

OPTIMAL LINEAR RANDOMIZED FILTRATION OF QUANTUM BOSON SIGNALS

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An equation defining the optimal linear randomized estimation $z = Ay + \eta$ of non-observables x based on indirect non-ideal measurement of the received quantum (boson, non-Gaussian) signal y is derived. Its solutions for several important cases are presented. The linear filtration of the stationary boson signals is discussed, and an example is given illustrating the stationary signal filtration at the output of the non-ideal wave communication line.

The progress of communications on super-high and optical frequencies where the quantum nature of the electromagnetic signal becomes essential, makes it necessary to develop the quantum theory of optimal signal processing [1]. Within the framework of the theory, the received signal y is described by a set of non-commuting observables $\{y_t, t \in T\}$ which are considered below as real: $y_t^* = y_t$. The fact that the observables are non-commuting with respect to the multiplication ($y_t y_{t'} \neq y_{t'} y_t$) is an adequate mathematical interpretation of their physical incompatibility (i.e. impossibility of their arbitrarily precise measurement). In the quantum physics, the non-commuting real observables are represented by linear Hermitian operators in the Hilbert space of quantum-mechanical states. Such a specific representation, however, is not required in this paper.

The non-commutating generalization of communication theory problems is naturally proceeded with the simplest optimization problem of boson¹ signal linear processing based on the linearity of the optimal estimation in the Gaussian case, proved in [2].

The problem of finding the optimal set of commuting observables $z^0 = \gamma^0(y)$ whose precise measurement gives estimates $\{z_1^0, \dots, z_m^0\} \equiv z^0$ minimizing the mean square losses (1.3) in the class of linear transforms $\gamma(y) =$

¹ The quantum signal y is referred to as boson if it is described by secondary commuting observables $\{y_t\}$, i.e. observables commuting with their own commutators $y_t y_{t'} - y_{t'} y_t$. In the quantum field theory, particles and fields adhering to the Bose–Einstein statistics are described by secondary commuting operators of generation and destruction. The electric and magnetic field strengths constitute an important, for the communication theory, example of secondary commuting real observables.

$= Ay + \eta$ of the received boson non-Gaussian signal y was formulated and studied in [3]. In general, however, the solution does not meet the correspondence principle, that is to say it does not turn into the solution of the corresponding classical (i.e. commutative) problem under approximation to the weak non-commutativity $\hbar \rightarrow 0$. This difficulty may be overcome by means of estimation algorithm randomization corresponding to the expansion the class of measurements under consideration from direct to indirect ones. The indirect measurements are done through the direct measurements of a set of commuting observables of an enlarged system comprising the initial system (accessible to observations) and an additional, usually independent one (measuring apparatus). Owing to such a randomization it becomes possible to consider joint quasi-measurements of non-commuting observables of the initial system [4, 5] realized through an indirect measurements. Making use of this concept, derive an equation for the optimal randomized algorithm of quantum signal linear filtration meeting the correspondence principle.

1. Derivation of the optimal linear randomized filtration equation

For the sake of simplicity, confine ourselves at first to the finite-dimensional signal $y = \{y_j, j = 1, \dots, n\}$ describing observable sequence of non-commuting physical values y_1, \dots, y_n . We have to find among transformations $z = \gamma(y)$ of linear kind

$$z = Ay + \eta \equiv \left\{ \sum_{j=1}^n a_{ij} y_j + \eta_i, i = 1, \dots, m \right\} \quad (1.1)$$

satisfying the commutativity condition

$$[z, z^T] = \|\| z_i z_j - z_j z_i \|\| = 0 \quad (1.2)$$

the optimal one $z^0 = A^0 y + \eta^0$ in the terms of the minimal mean square risk criterion customary to the linear theory

$$R = \frac{1}{2} M(z - x)^T G(z - x) \equiv \frac{1}{2} M \sum_{i,j} (z_i - x_i) g_{ij} (z_j - x_j). \quad (1.3)$$

Hereafter the following matrix notation will be used: z, z^T are column vector and row vector, respectively, consisting of commuting observables $\{z_1, \dots, z_m\}$ giving, as the result of joint precise measurement, numerical estimates $\{z_1, \dots, z_m\}$ of estimated $x^T = (x_1, \dots, x_m)$; $G = \|\| g_{ij} \|\|$ is a real symmetric positively defined matrix $m \times m$, and M is expectation.

For the quantum randomization of the linear estimation algorithm (1.1) given by the $m \times n$ matrix $A = \|\| a_{ij} \|\|$ and the m -dimensional vector $\eta = \{\eta_i, i = 1, \dots, m\}$, the components of the latter η_1, \dots, η_m should be regarded as non-commuting y -independent stochastic variables. Substituting (1.1) into (1.2) and taking into account that the independence condition for y and η implies their commutativity $[y, \eta^T] = 0$, write the commutativity condition for the observed variables z in the following matrix form:

$$A[y, y^T]A^T + [\eta, \eta^T] = 0. \quad (1.4)$$

One can easily see that this condition may be met only if the commutators $[y_i, y_j]$ and $[\eta_i, \eta_j]$ commute with y and η i.e. are c -numbers

$$[y, y^T] = \|\| c_{ij} \|\| \equiv C_y, [\eta, \eta^T] = \|\| c_{ij} \|\| = C_\eta. \quad (1.5)$$

The commutation matrix C_η of quantum observables η for each A is defined quite uniquely as

$$C_\eta = -AC_y A^T. \quad (1.6)$$

It follows from the fact that operators describing observables y and η act in different Hilbert state spaces of the main and an additional systems, and that equality of the commutators $[\eta, \eta^T] = -A[y, y^T]A^T$ may take place only if they are multiples of the unit operator. Further we shall confine ourselves only to the secondary commuting observables y having known commutation (imaginary, anti-symmetric) matrix C_y giving with (1.6) quantization of the vector η for each A . This quantum randomization of the linear estimation (1.1) enables us to look through arbitrary matrices A rather than only through those satisfying the condition $AC_y A^T = 0$ as it was done in [3] for non-randomized linear estimation.

