# QUANTUM NOISE, BITS AND JUMPS: UNCERTAINTIES, DECOHERENCE, TRAJECTORIES AND FILTERING.

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*In celebration of the 100th anniversary of the discovery of quanta*

ABSTRACT. It is shown that before the rise of quantum mechanics 75 years ago, the quantum theory had appeared first in the form of the statistics of quantum thermal noise and quantum spontaneous jumps which have never been explained by quantum mechanics. This led to numerous quantum paradoxes, some of them due to the great inventors of quantum theory such as Einstein and Schrödinger. They are reconsidered in this paper. The development of quantum measurement theory, initiated by von Neumann, indicated a possibility for resolution of this interpretational crisis by divorcing the algebra of the dynamical generators from the algebra of the actual observables. It is shown that within this approach quantum causality can be rehabilitated in the form of a superselection rule for compatibility of past observables with the potential future. This rule, together with the self-compatibility of measurements insuring the consistency of histories, is called the nondemolition principle. The application of this causality condition in the form of the dynamical commutation relations leads to the derivation of the generalized von Neumann reductions, usharp, instantaneous, spontaneous, and even continuous in time. This gives a quantum probabilistic solution, in the form of the dynamical filtering equations, of the notorious measurement problem which was tackled unsuccessfully by many famous physicists starting with Schrödinger and Bohr. The simplest Markovian quantum stochastic model for the continuous in time measurements involves a boundary-value problem in second quantization for input "offer" waves in one extra dimension, and a reduction of the algebra of "actual" observables to an Abelian subalgebra for the output waves.

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## 1. Introduction: The Common Thread of Mathematics and Physics

*In science one tries to tell people, in such a way as to be understood by everyone, something that no one even knew before. But in poetry, it's quite opposite* – Paul Dirac.

This review unravels the most obscure sides of Quantum Theory related to quantum noise, decoherence and measurement, explaining them using the example of quantum bits. These are the most elementary quantum systems such as spins $1/2$, which are now utilized in Quantum Information Theory and for Quantum Computations. The vast majority of papers on these new applications which have recently appeared in theoretical physics (see for example the Quantum Information Section in any issue of Phys Rev A) are mainly concerned with deterministic and mathematically well-defined quantum attributes such as unitary evolutions and entanglements in Hilbert space which are traditional in theoretical physics. They leave the probabilistic 'ill-defined' quantum causality, decoherence and measurement problems to quantum philosophers for vague speculations ignoring recent mathematical developments in solving these quantum probability questions. Surely the words "professional theoretical physicists ought to be able to do better" (cited from J Bell, [1], p. 173) should be also addressed to those mathematicians and computer scientists who invent the 'quantum' algorithms simply as the entangled unitary transformations if they really want to contribute to the prediction of the potential capabilities for Quantum Information Technologies in the new Quantum Century.

In order to utilize the results of such purely dynamical algorithms of quantum computations one should state rigorously and solve the dynamical measurement, quantum prediction and feedback control problems for quantum bits performing these computations, as it was done over 20 years ago for linear quantum dynamical systems (oscillators) in [2, 3]. The true Heisenberg principle which is explained in the last chapter of this paper does not leave a possibility for nondemolition observation of quantum computations without quantum noise: the measurement errors and dynamical perturbations satisfy the uncertainty relation, and the best what can be done for dynamical error correction is optimal filtering of the quantum noise. Without the mathematical formulation of interpretational problems of quantum theory, without use of quantum laws of large numbers in probability and information theory, not only it is impossible to prove some speculative conjectures about the enormous capabilities of quantum algorithms, but even the simplest traditional quantum attributes fail to differentiate this theory from classical theories.

Indeed, the unitary evolution and entanglement without quantum probabilistic interpretation are also the attributes of any classical linear Hamiltonian theory, and it is not clear why it is not possible or more practical to realize the quantum unitary algorithms or quantum logic in the classical form of waves and non-Boolean wave-logic in a cup of water instead of the waves of matter. One could utilize the oscillations of the complex amplitudes of Fourier components on the surface of the water, the interference of the water-waves and even the entanglements of the separated modes in the cup as the unitary evolution, interference and entanglement in any Hilbert subspace of the wave components, and would avoid the probabilistic uncertainties, decoherence and the measurement problem which could destroy the advantages of such 'quantum' computer.

Quantum theory is a mathematical theory which studies the most fundamental features of reality in a unified form of waves and matter, it raises and solves the

most fundamental riddles of Nature by developing and utilizing mathematical concepts and methods of all branches of modern mathematics, including probability and statistics. Indeed, as we shall see, it began with the discovery of new laws for 'quantum' numbers, the natural objects which are the foundation of pure mathematics. ('God made the integers; the rest is man's work' – Kronecker). Next it invented new applied mathematical methods for solving quantum mechanical matrix and partial differential equations. Next it married probability with algebra to obtain unified treatment of waves and particles in nature, giving birth to quantum probability and creating new branches of mathematics such as quantum logics, quantum topologies, quantum geometries, quantum groups. It inspired the recent creation of quantum analysis and quantum calculus, as well as quantum statistics and quantum stochastics.

Specialists in different narrow branches of mathematics and physics rarely understand quantum theory as a common thread which runs through everything. The creators of quantum mechanics, the theory invented for interpretation of the dynamical laws of fundamental particles, were unable to find a consistent interpretation of it since they were physicists with a classical mathematical education. After inventing quantum mechanics they spent much of their lives trying to tackle the Problem of Quantum Measurement, the greatest problem of quantum theory, not just of quantum mechanics, or even of unified quantum field theory, which would be the same 'thing in itself' as quantum mechanics of closed systems without such interpretation. As we shall see, the solution to this problem can be found in the framework of Quantum Probability and Stochastics as a part of a unified mathematics rather than physics. Most modern theoretical physicists have a broad mathematical education which tends to ignore two crucial aspects of this solution – information theory and statistical conditioning.

In order to appreciate the quantum drama which has been developing through the whole century, and to estimate possible consequences for it of the solution of quantum measurement problem in the new quantum technological age, it seems useful to give a brief account of the discovery of quantum theory and its probabilistic interpretation at the beginning of the 20th century. This is done in the Chapter 1 which starts with a short recall of how the quantum nature of thermal radiation was discovered by Max Planck giving the birth of quantum theory exactly 100 years ago [4]. Readers who are not interested in the subject of this review from the historical perspective are advised to start with the Chapter 2 dealing with the famous problems and paradoxes of the orthodox quantum theory from the modern quantum probabilistic point of view. The specialists who are familiar with this point of view and who want only to read a systematic review on the stochastic decoherence theory, consistent trajectories, continual measurements, quantum jumps and filtering, or would like to find the origin of these modern quantum theories, can go directly to the hard core of the review in Chapter 3. Because of the celebratory nature of this article, we refrain from pursuing the detailed implications for quantum electronics, although there are of course many, as well as many in other areas of modern experimental physics, see the for example the recent review papers [5, 6, 7] which contain many references to earlier relevant papers.

Thus the aim of this paper is to give a comprehensive review of recent development in modern quantum theory from the historical perspective of the discovery the deterministic quantum evolutions by Heisenberg and Schrödinger to the stochastic evolutions of quantum jumps and quantum diffusion in quantum noise. This is the

direction in which quantum theory would have developed by the founders if the mathematics of quantum stochastics had been discovered by that time. We shall give a brief account of this new mathematics (which plays the same role for quantum stochastics as did the classical differential calculus for Newtonian dynamics) and concentrate on its application to the dynamical solution of quantum measurement problems [8, 9, 10, 11, 12, 13, 14], rather than give the full account of all related theoretical papers. Among these we would like to mention the papers on quantum decoherence [15, 16], dynamical state reduction program [17, 18], consistent histories and evolutions [19, 20], spontaneous localization and events [21, 22], restricted and unsharp path integrals [23, 24] and their numerous applications to quantum countings, jumps and trajectories in quantum optics and atomic physics [25, 26, 27, 28, 29, 30, 31, 32]. Most of these papers develop a nonstochastic phenomenological approach which is based on a non-Hamiltonian "instrumental" linear master equation giving the statistics of quantum measurements, but is not well adapted for the description of individual and conditional behavior under the continuous measurements. Pearl and Gisen took an opposite deterministic nonlinear approach for the individual evolutions without considering the statistics of measurements in their earlier papers [17, 18]. In this review we shall concentrate on a more constructive modern stochastic approach [2, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42] which gives the output statistics recurrently as a result of the solution of a stochastic differential equation. This allows the direct application of quantum conditioning and filtering methods to tackle the dynamical problem of quantum individual dynamics under continual (trajectorial) measurement. Here we refer mainly to the pioneering and original papers in which the relevant quantum structures as mathematical notions and methods were first invented in the mathematical physics literature. There are also many excellent publications in the theoretical and applied physics literature which appeared during the 90's such as the state diffusion theory [43, 44], and especially in quantum optics [45, 46, 47, 48, 50, 49] where the nonlinear quantum stochastic equations for continual measurements have been used. However the transition from nonlinear to linear stochastic equations and the quantum stochastic unitary models for the underlying Hamiltonian microscopic evolutions remain unexplained in these papers. An exception occurred in [50, 51], where our quantum stochastic theory was well understood both at a macroscopic and microscopic level. Most of these papers treat very particular phenomenological models which are based only on the counting, or sometimes diffusive (homodyne and heterodyne) models of quantum noise and output process, and reinvent many notions such as quantum conditioning with respect to the individual trajectories, without references to the general quantum stochastic measurement and filtering (conditioning) theory which have been developed for these purposes in the 80's. This explains why a systematic review of this kind is needed, and, hopefully, will be appreciated.

