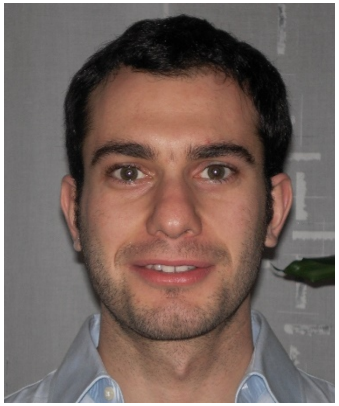
Thank you for your attention.

https://sites.google.com/site/dibernardogroup/

S. John Hogan

Mario Coraggio

Pietro De Lellis



Davide Liuzza