Substituting $z = Ay + \eta$ into (1.3) and averaging it, write the mean square risk allowing for independence of the observables η of x and y as the following sum:

$$R = \frac{1}{2} \text{Tr} GM(Ay + \eta - x)(Ay + \eta - x)^T = R_0(A) + R_\eta(A), \quad (1.7)$$

where

$$R_0(A) = \frac{1}{2} \text{Tr} G(K_x - AK_{xy}^T - K_{xy}A^T + AK_y A^T), \quad (1.8)$$

$$R_\eta(A) = \frac{1}{2} \text{Tr} G(K_\eta - (m_\eta - m_x + Am_y)(m_\eta - m_x + Am_y)^T). \quad (1.9)$$

Here m_x, m_y, m_η are expectations of x, y, η , respectively, and K_x, K_{xy}, K_y, K_η are respective symmetrized correlation matrices of the variables x, y , and η :

$$K_x = M \frac{1}{2} [(x - m_x), (x - m_x)]_+, \quad K_{xy} = M \frac{1}{2} [(x - m_x), (y - m_y)^T]_+,$$

$$K_y = M \frac{1}{2} [y - m_y), (y - m_y)^T]_+, \quad K_\eta = M \frac{1}{2} [(\eta - m_\eta), (\eta - m_\eta)^T]_+,$$

(the brackets $[\dots]_+$ define the anti-commutator $[y, y^T]_+ = ||y y_j + y_j y_i||$).

Risk (1.7) minimization in the class of linear transformations $Ay + \eta$ may be done in two steps: first, for a fixed A , the minimum of losses (1.9) is found having the sense of indirect measurement error of non-commuting $u = Ay$, and second, minimization with respect to A is done. Minimization of losses (1.9) in the terms of statistical states of the boson vector η having bounded below quantum entropy $S_\eta \geq S$ was carried out in [5]. The optimal state thus defined is Gaussian and characterized by the expectation vector m_η and symmetrized correlation matrix K_η

$$m_\eta = m_x - Am_y, \quad K_\eta = \frac{1}{2} C_\eta \operatorname{cth} \frac{1}{2\theta} GC_\eta. \quad (1.10)$$

Parameter $\theta \geq 0$ is defined from equation $S_\eta(\theta) = S$, where

$$S_\eta(\theta) = \frac{1}{2} \operatorname{Tr} \left[\frac{1}{2\theta} GC_\eta \operatorname{cth} GC_\eta - \ln \left| 2 \operatorname{sh} \frac{1}{2\theta} GC_\eta \right| \right]$$

is the entropy of boson Gaussian vector η corresponding to the temperature θ . The function $S_\eta(\theta)$ may be easily seen to be monotonically increasing, and the zero temperature $\theta = 0$ corresponds to the zero entropy $S = 0$. Below, for the sake of convenience, the temperature θ , rather than the entropy S , will be regarded as the independent parameter characterizing degree of linear estimation randomization. The randomized linear estimation (1.1) having the temperature θ is interpreted in [5] as indirect measurement of secondary commuting Ay by the optimal linear measuring apparatus with phase coordinates η and non-ideality degree given by the temperature θ .

The minimal indirect measurement error for fixed θ and with due regard to (1.10), (1.6) is as follows

$$R_\eta^\theta(A) = \frac{1}{2} \operatorname{Tr} GAC_y A^T \operatorname{cth} \frac{1}{2\theta} GAC_y A^T \equiv \frac{1}{2} \operatorname{Tr} f_\theta \left(\frac{1}{2} GAC_y A^T \right) \quad (1.11)$$

and increases monotonically with θ . The absolute error minimum (1.9) attained under absence of limitations on entropy and corresponding to the ideal indi-

rect measurement of the observables $u = Ay$ is obtained from (1.11) through the passage to the limit $\theta \rightarrow 0$:

$$R_\eta^0(A) = \lim_{\theta \rightarrow 0} \frac{1}{2} \operatorname{Tr} f_\theta \left(\frac{1}{2} GAC_y A^T \right) = \frac{1}{2} \operatorname{Tr} \left| \frac{1}{2} GAC_y A^T \right| \quad (1.12)$$

where the fact that the limit of $f_\theta(\alpha) = \alpha \operatorname{cth} \frac{1}{\theta} \alpha$ with respect to the parameter $\theta \rightarrow 0$ equals to the module $|\alpha|$ is allowed for. In the classical (i.e. commutative $C_y = 0$) case, the losses (1.12) turn into zero and the linear estimation (1.1) becomes for $\theta \rightarrow 0$ non-randomized, that is the vector η is identified with its expectation. For the quantum case, the correlation matrix K_η^θ defining the error (1.11) is not equal to zero even for $\theta = 0$, and in this sense the linear estimation is still randomized even under the ideal indirect measurement.²