## 2. The Quantum Century Begins

*The whole is more than the sum of its parts* – Aristotle.

This is the famous superadditivity law from Aristotle's *Metaphysics* which studies 'the most general or abstract features of reality and the principles that have universal validity'. Certainly in this broad definition quantum physics is the most essential part of metaphysics.

Quantum theory is one of the greatest intellectual achievements of the past century. Since the discovery of quanta by Max Planck exactly 100 years ago on the basis of spectral analysis of quantum thermal noise, and the wave nature of matter, it has produced numerous paradoxes and confusions even in the greatest scientific minds such as those of Einstein, de Broglie, Schrödinger, Bell, and it still confuses many contemporary philosophers and scientists who fail to accept the Aristotle's superadditivity law of Nature. Rapid development of the beautiful and sophisticated mathematics for quantum mechanics and the development of its interpretation by Bohr, Born, Heisenberg, Dirac, von Neumann and many others who abandoned traditional causality, were little help in resolving these paradoxes despite the astonishing success in the prediction of quantum phenomena. Both the implication and consequences of the quantum theory of light and matter, as well as its profound mathematical, conceptual and philosophical foundations are not yet understood completely by the majority of quantum physicists.

In order to appreciate the quantum drama which has been developing through the whole century, and to estimate possible consequences of it in the new quantum technological age, it seems useful to give a brief account of the discovery of quantum theory at the beginning of the 20th century.

**2.1. The Discovery of Quantum Noise.** In 1918 Max Planck was awarded the Nobel Prize in Physics for his quantum theory of blackbody radiation or as we would say now, quantum theory of thermal noise based on the hypothesis of energy discontinuity. Invented in 1900, it inspired an unprecedented revolution in both physical science and philosophy of the 20th century, with an unimaginable deep revision in our way of thinking. As Planck stated later:-

> *If anyone says he can think about quantum problems without getting giddy, that only shows he has not understood the first thing about them*

**2.1.1.** *Quantum statistics for simpletons and children.* It is hard to realize that just 100 years ago the existence of the fundamental quantum of action $\hbar$ had not been known. In autumn of 1900 Planck made two famous reports to Berlin Physics Society [4] about the discovery of his constant. Although $\hbar$ evaluated by Planck is very-very small, $\hbar \approx 10^{-34}\,\mathrm{kg\,m^2\,/\,s}$ when measured in conventional units, such as kilograms, meters and seconds (and in some places it is still being neglected), we can always take $\hbar = 1$ by choosing suitable units.

There is a broad literature on the analysis Planck's 1900 paper, see for example Landsberg 1981, [52], for the analysis of his arguments from the point of view of statistical thermodynamics and the literature cited in this paper. Our analysis is much more elementary, it is simply based on experimental data which had been known to Planck prior he derived his formula.

According to the Boltzmann law of classical statistics, an absolutely black body, which absorbs light of all colours, or frequencies, equally at a temperature $\tau > 0$, would radiate absolutely white light, consisting of the mixture of all colours (frequencies $\omega$) of equal energies $E(\tau, \omega) = k\tau$, where $k$ is the Boltzmann constant which can be also set to one by choosing the suitable units of measurement.

Every child knows that it is not true: a heated blackbody (a piece of burning coal) at lower temperatures radiates more red light, and becomes white only at a high temperatures $\tau$. Planck's concern was to combine the empirical formulas of

Rayleigh ($E_r$) and Wien ($E_w$),

$$E_r\left(\omega\right) = k\tau - \frac{1}{2}\hbar\omega, \qquad\qquad E_w\left(\omega\right) = \hbar\omega e^{-\hbar\omega/k\tau},$$

which approximate the spectral density of the energy radiated by a blackbody at the temperature $\tau$ at a high and low frequencies $\omega$ respectively. In October 1900 he announced the Planck's quantum radiation formula $E\left(\tau,\omega\right) = \hbar\omega\left(e^{\hbar\omega/k\tau} - 1\right)^{-1}$ which becomes classical $k\tau$ only at the limit $\omega \to \infty$. Thus the constant (classical, $E_c$) spectral distribution of the thermal "white noise" was replaced by the linear-exponential (quantum $E_q$) spectral distribution:

$$E_c\left(\omega\right) = k\tau, \qquad\qquad E_q\left(\omega\right) = \hbar\omega\left(e^{\hbar\omega/k\tau} - 1\right)^{-1}$$

2.1.2. *The A-level derivation of Planck's formula.* Within two months Planck made a complete theoretical deduction of his formula, renouncing classical physics and introducing the weird assumption that the radiated energy consist of discrete portions, or quanta which are proportional to $\omega$:

$$\varepsilon_n\left(\omega\right) = \hbar\omega n, \quad n = 0,1,2,\ldots .$$

By expanding $z/\left(1-z\right)$ into the power series $\left(1-z\right)\sum nz^n$ with respect to $z = e^{-\hbar\omega/k\tau}$ he noted that the right formula $E = E_q$ can be written as an averaging

$$\hbar\omega z\left(1-z\right)^{-1} = \sum_{n=0}^{\infty}\hbar\omega n p_n$$

of the discrete energy levels $\varepsilon_n (\omega)$ with the geometric probability distribution $p_n = (1 - z) z^n$. Thus he concluded that

$$E (\tau, \omega) = \hbar\omega N (\tau, \omega) ,$$

where

$$N (\tau, \omega) = \left(1 - e^{-\hbar\omega/k\tau}\right) \sum_{n=0}^{\infty} n e^{-\hbar\omega n/k\tau} = \left(e^{\hbar\omega/k\tau} - 1\right)^{-1}$$

is the average of the number of quanta for the Boltzmann probability distribution

$$p_n = \left(1 - e^{-\hbar\omega/k\tau}\right) e^{-\hbar\omega n/k\tau} .$$

of the discrete energy levels $\varepsilon_n (\omega)$. The theory met resistance, and Planck himself "tried at first to somehow weld the elementary quantum of action somehow onto the framework of classical theory. But in the face of all such attempts this constant showed itself to be obdurate" [53]. He continued these futile attempts for a number of years, and they cost him a great deal of efforts. He himself gave credit to Boltzmann, as the only way to get the right spectral density was to assume that it is the Boltzmann mean value of the discrete $\varepsilon_n$.

2.1.3. *The emergence of the quantum theory of light.* In 1905 Einstein, examining the photoelectric effect, proposed a quantum theory of light, only later realizing that Planck's theory made implicit use of this quantum light hypothesis. Einstein saw that the energy changes in a quantum material oscillator occur in jumps which are multiples of $\omega$. Einstein received Nobel prize in 1922 for his work on the photoelectric effect. In 1924 Einstein arranged for the a publication of another important paper by Bose, which had been initially rejected by a referee. Bose proposed a new notion for statistical independence of quantum particles by putting them into independent cells, and conjectured that there is no conservation of the number of photons. Time has shown that Bose was right on all these points.

Thus, quantum theory first emerged as the result of experimental data not in the form of quantum mechanics but in the form of statistical observations of quantum noise, the basic concept of quantum probability and quantum stochastic processes. The corpuscular nature of light seemed to contradict the Maxwell electromagnetic wave theory of light. In 1924 Einstein wrote:

> *There are therefore now two theories of light, both indispensable, and - as one must admit today, despite twenty years of tremendous effort on the part of theoretical physics - without any logical connection.*

2.2. **The Rise of Quantum Mechanics.** The solution to the paradox of the wave/corpuscular duality of light came unexpectedly when de Broglie made the even more bizarre conjecture of extending this duality to matter. As a young girl said later to Schrödinger, who discovered the quantum mechanics of wave matter:

> *Hey, you never even thought when you began that so much sensible stuff would come out of it*

. (quoted from the Preface in [57]).