Derive an equation defining the optimal matrix A^θ minimizing the risk $R^\theta(A) \equiv R(A) + R_\eta^\theta(A)$ under the fixed θ . The functional

$$R^\theta(A) = \frac{1}{2} \operatorname{Tr} \left[G(K_x - AK_{xy}^T - K_{xy}A^T + AK_y A^T) + f_\theta \left(\frac{1}{2} GAC_y A^T \right) \right] \quad (1.13)$$

is convex and by virtue of its analyticity allows differentiation by A^T . Equating variation

$$\delta R^\theta(A) = \operatorname{Tr} \left[G(AK_y - K_{xy}) + f'_\theta \left(\frac{1}{2} GAC_y A^T \right) \frac{1}{2} GAC_y \right] \delta A^T$$

to zero and taking into consideration independence of variations δA^T , obtain the following equation

$$G(AK_y - K_{xy}) + \frac{1}{2} f'_\theta \left(\frac{1}{2} GAC_y A^T \right) GAC_y = 0 \quad (1.14)$$

where $f'_\theta(\alpha) \equiv \frac{d}{d\alpha} f_\theta(\alpha)$ is a monotonic increasing odd function with values $|f'_\theta(\alpha)| \leq 1$:

$$f'_\theta(\alpha) = \operatorname{cth} \frac{\alpha}{\theta} \operatorname{cosech}^2 \frac{\alpha}{\theta} = \frac{\operatorname{sh}(2\alpha/\theta) - 2\alpha/\theta}{\operatorname{ch}(2\alpha/\theta) - 1}. \quad (1.15)$$

The matrix non-linear equation (1.14) is that desired. Its solution A^θ for each θ defines the optimal randomized linear estimation $u^\theta = Ay^\theta + \eta^\theta$ characterized by the temperature θ , the optimal Gaussian boson vector η^θ

² The ideal indirect measurement realized by measurement of linear superposition (1.1) of signal y and vacuum noise η is referred to as coherent, i.e. it may be described by a set of coherent measurement state vectors [1].

being described in the terms of the commutation matrix (1.6), the expectation vector and the correlation matrix (1.10) for $A = A^\theta$. The minimal risk defining the quality of the optimal linear filtration under indirect measurement is as follows:

$$R^\theta = \frac{1}{2} \text{Tr} \left[G(K_x - A^\theta K_{xy}) + g_\theta \left(\frac{1}{2} G A^\theta C_y A^{\theta T} \right) \right] \quad (1.16)$$

($g_\theta(\alpha) = f_\theta(\alpha) - \alpha f'_\theta(\alpha)$), and is readily obtainable by multiplying (1.14) by A^T from the right and substituting the obtained identity into (1.13). The second term in (1.16) is due to the estimation randomization and under $\theta \rightarrow \infty$ increases as $m\theta/2$ (where m is the dimensionality of the vectors x and z). Under $\theta \rightarrow 0$ it vanishes, and for the Gaussian case risk (1.16) reaches the lower bound obtained in [2] through minimization in the class of arbitrary indirect measurements. This proves equivalence of linear and Gaussian approximations in optimization of boson signal processing.

2. Study of the optimal linear filtration equation

By applying the matrix identity $f(DA^T)D = Df(A^T D)$ to (1.15) and to matrix D equal to $\frac{1}{2} GAC_y$, write solution of (1.14) as follows:

$$A^\theta = K_{xy}(K_y + B_\theta)^{-1} \quad (2.1)$$

where B_θ standing for the matrix $\frac{1}{2} C_y f'_\theta \left(\frac{1}{2} A^{\theta T} G A^\theta C_y \right)$ is, for fixed θ , the solution of

$$B = \frac{1}{2} C_y f'_\theta \left(\frac{1}{2} (K_y + B)^{-1} K_{yx} G K_{xy} (K_y + B)^{-1} C_y \right) \quad (2.2)$$

with respect to B . By introducing $\alpha = \varphi_\theta(\cdot)$ which is the inverse function of (1.15), rewrite this matrix equation in a more suitable form:

$$\varphi_\theta(2C_y^{-1}B) 2C_y^{-1}K_y = (1 + K_y^{-1}B)^{-1} K_y^{-1} K_{yx} G K_{xy} (1 + K_y^{-1}B)^{-1}. \quad (2.3)$$

In its domain of definition $-1 \leq \beta \leq 1$, $\varphi_\theta(\beta)$ is odd analytical function increasing monotonically from $-\infty$ to ∞ . The right part of (2.3) is monotonically decreasing non-negative bounded function of argument $2K_y^{-1}B$. Under such conditions, the right and left parts of (2.3) as independent functions of matrix B always have for $C_y \neq 0$ a single common point $B = B_\theta$ such that dependence of B_θ on $\theta > 0$ is monotonically non-increasing: $B_\theta \leq B_{\theta'}$ for $\theta \geq \theta'$ with the product $2C_y^{-1}B_\theta$ lying within the domain

$$-1 \leq 2C_y^{-1}B_\theta \leq 1. \quad (2.4)$$

Solution of (2.2) for $\theta \neq 0$ may be looked for in the form of a series $B_\theta = B^{(0)} + \frac{1}{\theta} B^{(1)} + \frac{1}{\theta^2} B^{(2)} + \dots$ converging rapidly for $\theta \gg 1$ (strong estimation randomization). Taking into consideration that $f'(\alpha) = \frac{1}{3} \left(\frac{2\alpha}{\theta} \right) + 0 \left(\frac{2\alpha}{\theta} \right)^\alpha$ find readily that

$$B^{(0)} = 0, B^{(1)} = \frac{1}{6} C_y A_0^T G A_0 C_y,$$

where $A_0 = K_{xy} K_y^{-1}$ is a matrix defining estimation (1.1) for the commutative case of $C_y = 0$. Under weak non-commutativity $C_y = \varepsilon S_y$ (where ε is a small parameter), $B^{(1)} = o(\varepsilon^2)$ and matrix A^θ has the classical form $A^\theta = A_0$ not only in the zero approximation, but in the first quasi-classical one for $\varepsilon \ll 1$ ($\theta \neq 0$) as well. Thus, the randomized linear estimation, as opposed to the non-randomized one of [3], satisfies the correspondence principle. Quantum corrections in (1.16) to the losses of the classical randomized estimation $R_\theta^\theta = \frac{1}{2} \text{Tr} (G(K_x - A K_{xy}^T) + \theta)$ are for $\theta \neq 0$ on the order of ε^2 .