**2.2.1.** *The discovery of matrix mechanics.* In 1912 Niels Bohr worked in the Rutherford group in Manchester on his theory of the electron in an atom. He was puzzled by the discrete spectra of light which is emitted by atoms when they are subjected to an excitation. He was influenced by the ideas of Planck and Einstein and addressed a certain paradox in his work. How can energy be conserved when some energy changes are continuous and some are discontinuous, i.e. change by quantum amounts? Bohr conjectured that an atom could exist only in a discrete set of stable energy states, the differences of which amount to the observed energy quanta. Bohr returned to Copenhagen and published a revolutionary paper on the hydrogen atom in the next year. He suggested his famous formula

$$E_m - E_n = \hbar\omega_{mn}$$

from which he derived the major laws which describe physically observed spectral lines. This work earned Niels Bohr the 1922 Nobel Prize about $10^5$ Swedish Kroner.

In 1925 a young German theoretical physicist, Heisenberg, gave a preliminary account of a new and highly original approach to the mechanics of the atom [54]. He was influenced by Niels Bohr and proposed to substitute for the position coordinate of an electron in the atom arrays

$$q_{mn}(t) = q_{mn}e^{i\omega_{mn}t}$$

indexed by Bohr's differences $E_m - E_n$. They would account for the random jumps $E_m \mapsto E_n$ in the atom corresponding to the spontaneous emission of the energy quanta $\hbar\omega_{mn}$ for the Planck's electro-magnetic oscillators

$$\frac{\mathrm{d}}{\mathrm{d}t}q_{mn}(t) = i\omega_{mn}q_{mn}(t).$$

His Professor, Max Born, was a mathematician who immediately recognized an infinite matrix algebra in Heisenberg's multiplication rule for the tables $Q = [q_{mn}]$. The classical momentum was also replaced by a similar matrix,

$$P(t) = \left[p_{mn}e^{i\omega_{mn}t}\right]$$

and $P$ and $Q$ matrices were postulated to follow a commutation law:

$$[Q(t), P(t)] = i\hbar I,$$

where $I$ is the unit matrix. The classical Hamiltonian equations of dynamical evolution were now replaced by

$$\frac{\mathrm{d}}{\mathrm{d}t}Q(t) = \frac{i}{\hbar}[H, Q(t)], \quad \frac{\mathrm{d}}{\mathrm{d}t}P(t) = \frac{i}{\hbar}[H, P(t)],$$

where $H = [E_n\delta_{mn}]$ is the diagonal Hamilton matrix [55]. Thus quantum mechanics was first invented in the form of *matrix mechanics*, emphasizing the possibilities of quantum transitions, or jumps between the stable energy states $E_n$ of an electron. In 1932 Heisenberg was awarded the Nobel Prize for his work in mathematical physics.

Conceptually, the new atomic theory was based on the positivism of Mach as it operated not with real space-time but with only observable quantities like atomic transitions. However many leading physicists were greatly troubled by the prospect of loosing reality and deterministic causality in the emerging quantum physics. Einstein, in particular, worried about the element of 'chance' which had entered physics. In fact, this worries came rather late since Rutherford had introduced a spontaneous effect when discussing radio-active decay in 1900.

2.2.2. *The discovery of wave mechanics.* In 1923 de Broglie, inspired by the works of Einstein and Planck, extended the wave-corpuscular duality also to material particles. He used the Hamilton-Jacobi theory which had been applied both to particles and waves. In 1928 de Broglie received the Nobel Prize for this work.

In 1925, Schrödinger gave a seminar on de Broglie's material waves, and a member of the audience suggested that there should be a wave equation. Within a few weeks Schrödinger found his celebrated wave equation, first in a relativistic, and then in the non-relativistic form [56]. Instead of seeking the classical solutions to the Hamilton-Jacobi equation

$$H\left(q, \frac{\hbar}{i}\frac{\partial}{\partial q}\ln\psi\right) = E$$

he suggested finding those wave functions $\psi$ which satisfy the linear equation

$$H\left(q, \frac{\hbar}{i}\frac{\partial}{\partial q}\right)\psi = E\psi$$

(It coincides with the former equation only if $H$ is linear with respect to $p$).

Schrödinger published his revolutionary *wave mechanics* in a series of six papers [57] in 1926 during a short period of sustained creative activity that is without parallel in the history of science. Like Shakespeare, whose sonnets were inspired by a dark lady, Schrödinger was inspired by a mysterious lady of Arosa where he took ski holidays during the Christmas 1925 but 'had been distracted by a few calculations'. This was the second formulation of quantum theory, which he successfully applied to the Hydrogen atom, oscillator and other quantum mechanical systems, solving the corresponding Sturm-Liouville problems. The mathematical equivalence between the two formulations of quantum mechanics was understood by Schrödinger in the fourth paper where he obtained the non-stationary wave equation written in terms of the Hamiltonian operator $H = H\left(q, \frac{\hbar}{i}\frac{\partial}{\partial q}\right)$ as

$$i\hbar\frac{\partial}{\partial t}\psi\left(t\right) = H\psi\left(t\right),$$

and he also introduced operators associated with each dynamical variable.

Unlike Heisenberg and Born's matrix mechanics, the general reaction towards wave mechanics was immediately enthusiastic. Plank described Schrödinger's wave mechanics as 'epoch-making work'. Einstein wrote: 'the idea of your work springs from true genius...'. Next year Schrödinger was nominated for the Nobel Prize, but he failed to receive it in this and five further consecutive years of his nominations by most distinguished physicists of the world, the reason behind his rejection being 'the highly mathematical character of his work'. Only in 1933 did he receive his prize, this time jointly with Dirac, and this was the first, and perhaps the last, time when the Nobel Prize for physics was given to true mathematical physicists.

Following de Broglie, Schrödinger initially thought that the wave function corresponds to a physical vibration process in a real continuous space-time because it was not stochastic, but he was puzzled by the failure to explain the blackbody radiation and photoelectric effect from this wave point of view. In fact the wave interpretation applied to light quanta leads back to classical electrodynamics, as his relativistic wave equation for a single photon coincides mathematically with the classical wave equation. However after realizing that the time-dependent $\psi$ is a complex function in his fourth 1926 paper [57], he admitted that the wave function

$\psi$ cannot be given a direct interpretation, and described the wave density $\psi\bar{\psi}$ as a sort of weight function for superposition of point-mechanical configurations.

Although Schrödinger was a champion of the idea that the most fundamental laws of the microscopic world are absolutely random even before he discovered wave mechanics, he failed to see the probabilistic nature of $\psi\bar{\psi}$. Indeed, his equation was not stochastic, and it didn't account for the random jumps $E_m \to E_n$ of the Bohr-Heisenberg theory but rather opposite, it did prescribe the preservation of the eigenvalues $E = E_n$. He even wrote:

> *If we have to go on with these dammed quantum jumps, then I'm sorry that I ever got involved.*

For the rest of his life Schrödinger was trying to find from time to time without a success a more fundamental equation which would be responsible for the energy transitions in the process of measurement of the quanta $\hbar\omega_{mn}$. As we shall see, he was right assuming the existence of such equation.

2.2.3. *Interpretations of quantum mechanics.* The creators of the rival matrix quantum mechanics were forced to accept the simplicity and beauty of Schrödinger's approach. In 1926 Max Born put forward the statistical interpretation of the wave function by introducing the statistical mean

$$\langle H \rangle = \int \bar{\psi}\left(x\right) H \psi\left(x\right) \mathrm{d}x$$

for a dynamical variable $H$ in the physical state, described by $\psi$. This was developed in Copenhagen and gradually was accepted by almost all physicists as the "Copenhagen interpretation". Born by education was a mathematician, and he would be the only mathematician ever to receive the Nobel Prize (in 1953, for his statistical studies of wave functions) were it not for the fact that he became a physicist, later Professor of Natural Philosophy at Edinburgh. Bohr, Born and Heisenberg considered electrons and quanta as unpredictable particles which cannot be visualized in the real space and time.

The most outspoken opponent of a/the probabilistic interpretation was Einstein. Albert Einstein admired the new development of quantum theory but was suspicious, rejecting its acausality and probabilistic interpretation. It was against his scientific instinct to accept statistical interpretation of quantum mechanics as a complete description of physical reality. There are famous sayings of his on that account:

> *'God is subtle but he is not malicious', 'God doesn't play dice'*

During these debates on the probabilistic interpretation of quantum mechanics of Einstein between Niels Bohr, Schrödinger often sided with his friend Einstein, and this may explain why he was distancing himself from the statistical interpretation of his wave function. Bohr invited Schrödinger to Copenhagen and tried to convince him of the particle-probabilistic interpretation of quantum mechanics. The discussion between them went on day and night, without reaching any agreement. The conversation, however deeply affected both men. Schrödinger recognized the necessity of admitting both wave and particles, but he never devised a comprehensive interpretation rival to Copenhagen orthodoxy. Bohr ventured more deeply into philosophical waters and emerged with his concept of complementarity:

> *Evidence obtained under different experimental conditions cannot be comprehended within a single picture, but must be regarded as complementary in the sense that only the totality of the phenomena exhausts the possible information about the objects.*

In his later papers Schrödinger accepted the probabilistic interpretation of $\psi\bar{\psi}$, but he did not consider these probabilities classically, instead he interpreted them as the strength of our belief or anticipation of an experimental result. In this sense the probabilities are closer to propensities than to the frequencies of the statistical interpretation of Born and Heisenberg. Schrödinger had never accepted the subjective positivism of Bohr and Heisenberg, and his philosophy is closer to that called representational realism. He was content to remain a critical unbeliever.

There have been many other attempts to retain the deterministic realism in the quantum world, the most extravagant among these being the ensemble-world interpretations of Bohm [59] and Everett [60]. The first interpretational theory, known as the pilot-wave theory, is based on the conventional Schrödinger equation which is used to define the flow of a classical fluid in the configuration space. The predictions of this classical macroscopic theory coincide with the statistical predictions of the orthodox quantum theory only for the ensembles of coordinate-like observables with the initial probability distribution over the many worlds given by the initial pilot wave. Other observables like momenta which are precisely determined at each point by the velocity of this fluid, have no uncertainty under the fixed coordinates. This is inconsistent with the prediction of quantum theory for individual systems, and there is no way to incorporate the stochastic dynamics of sequentially monitored individual quantum particles in the single world into this fluid dynamics. Certainly this is a variation of the de Broglie-Schrödinger old interpretation, and it doesn't respect the Bell's first principle for the interpretational theories "that it should be possible to formulate them for small systems" [1], p. 126.

The Everett's many-world interpretation also assumes that the classical configurations at each time are distributed in the comparison class of possible worlds worth probability density $\psi\bar{\psi}$. However no continuity between present and past configurations is assumed, and all possible outcomes of their measurement are realized every time, each in a different edition of the continuously multiplying universe. The observer in a given brunch of the universe is aware only of what is going on in that particular branch, and this results in the reduction of the wave-function. This would be macroscopically equivalent to the pilot-wave theory if the de Broglie-Schrödinger-Bohm fluid dynamics could be obtained as the average of wave equations over all brunches. Every statistician experienced in the classical continuous measurements would recognize in this continuously branching model the many-world re-interpretation for a stochastic Markov process, and would apply the well-developed stochastic differential calculus to analyze this dynamical model. However no such equation for a continuously monitored branch was suggested in this theory, although there should be many if at all, corresponding to many possible choices of classical configurations (e.g. positions or momenta) for a single many-world Schrödinger equation.

Certainly there are some advantages living in many worlds from the philosophical point of view, from the practical point of view, however, to have an infinite number (continuum product of continua?) of real worlds at the same time seems not better than to have none. As Bell wrote in [1], p. 134: "to have multiple universes,

to realize all possible configurations of particles, would have seemed grotesque". Even if such a weighted many-world dynamical theory had been developed to a satisfactory level, it would be immediately reformulated as a stochastic evolutionary theory in our single world with well-established mathematical language and statistical interpretation. In fact, the stochastic theory of continuously observed quantum systems has been already derived, not just developed, in full generality and rigor in quantum probability, and it will be presented in the last section. But first I shall demonstrate the underlying ideas on the elementary single-transition level.

## 3. Quantum Uncertainties and Paradoxes

*How wonderful we have met with a paradox, now we have some hope of making progress* - Niels Bohr.

In 1932 von Neumann put quantum theory on firm theoretical basis by setting the mathematical foundation for new, quantum, probability theory, the quantitative theory for counting non commuting events of quantum logics. This noncommutative probability theory is based on essentially more general axioms than the classical (Kolmogorovian) probability of commuting events, which form common sense Boolean logic, first formalized by Aristotle. It has been under extensive development during the last 30 years since the introduction of algebraic and operational approaches for treatment of noncommutative probabilities, and currently serves as the mathematical basis for quantum information and measurement theory.

Here we shall demonstrate the main ideas of quantum probability and quantum paradoxes arising from the application of classical probability theory to quantum phenomena on the simple examples of a single quantum system such as quantum bit (qubit, or q-bit). One can identify it with a single spin 1/2 in the famous EPR paradox, or with an unstable atom in a single state as in the Schrödinger's quantum measurement model with his cat. The most recent mathematical development of these models and methods leads to a profound quantum filtering and control theory in quantum open systems presented in the Section 3 which has found numerous applications in quantum statistics, optics and spectroscopy, and is an appropriate tool for the solution of the dynamical decoherence problem for quantum communications and computations.

### 3.1. Uncertainties and Quantum Logics.
Bohr was concerned with the paradox of spontaneous emission. He addressed the question: How does the electron know when to emit radiation? Bohr, Born and Heisenberg abandoned causality of traditional physics in the most positivistic way. Max Born said:

*If God made the world a perfect mechanism, ... we need not solve innumerable differential equations, but can use dice with fair success.*

3.1.1. *Heisenberg uncertainty relations.* In 1927 Heisenberg derived [62] his famous uncertainty relation

$$\Delta Q \Delta P \geq \hbar/2, \quad \Delta T \Delta E \geq \hbar/2$$

which gave mathematical support to the revolutionary complementary principle of Bohr. The first relation was easily proved in the Schrödinger representations $Q = x$,

$P = \frac{\hbar}{i}\frac{\partial}{\partial x}$ in terms of the standard deviations

$$\Delta Q = \left(\langle Q^2 \rangle - \langle Q \rangle^2\right)^{1/2}, \quad \Delta P = \left(\langle P^2 \rangle - \langle P \rangle^2\right)^{1/2}.$$

The second relation, which was first stated by analogy of $t$ with $x$ and of $E$ with $P$, can be proved [63, 64] in terms of the optimal measurement of the initial time as an unknown parameter $\tau$ of the Schrödinger's state $\psi(t - \tau)$ which is realized by the measurement of self-adjoint operator $T = t$ in an extended representation where the Hamiltonian $H$ is given by the operator $E = i\hbar\frac{\partial}{\partial t}$. As Dirac stated:

> *Now when Heisenberg noticed that, he was really scared.*

Einstein launched an attack on the uncertainty relation at the Solvay Congress in 1927, and then again in 1930, by proposing cleverly devised thought experiments which would violate this relation. Most of these imaginary experiments were designed to show that interaction between the microphysical object and the measuring instrument is not so inscrutable as Heisenberg and Bohr maintained. He suggested, for example, a box filled with radiation with a clock. The clock is designed to open a shutter and allow one photon to escape. By weighing the box the photon energy and the time of escape can both be measured with arbitrary accuracy.

After proposing this argument Einstein is reported to have spent a happy evening, and Niels Bohr an unhappy one. After a sleepless night he showed next morning that Einstein was wrong. Mathematically his solution can be expressed by the following formula of 'signal plus noise'

$$X = t + Q, \quad Y = i\hbar\frac{\partial}{\partial t} + P$$

for the measuring quantity $X$, the pointer coordinate of the clock, and the observable $Y$ for indirect measurement of photon energy $E = i\hbar\frac{\partial}{\partial t}$ in the Einstein experiment, where $Q$ and $P$ are the position and momentum operators of the compensation weight under the box. Due to the initial independence of the weight, the commuting observables $X$ and $Y$ have even greater uncertainty

$$\Delta X \Delta Y = \Delta T \Delta E + \Delta Q \Delta P \geq \hbar$$

than that predicted by Heisenberg uncertainty $\Delta T \Delta E \geq \hbar/2$.

3.1.2. *Nonexistence of hidden variables.* Einstein hoped that eventually it would be possible to explain the uncertainty relations by expressing quantum mechanical observables as functions of some hidden variables $\lambda$ in deterministic physical states such that the statistical aspect will arise as in classical statistical mechanics by averaging these observables over $\lambda$.