In the case of the ideal indirect measurement $\theta = 0$, the matrix $B_0 \equiv \lim_{\theta \rightarrow 0} B_\theta$ realizes, evidently, the boundaries of the inequality (2.4). Consider solution of (2.2) at $\theta = 0$ for the following important case. Let the signal y having non-degenerate commutation matrix of rank $n = 2s$ consist of two groups of physically different observables $\{p_1, \dots, p_s\} \equiv p$ and $\{q_1, \dots, q_s\} \equiv q$ described, respectively, by matrices C_y and K_y having the following form

$$C_y = \begin{pmatrix} 0 & -iC \\ iC & 0 \end{pmatrix} \equiv C \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, K_y = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} = K \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.5)$$

where the $s \times s$ matrices C and K are real, symmetric and positive definite. (Otherwise, one would have to obtain such a signal $\tilde{y} = (p, q)$ by means of non-degenerate real transform $\tilde{y} = Ty$ which always may be done [3].) If in doing so the matrix $K_{yx} G K_{xy}$ also breaks down into direct product

$$K_{yx} G K_{xy} = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} \equiv D \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (2.6)$$

solution B_0 of (2.2) for $\theta \rightarrow 0$ is as follows:

$$B_0 = \frac{1}{2} \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \equiv \frac{1}{2} C \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (2.7)$$

Indeed, substitute (2.5) and (2.6) into (2.2) and search its solution in the form of $B = H \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Taking into account the following identity:

$$\begin{aligned} f'_\theta \left(\frac{1}{2} (K+H)^{-1} D(K+H) C \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right) = \\ = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes f'_\theta \left(\frac{1}{2} (K+H)^{-1} D(K+H)^{-1} C \right) \end{aligned}$$

which holds because function $f_\theta(\cdot)$ is odd, obtain the following equation defining matrix H :

$$H = \frac{1}{2} C f'_\theta \left(\frac{1}{2} (K+H)^{-1} D(K+H)^{-1} C \right).$$

For $\theta \rightarrow 0$ function f'_θ turns into the signature of the product of two positive definite matrices that is equal to the identity matrix. Thus $H = 1/2 C$.

Condition (2.6) is met if matrices K_{xy} and G have the following form:

$$K_{xy} = k \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad G = g \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In this case the estimated observables x and their estimates z fall into two groups $x = \{x_p, x_q\}$, $z = \{z_p, z_q\}$ conveniently joined with complexification:

$$a = \frac{1}{\sqrt{2}} (x_p + ix_q), \quad c = \frac{1}{\sqrt{2}} (z_p + iz_q)$$

by halving the dimensionality of the linear space. The commuting optimal complex estimates $c = \{c_j\}$ under the ideal indirect measurement $\theta = 0$ are as follows:

$$c = k \left(K + \frac{1}{2} C \right)^{-1} b + \beta$$

where $b = \frac{1}{\sqrt{2}} (p + iq)$ is a complex boson s -dimensional vector described by commutators $[b, b^+]$ and correlators $\frac{1}{2} M[b, b^+]$ equal to positive definite matrices C and K ; and $\beta = \frac{1}{\sqrt{2}} (\eta_p + i\eta_q)$ is an independent coherent noise with the following commutation and correlation matrices:

$$\begin{aligned} C_\beta &= - \left(K + \frac{1}{2} C \right)^{-1} k^T C k \left(K + \frac{1}{2} C \right), \\ K_\beta &= \frac{1}{2} \left(K + \frac{1}{2} C \right)^{-1} k^T C k \left(K + \frac{1}{2} C \right)^{-1}. \end{aligned}$$

The appendix gives solution of (2.3) for the two-dimensional signal in the general case where condition (2.6) is not met. A critical point exists in such cases where the ideal indirect measurement degenerates into a direct measurement of optimal linear combination ($b_1 p + b_2 q$).

3. Optimal linear filtration of stationary boson signals

The vectors x, y were assumed above to be finite-dimensional. Since all results were written in the matrix form, their generalization to the infinite-dimensional and continual cases is evident. Consider such a generalization for the stationary case.

Let the received boson signal $y = \{y_t\}$ and the estimated useful signal $x = \{x_t\}$ be stationary and stationarily correlated signals (either discrete, or continuous). Confine ourselves to the case where y_t is one-dimensional (x_t may be multi-dimensional: $x_t = \{x_{jt}(t)\}$), and assume that expectations m_x and m_y are equal to zero.