Von Neumann's monumental book [65] on the mathematical foundations of quantum theory was therefore a timely contribution, clarifying, as it did, this point. Inspired by Lev Landau, he introduced, for the unique characterization of the statistics of a quantum ensemble, the statistical density operator $\rho$ which eventually, under the name normal, or regular state, became a major tool in quantum statistics. He considered the linear space $\mathcal{L}$ of all bounded Hermitian operators $L = L^*$ as potential obsrvables in a quantum system described by a Hilbert space $\mathbb{H}$ of all (not yet normalized to one) wave functions $\psi$. Although von Neumann considered any complete inner product complex linear space as the Hilbert space, it is sufficient to reproduce his analysis for a finite-dimensional $\mathbb{H}$. He defined the expectation $\langle L \rangle$ of each $L \in \mathcal{L}$ in a state $\rho$ by the continuous (ultra-weakly continuous if $\mathbb{H}$

is infinite-dimensional) linear functional $\langle L \rangle = \mathrm{Tr}L\rho$, where $\mathrm{Tr}$ denotes the linear operation of trace applied to the product of all operators on the right. He noted that in order to have positive probabilities for quantum mechanical events $E = 1$ as expectations $\langle E \rangle$ of yes-no observables $E \in \mathcal{L}$ with $\{0, 1\}$ spectrum, and probability one for the identity event $I = 1$ described by identity operator $I$ (the multiplication by 1),

$$\mathrm{Pr}\,\{E = 1\} = \mathrm{Tr}E\rho \geq 0, \quad \mathrm{Pr}\,\{I = 1\} = \mathrm{Tr}\rho = 1,$$

the statistical operator $\rho$ must be positive-definite and have trace one. Then he proved that any *physically continuous additive* functional $L \mapsto \langle L \rangle$ is regular, i.e. has such trace form. He applied this technique to the analysis of the completeness problem of quantum theory, i.e. whether it constitutes a logically closed theory or whether it could be reformulated as an entirely deterministic theory through the introduction of hidden parameters (additional variables which, unlike ordinary observables, are inaccessible to measurements). He came to the conclusion that

> the present system of quantum mechanics would have to be objectively false, in order that another description of the elementary process than the statistical one may be possible

(quoted on page 325 in [65])

To prove this theorem, von Neumann showed that there is no such state which would be dispersion-free simultaneously for all quantum events $E \in \mathcal{L}$ described by Hermitian projectors $E^2 = E$. For each such state, he argued,

$$\langle E^2 \rangle = \langle E \rangle = \langle E \rangle^2$$

for all events $E$ would imply that $\rho = O$, which cannot be statistical operator as $\mathrm{Tr}O = 0 \neq 1$. Thus no state can be considered as a mixture of dispersion-free states, each of them associated with a definite value of hidden parameters. There are simply no such states, and thus, no hidden parameters. In particular this implies that the statistical nature of pure states, which are described by one-dimensional projectors $\rho = P_\psi$ corresponding to wave functions $\psi$, cannot be removed by supposing them to be a mixture of dispersion-free substates.

It is widely believed that in 1966 John Bell showed that von Neuman's proof was in error, or at least his analysis left the real question untouched [66]. To discredit the von Neumann's proof he constructed an example of dispersion-free states parametrized for each quantum state $\rho$ by a real parameter $\lambda$ for a single quantum bit corresponding to the two dimensional $\mathbb{H} = \mathfrak{h}$ (we use the little $\mathfrak{h} \simeq \mathbb{C}^2$ in order to emphasize that this is the simplest quantum system). He succeeded to do this by weakening the assumption of the additivity for such states, requiring it only for the commuting observables in $\mathcal{L}$, and by abandoning the linearity of the constructed expectations in $\rho$ described by spin polarization vector $\mathbf{r}$. There is no reason, he argued, to keep the linearity in $\rho$ for the observable eigenvalues determined by $\lambda$ and $\rho$, and to demand the additivity for non-commuting observables as they are not simultaneously measurable: The measured eigenvalues of a sum of noncommuting observables are not the sums of the eigenvalues of this observables measured separately. For each spin-projection $L = \sigma(\mathbf{l})$ given by a 3-vector $\mathbf{l}$ Bell found a family $s_\lambda(\mathbf{l})$ of dispersion-free values $\pm l$, $l = |\mathbf{l}|$, parameterized by $|\lambda| \leq 1/2$, which reproduce the expectation $\langle \sigma(\mathbf{l}) \rangle = \mathbf{l} \cdot \mathbf{r}$ in the pure quantum state when uniformly averaged over the $\lambda$. However his example does not contradict the von Neumann theorem even if the latter is strengthened by the restriction of the additivity only to

the commuting observables: The constructed dispersion-free expectation function $L \mapsto \langle L \rangle_\lambda$ is not *physically continuous* on $\mathcal{L}$ because the value $\langle L \rangle_\lambda = s_\lambda(\mathbf{1})$ is one of the eigenvalues $\pm 1$ for each $\lambda$, and it covers both values when the directional vector $\mathbf{l}$ rotates continuously over the three-dimensional sphere. A function $\mathbf{l} \mapsto \langle \sigma(\mathbf{l}) \rangle_\lambda$ on the continuous manifold (sphere) with discontinuous values can be continuous only if it is constant, but this is ruled out by the demand to reproduce the expectation $\langle \sigma(\mathbf{l}) \rangle = \mathbf{l} \cdot \mathbf{r}$ which is variable in $\mathbf{l}$ by averaging over $\lambda$ the constants $\langle \sigma(\mathbf{l}) \rangle_\lambda$ in $\mathbf{l}$. Measurements of the projections of spin on the physically close directions should be described by close expected values in any physical state specified by $\lambda$, otherwise it cannot have physical meaning! More detailed critical analysis of the Bell's arguments is given in the Appendix 1.

Since then there were innumerable attempts to introduce hidden variables in ever more sophisticated forms, perhaps not yet discovered, which would determine the complementary variables if the hidden variables were measured precisely. In higher dimensions of $\mathbb{H}$ all these attempts are ruled out by Gleason's theorem [67] who proved that there is no even one additive zero-one probability function if dim $\mathbb{H} > 2$.

3.1.3. *Complementarity and common sense.* In view of the decisive importance of this analysis for the foundations of quantum theory, Birkhoff and von Neumann [68] setup a system of formal axioms for the calculus of logicotheoretical propositions concerning results of possible measurements in a quantum system. They started by formalizing the calculus of *quantum propositions* $E \in \mathcal{E}$ corresponding to the idealized events described by orthoprojectors $E$ on a Hilbert space $\mathbb{H}$, the projective operators $E = E^2$ which are orthogonal to their complements $E^\perp = I - E$ in the sense $E^* E^\perp = O$, where $O$ denotes the multiplication by 0. This is equivalent to

$$E^* = E^* E = E,$$

i.e. the set of propositions $\mathcal{E}$ is the set of all Hermitian projectors $E \in \mathcal{L}$ which are the only observables with two eigenvalues $\{1, 0\}$ ("yes" and "no"). Such calculus coincides with the calculus of linear subspaces $\mathfrak{e} \subseteq \mathbb{H}$ including 0-dimensional subspace $\mathbb{O}$, in the same sense as the common sense propositional calculus of classical events coincides with the calculus in a Boolean algebra of subsets $E \subseteq \Omega$ including empty subset $\emptyset$. The subspaces $\mathfrak{e}$ are defined by the condition $\mathfrak{e}^{\perp\perp} = \mathfrak{e}$, where $\mathfrak{e}^\perp$ denotes the orthogonal complement $\{\chi \in \mathbb{H} : \langle \chi | \psi \rangle = 0, \psi \in \mathfrak{e}\}$ of $\mathfrak{e}$, and they uniquely define the propositions $E$ as the orthoprojectors $P(\mathfrak{e})$ onto the ranges

$$\mathfrak{e} = \mathrm{range} E := E\mathbb{H}$$

of $E \in \mathcal{E}$. In this calculus the logical ordering $E \leq F$ implemented by the algebraic relation $EF = E$ coincides with

$$\mathrm{range} E \subseteq \mathrm{range} F,$$

the conjunction $E \wedge F$ corresponds to the intersection,

$$\mathrm{range}(E \wedge F) = \mathrm{range} E \cap \mathrm{range} F,$$

however the disjunction $E \vee F$ is represented by the linear sum $\mathfrak{e} + \mathfrak{f}$ of the corresponding subspaces but not their union

$$\mathrm{range} E \cup \mathrm{range} F \subseteq \mathrm{range}(E \vee F),$$

and the smallest orthoprojector $O$ corresponds to zero-dimensional subspace $\mathbb{O} = \{0\}$ but not the empty subset $\emptyset$ (which is not linear subspace). Note that although

range $(E + F) = \mathfrak{e} + \mathfrak{f}$  for any $E, F \in \mathcal{E}$, the operator $E + F$ is not the ortho-projector $E \vee F$ corresponding to $\mathfrak{e} + \mathfrak{f}$ unless $EF = 0$. This implies that the distributive law characteristic for propositional calculus of classical logics no longer holds. However it still holds for compatible propositions described by commutative orthoprojectors due to the orthomodularity property

$$E \leq I - F \leq G \Longrightarrow (E \vee F) \wedge G = E \vee (F \wedge G).$$

Given the regular state $\langle E \rangle = \text{Tr} E\rho$ on $\mathcal{E}$ one can also inroduce the probability measure

$$\text{P} (\mathfrak{e}) = \text{Pr} \{P (\mathfrak{e}) = 1\} = \langle P (\mathfrak{e}) \rangle$$

which is additive but only for orthogonal $\mathfrak{e}$ and $\mathfrak{f}$:

$$\mathfrak{e} \perp \mathfrak{f} \Rightarrow \text{P} (\mathfrak{e} + \mathfrak{f}) = \text{P} (\mathfrak{e}) + \text{P} (\mathfrak{f}).$$

Two propositions $E, F$ are called complementary if $E \vee F = I$, orthocomplementary if $E + F = I$, incompatible or disjunctive if $E \wedge F = 0$, and contradictory or orthogonal if $EF = 0$. As in the classical, common sense case, logic contradictory propositions are incompatible. However *incompatible propositions are not necessary contradictory* as can be easily seen for any two nonorthogonal but not coinciding one-dimensional subspaces. In particular, in quantum logics there exist complementary incompatible pairs $E, F$, $E \vee F = I$, $E \wedge F = 0$ which are not ortho-complementary in the sense $E + F \neq I$, i.e. $EF \neq 0$ (this would be impossible in the classical case). This is a rigorous logico-mathematical proof of Bohr's complementarity.

As an example, we can consider the proposition that a quantum system is in a stable energy state $E$, and an incompatible proposition $F$, that it collapses at a given time $t$, say. The incompatibility $E \wedge F = 0$ follows from the fact that there is no state in which the system would collapse preserving its energy, however these two propositions are not contradictory (i.e. not orthogonal, $EF \neq 0$): the system might not collapse if it is in other than $E$ stationary state (remember the Schrödinger's earlier belief that the energy law is valid only on average, and is violated in the process of radiation).

In 1952 Wick, Wightman, and Wigner [69] showed that there are physical systems for which not every orthoprojector corresponds to an observable event, so that not every one-dimensional orthoprojector $\rho = P_\psi$ corresponding to a wave function $\psi$ is a pure state. This is equivalent to the admission of some selective events which are dispersion-free in all pure states. Jauch and Piron [70] incorporated this situation into quantum logics and proved in the context of this most general approach that the hidden variable interpretation is only possible if the theory is observably wrong, i.e. if incompatible events are in fact compatible or contradictory.

Bell criticized this as well as the Gleason's theorem, but this time his arguments were not based even on the classical ground of usual probability theory. Although he explicitly used the additivity of the probability on the orthogonal events in his counterexample for $\mathbb{H} = \mathbb{C}^2$, he questioned : 'That so much follows from such apparently innocent assumptions leads us to question their innocence'. (p.8 in [1]). In fact this was equivalent to questioning the additivity of classical probability on the corresponding disjoint subsets, but he didn't suggest any other complete system of physically reasonable axioms for introducing such peculiar "nonclassical" hidden variables, not even a single counterexample to the orthogonal nonadditivity

for the simplest case of quantum bit $\mathbb{H} = \mathfrak{h}$. Thus Bell implicitly rejected classical probability theory in the quantum world, but he didn't want to accept quantum probability as the only possible theory for explaining the microworld. Even if such attempt was successful for a single quantum system (as he possibly thought his unphysical discontinuous construction in the case $\dim \mathbb{H} = 2$ was), it would satisfy only the classical composition law. This would not allow extension of the dispersion-free product-states to a composed quantum system because of their nonadditivity on the space $\mathcal{L}$ of all composed observables in the Hilbert space $\mathbb{H} \otimes \mathfrak{g}$ with $\mathfrak{g} = \mathbb{C}^n$ for $n > 1$. As it is shown in the Appendix 1, the quantum composition law together with the orthoadditivity excludes the hidden variable possibility also for $\mathbb{H} = \mathbb{C}^2$. Otherwise it would contradict Gleason's theorem because $\dim(\mathfrak{h} \otimes \mathfrak{g}) = 2n > 2$, not to mention a hidden variable reproduction of nonseparable, entangled states. Thus the quantum composition principle justifies the von Neumann's additivity assumption for the states on the whole operator algebra $\mathcal{B}(\mathbb{H})$ of each Hilbert space $\mathbb{H}$.

**3.2. Entanglement of Quantum Bits.** Heisenberg derived from the uncertainty relation that 'the nonvalidity of rigorous causality is necessary and not just consistently possible'. Max Born even stated:

> *One does not get an answer to the question, what is the state after collision? but only to the question, how probable is a given effect of the collision?*

The general scientific consensus in the physical world is that no positive solution exists to these negative statements at present. And this will be so unless we formulate these problems in a rigorous way and disagree with the notorious saying [71] that in mathematics "we never know what we are talking about" ( B. Russel should have better said that "we never know what *mathematicians* are talking about": he was not a true mathematician, the mathematicians know precisely what they are talking about).

3.2.1. *Spooky action at distance.* After his defeat on uncertainty relations Einstein seemed to have become resigned to the statistical interpretation of quantum theory, and at the 1933 Solvay Congress he listened to Bohr's paper on completeness of quantum theory without objections. Then, in 1935, he launched a brilliant and subtle new attack in a paper [72] with two young co-authors, Podolski and Rosen, which is known as the EPR paradox that has become of major importance to the world view of physics. They stated the following requirement for a complete theory as a seemingly necessary one:

> *Every element of physical reality must have a counterpart in the physical theory.*

The question of completeness is thus easily answered as soon as soon as we are able to decide what are the elements of the physical reality. EPR then proposed a sufficient condition for an element of physical reality:

> *If, without in any way disturbing the system, we can predict with certainty the value of a physical quantity, then there exists an element of physical reality corresponding to this quantity.*

Then they designed a thought experiment the essence of which is that two quantum "bits", particle spins of two electrons say, are brought together to interact, and after separation an experiment is made to measure the spin orientation of one

of them. The state after interaction is such that the measurement result $\tau = \pm\frac{1}{2}$ of one particle uniquely determination the spin $z$-orientation $\sigma = \mp\frac{1}{2}$ of the other particle. EPR apply their criterion of local reality: since the value of $\sigma$ can be predicted by measuring $\tau$ without in any way disturbing $\sigma$, it must correspond to an existing element of physical reality. Yet the conclusion contradicts a fundamental postulate of quantum mechanics, according to which the sign of spin is not an intrinsic property of a complete description of the spin but is evoked only by a process of measurement. Therefore, EPR conclude, quantum mechanics must be incomplete, there must be hidden variables not yet discovered, which determine the spin as an intrinsic property. It seems Einstein was unaware of the von Neumann's theorem, although they both had positions at the Institute for Advanced Studies at Princeton (being among the original six mathematics professors appointed there in 1933).

Bohr carefully replied to this challenge by rejecting the assumption of local physical realism as stated by EPR [73]: 'There is no question of a mechanical disturbance of the system under investigation during the last critical stage of the measuring procedure. But even at this stage there is essentially a question of *an influence on the very conditions which define the possible types of predictions regarding the future behavior of the system*'. This influence became notoriously famous as Bohr's *spooky action at a distance*. He had obviously meant the semi-classical model of measurement, when one can statistically infer the state of one (quantum) part of a system immediately after observing the other (classical) part, whatever the distance between them. In fact, there is no paradox of "spooky action at distance" in the classical case. The statistical inference, playing the role of such immediate action, is simply based on the Bayesian selection rule of a posterior state from the prior mixture of all such states, corresponding to the possible results of the measurement. Bohr always emphasized that one must treat the measuring instrument classically (the measured spin, or another bit interacting with this spin, as a classical bit), although the classical-quantum interaction should be regarded as purely quantum. The latter follows from non-existence of semi-classical Poisson bracket (i.e. classical-quantum potential interaction). Schrödinger clarified this point more precisely then Bohr, and he followed in fact the mathematical pattern of von Neumann measurement theory. EPR paradox is related to so called Bell inequality the probabilistic roots of which was evidentiated in [74].

3.2.2. *Releasing Schrödinger's cat.* Motivated by EPR paper, in 1935 Schrödinger published a three part essay [75] on 'The Present Situation in Quantum Mechanics'. He turns to EPR paradox and analyses completeness of the description by the wave function for the entangled parts of the system. (The word *entangled* was introduced by Schrödinger for the description of nonseparable states.) He notes that if one has pure states $\psi(\sigma)$ and $\varphi(\tau)$ for each of two completely separated bodies, one has maximal knowledge, $\chi(\sigma, \tau) = \psi(\sigma)\varphi(\tau)$, for two taken together. But the converse is not true for the entangled bodies, described by a non-separable wave function $\chi(\sigma, \tau) \neq \psi(\sigma)\varphi(\tau)$:

> *Maximal knowledge of a total system does not necessary imply maximal knowledge of all its parts, not even when these are completely separated one from another, and at the time can not influence one another at all.*

To make absurdity of the EPR argument even more evident he constructed his famous burlesque example in quite a sardonic style. A cat is shut up in a

steel chamber equipped with a camera, with an atomic mechanism in a pure state $\rho_0 = P_\psi$ which triggers the release of a phial of cyanide if an atom disintegrates spontaneously (it is assumed that it might not disintegrate in a course of an hour with probability $\operatorname{Tr}(EP_\psi) = 1/2$). If the cyanide is released, the cat dies, if not, the cat lives. Because the entire system is regarded as quantum and closed, after one hour, without looking into the camera, one can say that the entire system is still in a pure state in which the living and the dead cat are smeared out in equal parts.