By means of the following spectral representation

$$y_t = \int_0^{N/2} \sqrt{2} \cos 2\pi vt dP(v) + \int_0^{N/2} \sqrt{2} \sin 2\pi vt dQ(v) = \int_{-N/2}^{N/2} e^{-2\pi i vt} dY(v) \quad (3.1)$$

($N = 1$ if t has integral values $-\infty, \dots, -1, 0, 1, \dots, \infty$ and $N = \infty$ if t is continuous: $t \in (-\infty, \infty)$) come to the pairs $(dP(v), dQ(v))$ which may be easily complexified as follows:

$$dY(v) = \begin{cases} \frac{1}{\sqrt{2}} (dP(v) + i dQ(v)), & v \geq 0 \\ \frac{1}{\sqrt{2}} (dP(-v) + i dQ(-v)), & v < 0. \end{cases}$$

The spectral components $dY(v)$ under different v are orthogonal

$$[dY(v), dY(v')^*]_- = 0, \quad M[dY(v), dY(v')^*]_+ = 0 \quad v \neq v',$$

and under similar v are characterized by the following commutation and correlation measures:

$$dC_y(v) = [dY(v), dY(v)^*]_-, \quad dK_y(v) = \frac{1}{2} M[dY(v), dY(v)^*]_+, \quad \neq (3.2) dv$$

defining spectral expansions

$$C_y(\tau) = \int_{-N/2}^{N/2} e^{2\pi i v \tau} dC_y(v), \quad K_y(\tau) = \int_{-N/2}^{N/2} e^{2\pi i v \tau} dK_y(v)$$

of the commutation $C_y(t-t') = [y_t, y_{t'}]_-$ and correlation $K_y(t-t') = \frac{1}{2} M[y_t, y_{t'}]_+$ functions of the stationary (in a wide sense) boson signal y_t .

Since all the matrices involved into (2.3) are assumed to be dependent only on the difference between indices t, t' , its solution in the spectral representation boils down to the solution of independent algebraic equations with respect to one unknown (for each ν) that may be found graphically. Indeed, by introducing the derivatives

$$\begin{aligned} \varepsilon(\nu) &= \frac{1}{2} dC_y(\nu)/dK_y(\nu), \quad \gamma(\nu) = dG_y(\nu)/dK_y(\nu), \\ \beta(\nu) &= dB(\nu)/dK_y(\nu), \end{aligned} \quad (3.3)$$

where $dG_y(\nu)$ is the spectral measure of matrix $K_{yx}GK_{xy} = \|\| g_y(t-t') \|\|$, and $dB(\nu)$ is the spectral measure of the desired $B = \|\| b(t-t') \|\|$, obtain

$$\varphi_\theta \left(\frac{\beta(\nu)}{\varepsilon(\nu)} \right) = \varepsilon(\nu) \gamma(\nu) / (1 + \beta(\nu))^2 \quad (3.4)$$

with respect to the new unknowns $\beta(\nu)$, $|\nu| \leq N/2$. The solution $\beta_\theta(\nu)$ of (3.4) for each ν , θ lies within $0 \leq \beta_\theta(\nu) \leq |\varepsilon(\nu)|$ and increases monotonically from zero to $|\varepsilon(\nu)| \leq 1$ for $\theta \rightarrow 0$.

The spectral density $a^\theta(\nu)$ of the matrix $A^\theta = \|\| a^\theta(t-t') \|\|$ defining the optimal randomized linear estimation (1.1) is as below:

$$a^\theta(\nu) = \frac{1}{1 + \beta_\theta(\nu)} dK_{xy}(\nu)/dK_y(\nu) = \frac{1}{1 + \beta_\theta(\nu)} a_0(\nu) \quad (3.5)$$

where $dK_{xy}(\nu)$ is the spectral measure of correlation matrix $K_{xy} = \|\| \frac{1}{2} M[x_t, y_{t'}]_+ \|\|$.

The frequency characteristic of the optimal linear filter (without regard to the realizability in the physical causal sense) is defined by (1.5). This formula enables one also to write readily the spectral measures $dC_\eta(\nu)$, $dK_\eta(\nu)$ of the "stationary self-noise $\eta(t)$ of the optimal measuring apparatus" having temperature θ :

$$\begin{aligned} dC_\eta(\nu) &= \frac{1}{(1 + \beta_\theta(\nu))^2} a_0(\nu) a_0(\nu)^+ dC_y(\nu); \\ dK_\eta(\nu) &= \frac{\text{cth} \varphi(\beta_\theta(\nu)/\varepsilon(\nu))}{2(1 + \beta_\theta(\nu))^2} a_0(\nu) a_0(\nu)^+ + dC_y(\nu) \end{aligned} \quad (3.6)$$

where $\varphi(\cdot)$ is the inverse of the derivative $\frac{d}{d\alpha} (\alpha \text{cth} \alpha)$. By substituting (3.5) into (1.16) and taking into consideration (3.4), obtain the optimal density

of optimal filtration losses:

$$R^\theta(\nu) = R_0(\nu) + \frac{1}{2} \left[\frac{\gamma(\nu) \beta_\theta(\nu)}{1 + \beta_\theta(\nu)} + \varphi \left(\frac{\beta_\theta(\nu)}{\varepsilon(\nu)} \right) \left(\text{cth} \varphi \left(\frac{\beta_\theta(\nu)}{\varepsilon(\nu)} \right) - \frac{\beta_\theta(\nu)}{\varepsilon(\nu)} \right) \right], \quad (3.7)$$

where $R_0(\nu)$ is the spectral density of losses of the non-randomized classic estimation $A_0 y$ without allowance for the losses introduced by the indirect measurement. In this formula (as, incidentally, in (3.5) and (3.6)), parameter $\beta(\nu) = \beta_\theta(\nu)$ may be regarded as independent because of the one-to-one monotonic correspondence between θ and β . For the ideal indirect measurement one should assume that $\beta = |\varepsilon(\nu)|$, and the last term in (3.7) vanishes. In this case, losses (3.7) are greater than losses $R_0(\nu)$ for the "energy" of vacuum fluctuations $\frac{1}{2} \gamma(\nu) \varepsilon(\nu) / (1 + \varepsilon(\nu))$. The zero value of parameter β corresponds to the strongly non-ideal optimal indirect measurement.

Example. Let us apply the results obtained to the case of stationary signal filtration at the output of a wave transmission line with strong attenuation and temperature T . The wave y_t received at the output of the line (e.g. travelling wave of voltage) is represented by superposition $x_t + \xi_t$ of the stationary classical signal x_t and the boson equilibrium noise ξ_t having the following [6] spectral commutation $\mathcal{C}_\xi(\nu) = dK_\xi(\nu)/d\nu$ and correlation $k_\xi(\nu) = dK_\xi(\nu) d\nu$ densities:

$$c_\xi(\nu) = 2r(\nu) h\nu, \quad k_\xi(\nu) = r(\nu) h\nu \text{cth} \frac{1}{2} \frac{h\nu}{kT}$$

where h is the Plank constant, k is the Boltzmann constant, and $r(\nu)$ is the wave resistance at frequency ν . Taking into consideration that $dK_{xy}(\nu) = k_x(\nu) d\nu$, $dK_y(\nu) = (k_x(\nu) + k_\xi(\nu)) d\nu$ ($k_x(\nu)$ being the spectral intensity of a process x_t), obtain the following form of the frequency response (3.5) of the optimal filter under the ideal indirect measurement:

$$a^0(\nu) = k_x(\nu) / (k_x(\nu) + 2r(\nu) h\nu (1 - e^{-h\nu/kT})^{-1}).$$

The spectral density of filtration losses is as follows:

$$R^0(\nu) = \frac{1}{2} k_x(\nu) \left/ \left(1 + \frac{k_x(\nu)}{2r(\nu) h\nu} (1 - e^{-h\nu/kT}) \right) \right.,$$

for weak signals $k_x(\nu) \lesssim 2r(\nu) h\nu$ at high frequencies $h\nu \gtrsim kT$ it differs significantly from the corresponding classical limit

$$R_0(\nu) = \frac{1}{2} k_x(\nu) / (1 + k_x(\nu) / 2r(\nu) kT).$$

In the case where the spectral density $k_x(\nu)$ of the useful signal becomes equal to zero at $|\nu| \geq N/2$, $N < \infty$, the stationary estimation $z^0 = A^0 y + \eta$ at the output of the optimal filter has the following spectral intensity

$$|k_x(\nu)|^2 \left/ \left(k_y(\nu) + \frac{1}{2} C_y(\nu) \right) \right.$$

equal to zero at $|\nu| \geq N/2$. In this case, precise measurement of optimal values realizing the ideal indirect measurement of observables $u^0 = A^0 y$ may be done at discrete times $t_j = j\Delta$ ($\Delta \leq 1/N$), and the optimal estimates may be reestablished through the usual Kotelnikov interpolation formula.

Appendix

Optimal linear measurement of a pair of secondary commuting observables

Let y be a pair of conjugate observables p and q whose commutation matrix is characterized only by one parameter \hbar : $C_y = \hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \hbar \sigma_2$.

Decompose the arithmetic roots of positively defined 2×2 matrices $\sqrt{K_y^{-1}} K_{yx} G K_{xy} \sqrt{K_y^{-1}}$, $\sqrt{K_y^{-1}} B \sqrt{K_y^{-1}}$ with respect to the Pauli matrices:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Obtain

$$\sqrt{K_y^{-1}} K_{yx} G K_{xy} \sqrt{K_y^{-1}} = (\kappa + k)^2, \sqrt{K_y^{-1}} B \sqrt{K_y^{-1}} = (\alpha - a)^2 \quad (\text{A.1})$$

where $\kappa + k \equiv \kappa 1 + \sum_{i=1}^3 k_i \sigma_i$; $\alpha + a \equiv \alpha 1 + \sum_{i=1}^3 a_i \sigma_i$ ($k_2 = 0$, $a_2 = 0$ because the matrices of (A.1) are symmetrical). By multiplying the left part of (2.3) by $\sqrt{K_y}(1 + K_y^{-1}B)$, and the right one by $(1 + K_y^{-1}B)\sqrt{K_y^{-1}}$, and allowing for decompositions (A.1), we can write it as follows:

$$(1 + (\alpha + a)^2) \varphi_\theta \left(\frac{1}{\varepsilon} \sigma_2 (\alpha + a)^2 \right) \frac{1}{\varepsilon} \sigma_2 (1 + (\alpha + a)^2) = (\kappa + k)^2. \quad (\text{A.2})$$

Here formula $A \sigma_2 A^T = \det A \sigma_2$ is used which leads to $\sqrt{K_y} 2 C_y^{-1} \sqrt{K_y} = \frac{2}{\hbar} \sqrt{\det K_y} \sigma_2 \equiv \frac{1}{\varepsilon} \sigma_2$. Taking into consideration the matrix identity $A \varphi(\sigma_2 A^2) = \varphi(A \sigma_2 A) A$ and the fact that for any symmetric matrix A and any odd function φ the equality $\varphi(A \sigma_2 A) = \varphi(\det A \sigma_2) = \varphi(\det A) \sigma_2$ holds, rewrite (A.2) as follows:

$$\varphi_\theta \left(\frac{\beta}{\varepsilon} \right) \frac{\beta}{\varepsilon} \left(\frac{\alpha - a}{\beta} + \alpha + a \right)^2 = (\kappa + k)^2$$

where $\beta = \alpha^2 - a^2 = \sqrt{\det K_y^{-1} B}$. By solving it with respect to α and a , obtain:

$$\alpha = \frac{1}{\beta + 1} \sqrt{\frac{\beta \varepsilon}{\varphi_\theta(\beta/\varepsilon)}} \kappa, \quad a = \frac{1}{\beta - 1} \sqrt{\frac{\beta \varepsilon}{\varphi_\theta(\beta/\varepsilon)}} k \quad (\text{A.3})$$

where parameter β is defined for each θ through the condition $\alpha^2 - a^2 = \beta$, thus leading to the following algebraic equation:

$$\frac{1}{\varepsilon} \varphi_\theta \left(\frac{\beta}{\varepsilon} \right) = \frac{\kappa^2}{(1 + \beta)^2} - \frac{k^2}{(1 - \beta)^2} = \frac{1}{(1 - \beta^2)^2} ((1 + \beta^2) d - 2\beta S). \quad (\text{A.4})$$