Schrödinger resolves this paradox by noting that the cat is a macroscopic object, the states of which (alive or dead) could be distinguished by a macroscopic observation as distinct from each other whether observed or not. He calls this 'the principle of state distinction' for macroscopic objects, which is in fact the postulate that the directly measurable system (consisting of the cat) must be classical:

> It is typical in such a case that an uncertainty initially restricted to an atomic domain has become transformed into a macroscopic uncertainty which can be resolved through direct observation.

The dynamical problem of the transformation of the atomic, or "coherent" uncertainty, corresponding to a probability amplitude $\psi(\sigma)$, into a macroscopic uncertainty, corresponding to a mixed state $\rho$, is called quantum *decoherence* problem. In order to make this idea clear, let us give the solution of the Schrödinger's elementary decoherence problem in the purely mathematical way. Instead of the values $\pm 1/2$ for the spin-variables $\sigma$ and $\tau$ we shall use the values $\{0, 1\}$ corresponding to the states of a classical "bit", the simplest nontrivial system in classical probability or information theory.

Consider the atomic mechanism as a quantum "bit" with Hilbert space $\mathfrak{h} = \mathbb{C}^2$, the pure states of which are described by $\psi$-functions of the variable $\sigma \in \{0, 1\}$ (if atom is disintegrated, $\sigma = 1$, if not, $\sigma = 0$) with scalar (complex) values $\psi(\sigma)$ defining the probabilities $|\psi(\sigma)|^2$ of the quantum elementary propositions corresponding to $\sigma = 0, 1$. The Schrödinger's cat is a classical bit with only two pure states $\tau \in \{0, 1\}$ which can be identified with the probability distributions $\delta_0(\tau)$ when alive ($\tau = 0$) and $\delta_1(\tau)$ when dead ($\tau = 1$). These and other (mixed) states can also be described by the complex amplitudes $\varphi(\tau)$, however they are uniquely defined by the probabilities $|\varphi(\tau)|^2$ up to a phase function of $\tau$, the phase multiplier of $\varphi \in \mathfrak{g}$, $\mathfrak{g} = \mathbb{C}^2$ commuting with all cat observables $c(\tau)$, not just up to a phase constant as in the case of the atom (only constants commute with all atomic observables $A \in \mathcal{L}(\mathfrak{h})$). Initially the cat is alive, so its amplitude (uniquely defined up to the phase factor by the square root of probability distribution $\delta_0$ on $\{0, 1\}$) is $\delta(\tau)$ that is 1 if $\tau = 0$, and 0 if $\tau = 1$.

The only meaningful classical-quantum reversible interaction affecting not the atom but the cat after the hour, is described by unitary transformation

$$U[\psi \otimes \varphi](\sigma, \tau) = \psi(\sigma)\varphi((\tau \triangle \sigma)),$$

as both bits were quantum, where $\tau \triangle \sigma = |\tau - \sigma| = \sigma \triangle \tau$ is the difference (mod 2) on $\{0, 1\}$. Applied to the initial product-state $\psi \otimes \delta$ it has the resulting probability amplitude

$$\chi(\sigma, \tau) = \psi(\sigma)\delta(\tau \triangle \sigma) = 0 \quad \text{if} \quad \sigma \neq \tau.$$

Despite the fact that the initial state was pure, $\chi_0 = \psi \otimes \delta$ corresponding to the Cartesian product $(\psi, 0)$ of the initial pure states $\psi \in \mathfrak{h}$ and $\tau = 0$, the reversible

unitary evolution $U$ induces in $\mathbb{H} = \mathfrak{h} \otimes \mathfrak{g}$ the mixed state $\chi \in \mathbb{H}$ for the quantum-classical system "atom+cat" described by the wave function $\chi(\sigma, \tau)$.

Indeed, the observables of such a system are operator-functions $X$ of $\tau$ with values $X(\tau)$ in $\sigma$-matrices, represented as block-diagonal $(\sigma, \tau)$-matrices $\hat{X} = [X(\tau)\delta_{\tau'}^{\tau}]$ of the multiplication $X(\tau)\chi(\cdot, \tau)$ at each point $\tau \in \{0, 1\}$. This means that the amplitude $\chi$ induces the same expectations

$$\left\langle \hat{X} \right\rangle = \sum_{\tau} \chi(\tau)^{\dagger} X(\tau) \chi(\tau) = \sum_{\tau} \mathrm{Tr} X(\tau) \varrho(\tau) = \mathrm{Tr} \hat{X} \hat{\varrho}$$

as the block-diagonal density matrix $\hat{\varrho} = [\varrho(\tau)\delta_{\tau'}^{\tau}]$ of the multiplication by

$$\varrho(\tau) = E(\tau) P_{\psi} E(\tau) = \pi(\tau) P_{E(\tau)\psi}$$

where $\pi(\tau) = |\psi(\tau)|^2$, $E(\tau) = P_{\delta_{\tau}}$ is the projection operator

$$[E(\tau)\psi](\sigma) = \delta(\tau \triangle \sigma)\psi(\sigma) = \psi(\tau)\delta_{\tau}(\sigma),$$

and $P_{E(\tau)\psi} = P_{\delta_t}$ is also the projector onto $\delta_{\tau}(\cdot) = \delta(\cdot \triangle \tau)$. The $4 \times 4$-matrix $\hat{\varrho}$ is a mixture of two orthogonal projectors $P_{\delta_{\tau}} \otimes P_{\delta_{\tau}}$, $\tau = 0, 1$:

$$\hat{\varrho} = [P_{\delta_{\tau}}\delta_{\tau'}^{\tau}\pi(\tau)] = \sum_{\tau=0}^{1} \pi(\tau) P_{\delta_{\tau}} \otimes P_{\delta_{\tau}}.$$

3.2.3. *Von Neumann's projection postulate.* Inspired by Bohr's complementarity principle, von Neumann proposed even earlier the idea that every quantum measuring process involves an unanalysable element. He postulated [65] that, in addition to the continuous causal propagation of the wave function generated by the Schrödinger equation, during a measurement, due to an action of the observer on the object, the function undergoes a discontinuous, irreversible instantaneous change. Thus, just prior to a measurement of the event $F$, disintegration of the atom, say, the quantum pure state $P_{\psi}$ changes to the mixed one

$$\rho = \lambda P_{E\psi} + \mu P_{F\psi} = E\rho E + F\rho F,$$

where $E = I - F$ is the orthocomplement event, and $\lambda = \mathrm{Tr} E\rho$, $\mu = \mathrm{Tr} F\rho$ are the probabilities of $E$ and $F$. Such change is projective as shown in the second part of this equation, and it is called the von Neumann projection postulate.

This linear irreversible decoherence process should be completed by the nonlinear, acausal random jump to one of the pure states

$$\rho \mapsto P_{E\psi}, \text{ or } \rho \mapsto P_{F\psi}$$

depending on whether the tested event $F$ is false (the cat is alive, $\psi_0 = \lambda^{-1/2} E\psi$), or true (the cat is dead, $\psi_1 = \mu^{-1/2} F\psi$). This final step is the posterior prediction, called *filtering* of the decoherent mixture of $\psi_0$ and $\psi_1$ by selection of only one result of the measurement, and is an unavoidable element in every measurement process relating the state of the pointer of the measurement (in this case the cat) to the state of the whole system. This assures that the same result would be obtained in case of immediate subsequent measurement of the same event $F$. The resulting change of the initial wave-function $\psi$ is described up to normalization by one of the projections

$$\psi \mapsto E\psi, \quad \psi \mapsto F\psi$$

and is sometimes called the Lüders projection postulate [76].

Although unobjectionable from the purely logical point of view the von Neumann theory of measurement soon became the target of severe criticisms. Firstly it seams radically subjective, postulating the spooky action at distance (the filtering) in a purely quantum system instead of deriving it. Secondly the analysis is applicable to only the idealized situation of discrete instantaneous measurements.