The following notation is introduced here:

$$d = \kappa^2 - k^2 = \sqrt{\det K_y^{-1} K_{yx} G K_{xy}},$$

$$S = \kappa^2 + k^2 = \frac{1}{2} \text{Tr} K_y^{-1} K_{yx} G K_{xy}. \quad (\text{A.5})$$

The obtained formulas (A.3) together with (A.4) having one unknown β define the matrix $B_\theta = \sqrt{K_y}(\alpha + a)^2 \sqrt{K_y}$ and lead to the solution of (1.14) which in the case under consideration may be represented as:

$$A^\theta = \frac{1}{1 - \beta^2} K_{xy} (K_y^{-1} - \beta d (K_{yx} G K_{xy})^{-1}). \quad (\text{A.6})$$

Thus, in the two-dimensional case, the matrix B_θ defining the optimal linear estimation is defined by a single parameter determined through solution of non-linear algebraic equation (A.4). Under $S = d = \gamma$, the right part of the equation becomes simpler, and it takes the form of $\varphi_\theta(\beta/\varepsilon) = \varepsilon \gamma (1 + \beta)^2$ similar to that of (3.4) defining the optimal measurement of one-dimensional complex amplitude of frequency ν . Under $S \neq d$, the right-hand part of (A.4) decreases monotonically within the interval $0 \leq \beta < 1$ and has there a positive root:

$$\beta^0 = \frac{1}{d} (S - \sqrt{S^2 - d^2}).$$

It will be recalled that the function $\varphi_\theta(\beta/\varepsilon)$ increasing monotonically from $-\infty$ to ∞ is odd and has the domain of definition $|\beta/\varepsilon| < 1$. Thus (A.4) for each $\theta \geq 0$ has a single monotonically non-increasing solution $\beta = \beta_\theta$ lying within the domain:

$$0 \leq \beta_\theta \leq \min(|\varepsilon|, \beta^0) = \begin{cases} |\varepsilon| & \text{for } |\varepsilon| < \beta^0, \\ \beta^0 & \text{for } |\varepsilon| \geq \beta^0. \end{cases}$$

For $\theta \rightarrow \infty$, the lower bound ($\beta_\infty = 0$) of the inequality is reached, and for $\theta \rightarrow 0$, $\beta_0 = \min(|\varepsilon|, \beta^0)$ is reached.

By substituting (A.6) into (1.16), write the risk of optimal estimation for the case under consideration:

$$R^0 = \frac{1}{2} \text{Tr} GK_x - \frac{1}{1 - \beta^2} (S - \beta d) = R_0 + \frac{\beta}{1 - \beta^2} (d - \beta S) \quad (\text{A.7})$$

where R_0 is the estimation risk without allowance for measurement losses. Comparison of the equality (A.7) of the optimal estimation allowing randomized solutions with that of non-randomized estimation

$$R^* = R_0 + \frac{\beta^0}{1 - \beta^{02}} (d - \beta^0 S) = R_0 - \frac{1}{2} (S - \sqrt{S^2 - d^2})$$

based on measurement of the optimal combination $b_1^0 p + b_2^0 q$ [3] shows that within the domain $|\varepsilon| < \beta^0$ the randomization with temperature $\theta = 0$ gives essential advantage (see Fig. 1). For $|\varepsilon| \geq \beta^0$ (essentially quantum domain) losses (A.7) become under ideal measurement ε -independent and coincide with the non-randomized estimation losses. It means that at the point $|\varepsilon| = \beta^0$ the randomized estimation based on the ideal measurement undergoes the "phase transition of the second kind" and degenerates into the non-randomized

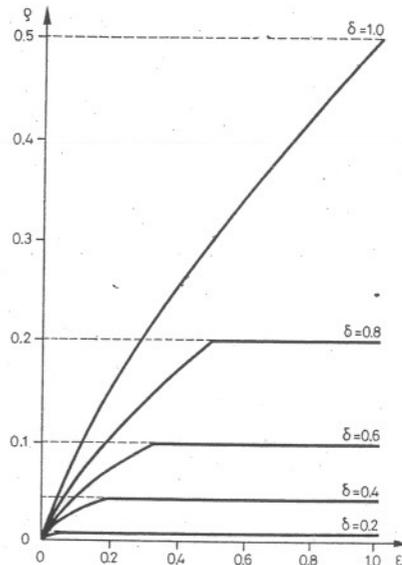


Fig. 1. Dependence of optimal ideal indirect measurement losses $q^0 = (R^0 - R_0)/S$ on ε under various values of $d/S = \delta$. Corresponding levels of ε -independent losses $q^* = (R^* - R_0)/S$ of the optimal direct measurement are shown by dashed lines

estimation. Indeed, (A.6) may be represented as

$$A^0 = K_{xy} \left(\frac{1 - \beta \sqrt{\gamma_2/\gamma_1}}{1 - \beta^2} b_{\gamma_1} b_{\gamma_1}^T + \frac{1 - \beta \sqrt{\gamma_1/\gamma_2}}{1 - \beta^2} b_{\gamma_2} b_{\gamma_2}^T \right), \quad (\text{A.8})$$

where $b_{\gamma_1}, b_{\gamma_2}$ are eigenvectors of matrix $K_y K_{yx} G K_{xy}$ corresponding to its eigenvalues γ_1, γ_2 and normalized with respect to weight K_y : $b_{\gamma_i} K_y b_{\gamma_i}^T = \delta_{ij}$. Taking into consideration that $\beta = \min(\sqrt{\gamma_1/\gamma_2}, \sqrt{\gamma_2/\gamma_1})$, obtain that matrix (A.8) within the domain $|\varepsilon| \geq \beta^0$ under ideal measurement $\theta = 0$ degenerates and has the rank $r(A^0) = 1$:

$$A^0 = K_{xy} b^0 b^{0T}$$

where b^0 is that of vectors $b_{\gamma_1}, b_{\gamma_2}$ which corresponds to the greater eigenvalue $\gamma^0 = \max(\gamma_1, \gamma_2)$. Observables $A_y^0 = K_{xy} b^0 b^{0T} y$ are linearly dependent on the only observable $b^{0T} y = b_1^0 p + b_2^0 q$ which may be measured without randomization because a single observable may be measured directly.

For $S = d, \beta_0 = 1$ and measurement randomization is advantageous along all the domain of the parameter $\varepsilon = \hbar/2 \sqrt{\det K_y} \leq 1$ values. This takes place if condition (2.6) is met and, specifically, in the stationary case allowing complexification.

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Оптимальная линейная рандомизированная фильтрация квантовых бозонных сигналов

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Линейная рандомизированная фильтрация бозонного действительного сигнала $y = \{y_t, t \in T\}$ описывается матрицей $\|A_{ts}\|$ и независимым от y бозонным шумом $\eta = \{\eta_s, s \in S\}$, которые определяют линейную оценку $z = Ay + \eta$ оцениваемого сигнала

$x = \{x_s, s \in S\}$. Поскольку измеряемые $z = \{z_s, s \in S\}$ должны коммутировать: $[z_s, z_{s'}] = 0$, c -числовая матрица C_η коммутаторов $[\eta_s, \eta_{s'}]$ равна матрице $AC_y A^T$ с противоположным знаком ($C_y = \|[y_t, y_{t'}]\|$ — коммутационная матрица принимаемого сигнала y). Оптимальное состояние бозонной системы, описываемой наблюдаемыми η , минимизирующее при фиксированной энтропии среднеквадратичные потери косвенного измерения $\frac{1}{2} \langle (z - x)^T G (z - x) \rangle$, есть гауссово.

Выводимое уравнение, определяющее оптимальное A , отличается от уравнения $AK_y - K_{xy} = 0$ в классическом (коммутативном) случае наличием нелинейного члена:

$$AK_y - K_{xy} + \frac{1}{2} f'_\theta \left(\frac{1}{2} AC_y A^T G \right) AC_y = 0 \quad (1)$$

(K_y, K_{xy} — матрицы корреляторов: $K_y = \frac{1}{2} \|[y'_t y_{t'} + y_t y_{t'}]\|$, $K_{xy} = \|[x_s y_t]\|$, а функции $f'_\theta(\cdot)$ есть производная от $f_\theta(\alpha) = \alpha \text{th}(\alpha/\theta)$ по α). Параметр $\theta \geq 0$ есть «температура» шум η , и характеризует степень рандомизации, т. е. степень неидеальности квазиизмерения некоммутирующих A_y .

Решение A^θ уравнения (1) удовлетворяет принципу соответствия: $A^\theta \rightarrow A_0$ при $C_y \rightarrow 0$ ($A_0 = K_{xy} K^{-1}$ есть решение уравнения (1) в классическом случае $C_y = 0$), и может быть найдено при $\theta \neq 0$ методом последовательных приближений: $A^\theta = A_0 + \frac{1}{\theta} A_1 + \frac{1}{\theta^2} A_2 + \dots$

В идеальном случае $\theta = 0$ оптимальное квазиизмерение становится когерентным и в существенно квантовых областях может вырождаться в прямое измерение. Последнее не имеет места, если

$$K_{xy} = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, \quad K_y = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}, \quad C_y = \begin{pmatrix} 0 & -iC \\ iC & 0 \end{pmatrix} \quad (2)$$

(R, C — положительно определенные матрицы). В этом случае сигналы $x = \{u_s, v_s\}$, $y = \{p_t, q_t\}$ допускают простое комплексное представление: $a = \frac{1}{\sqrt{2}}(u + iv)$, $b = \frac{1}{\sqrt{2}}(p + iq)$ и оптимальная комплексная оценка $c = \{c_s\}$ оцениваемого комплексного $a = \{a_s\}$ имеет вид:

$$c = K \left(R + \frac{1}{2} C \right)^{-1} b + \beta,$$

где $\beta = \{\beta_s\}$ есть комплексный шум, пропорциональный операторам рождения бозон оптимальной вспомогательной системы, находящейся в когерентном состоянии.

В частности, (2) имеет место, если y_t есть стационарный случайный бозонный процесс, стационарно коррелированный с x_s . В идеальном случае $\theta = 0$ оптимальная спектральная характеристика линейного фильтра есть $A^0(\nu) = A_0(\nu)/(1 + \varepsilon(\nu))$, где $0 \leq \varepsilon(\nu) \leq 1$, а $A_0(\nu)$ есть оптимальная спектральная характеристика в классическом случае $\varepsilon(\nu) = 0$. Если y_t есть принимаемая бегущая волна напряжения в линии передачи с затуханием и температурой T , то $\varepsilon(\nu) = \text{th}(h\nu/2kT)$, где h, k есть постоянные Планка и Больцмана.

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