However as we already mentioned when discussing the EPR paradox, the process of filtering is free from conceptual difficulty if it is understood as the statistical inference about a mixed state in an extended stochastic representation of the quantum system as a part of a semiclassical one, based upon the results of observation in its classical part. In order to demonstrate this, we can return to the dynamical model of Schrödinger's cat, identifying the quantum system in question with the Schrödinger's atom. The event $E$ (the atom exists) corresponds then to $\tau = 0$ (the cat is alive), $E = E(0)$, and the complementary event is $F = E(1)$. This model explains that the origin of the von Neumann irreversible decoherence $P_\psi \mapsto \rho$ of the atomic state is in the ignorance of the result of the measurement described by the partial tracing over the cat's Hilbert space $\mathfrak{g} = \mathbb{C}^2$:

$$\rho = \mathrm{Tr}_{\mathfrak{g}} \hat{\varrho} = \sum_{\tau=0}^{1} \pi(\tau) P_{\delta_\tau} = \varrho(0) + \varrho(1),$$

where $\varrho(\tau) = |\psi(\tau)|^2 P_{\delta_\tau}$. It has entropy $S(\rho) = \mathrm{Tr}\rho \log \rho^{-1}$ of the compound state $\hat{\varrho}$ of the combined semi-classical system prepared for the indirect measurement of the disintegration of atom by means of cat's death:

$$S(\rho) = -\sum_{\tau=0}^{1} |\psi(\tau)|^2 \log |\psi(\tau)|^2 = S(\hat{\varrho})$$

It is the initial coherent uncertainty in the pure quantum state of the atom described by the wave-function $\psi$ which is equal to one bit if initially $|\psi(0)|^2 = 1/2 = |\psi(1)|^2$.

This dynamical model of the measurement which is due to von Neumann, also interprets the filtering $\rho \mapsto \rho_\tau$ simply as the conditioning

$$\rho_\tau = \varrho(\tau) / \pi(\tau) = P_{\delta_\tau}$$

of the joint classical-quantum state $\varrho(\cdot)$ by the Bayes formula which is applicable due to the commutativity of actually measured observable (the life of cat) with any other observable of the combined semi-classical system.

Thus the atomic decoherence is derived from the unitary interaction of the quantum atom with the classical cat. The spooky action at distance, affecting the atomic state by measuring $\tau$, is simply the result of the statistical inference (prediction after the measurement) of the atomic posterior state $\rho_\tau = P_{\delta_\tau}$: the atom disintegrated if and only if the cat is dead. A formal derivation of the von-Neumann-Lüders projection postulate and the decoherence in the general case by explicit construction of unitary transformation in the extended semi-classical system is given in [77, 78].

## 4. Decoherence, Measurement and Filtering

*In mathematics you don't understand things. You just get used to them* - John von Neumann.

In this Chapter I present the author's views on the solution to quantum measurement problem which might not coincide with the present scientific consensus

that this problem is unsolvable, or at least unsolved. It will be shown that there exists such solution along the line suggested by the great founders of quantum theory Schrödinger, Heisenberg and Bohr. In fact it was envisaged by von Neumann in his Mathematical Foundation of Quantum Theory [65], and by more recent quantum philosopher J Bell in [1]. However while the dynamical consideration of the measurement process in [65] is absent at all, the continuous in time model of measurement suggested by Bell is simply wrong (the dynamical equation (5) he suggested in [1], p.176, doesn't preserve the positivity of transition probabilities for the stochastic process).

The differential analysis of the appropriate models is based on Itô stochastic calculus and its quantum generalization. The discovery of quantum thermal noise and its white-noise approximations lead to a profound revolution not only in modern physics but also in contemporary mathematics comparable with the discovery of differential calculus by Newton (for a feature exposition of this, accessible for physicists, see [79], the complete theory, which was mainly developed in the 80's [2, 80, 81, 82], is sketched in the Appendix 2) .

4.1. **Beables and Nondemolition Principle.** Schrödinger like Einstein was deeply concerned with the loss of reality and causality in the positivistic treatment of quantum measuring process by Heisenberg and Born. Schrödinger's remained unhappy with Bohr's reply to the EPR paradox, Schrödinger's own analysis was:

> *It is pretty clear, if reality does not determine the measured value, at least the measurable value determines reality.*

Our approach resolves the famous paradoxes of quantum measurement theory in a constructive way by giving exact nontrivial models for the statistical analysis of quantum observation processes determining the reality underlying these paradoxes. Conceptually it is based upon a new idea of quantum causality called the Nondemolition Principle [77] which divides the world into the classical past, forming the consistent histories, and the quantum future, the state of which is predictable for each such history.

4.1.1. *Compatibility and time arrow.* Von Neumann's projection postulate and its dynamical realization can be generalized to include cases with continuous spectrum of values. In fact there many such developments, we will only mention here the most general operational approach to quantum measurements of Ludwig [83], and its mathematical implementation by Davies and Lewis [84] in the "instrumental" form. The stochastic realization of the corresponding completely positive reduction map $\rho \mapsto \varrho\left(\cdot\right)$, resolving the corresponding instantaneous quantum measurement problem, can be found in [85, 77]. Because of the crucial importance of these realizations for developing understanding of the mathematical structure and interpretation of modern quantum theory, we need to analyze the mathematical consequences which can be drawn from such schemes.

The generalized reduction of the wave function $\psi\left(x\right)$, corresponding to a complete measurement with discrete or continuous data $y$, is described by a function $V\left(y\right)$ whose values are linear operators $\mathfrak{h} \ni \psi \mapsto V\left(y\right)\psi$ which are, in general, not isometric on the given Hilbert space $\mathfrak{h}$, $V\left(y\right)^{*}V\left(y\right) \neq I$, but have the following normalization condition. The resulting wave-function

$$\chi\left(x,y\right) = \left[V\left(y\right)\psi\right]\left(x\right)$$

is normalized with respect to a given measure $\mu$ on $y$ in the sense

$$\iint |\chi(x,y)|^2 \, \mathrm{d}\mu \mathrm{d}\lambda = \int |\psi(x)|^2 \, \mathrm{d}\lambda$$

for any probability amplitude $\psi$ (normalized with respect to a measure $\lambda$). This can be written as $V^\dagger V = I$ in terms of the integral

$$\int_y V(y)^* V(y) \, \mathrm{d}\mu = I, \quad \text{or} \quad \sum_y V(y)^* V(y) = I.$$

with respect to the base measure $\mu$ which is taken in the discrete case, such as the case of two-point variables $y = \tau$ (EPR paradox, or Schrödinger cat with the projection-valued $V(\tau) = E(\tau)$), to be the counting measure. As in that simple example the realization of such $V$ can be always constructed [77] in terms of a unitary transformation $U$ on an extended Hilbert space $\mathfrak{h} \otimes \mathfrak{g}$ and a normalized wave function $\varphi_0 \in \mathfrak{g}$ such that

$$U[\psi \otimes \varphi_0](x,y) = \chi(x,y)$$

for any $\psi$. The additional system described by "the pointer coordinate $y$ of the measurement apparatus" can be regarded as classical (like the cat) as the actual observables in question are the measurable functions $g(y)$ represented by commuting operators $\hat{g}$ of multiplication by these functions. They are appropriate candidates for Bell's "beables", [1], p.174, as such commuting observables, extended to the quantum part as $I \otimes \hat{g}$, are compatible with any possible (future) event, represented by an orthoprojector $F \otimes I$. The probabilities (or, it is better to say, the propensities) of all such events are the same in all states whether an observable $\hat{g}$ was measured but the result is not read, or it was not measured at all. In this sense the measurements of $\hat{g}$ are called *nondemolition* with respect to the future observables $F$, they do not demolish the picture of the possibilities, or propensities of $F$. But they are not necessary compatible with the initial operators $F \otimes I$ of the quantum system under the question in the present representation $U(F \otimes I)U^*$ corresponding to the actual states $\chi = U(\psi \otimes \varphi_0)$.

Indeed, the Heisenberg operators

$$G = U^*(I \otimes \hat{g})U$$

of the nondemolition observables in general do not commute with the past operators $F \otimes I$ on the initial states $\chi_0 = \psi \otimes \varphi_0$. One can see this from the example of the Schrödinger cat. The "cat observables" in Heisenberg picture are represented by commuting operators $G = [g(\sigma + \tau)\delta^\sigma_{\sigma'}\delta^\tau_{\tau'}]$ of multiplication by $g(\sigma + \tau)$, where the sum $\sigma + \tau = |\sigma - \tau|$ is modulo 2. They do not commute with $F \otimes I$ unless $F$ is also a diagonal operator $\hat{f}$ of multiplication by a function $f(\sigma)$ in which case

$$[F, G]\chi_0(\sigma, \tau) = [f(\sigma), g(\sigma + \tau)]\chi_0(\sigma, \tau) = 0, \quad \forall \chi_0.$$

However the restriction of the possibilities in a quantum system to only the diagonal operators $F = \hat{f}$ which would eliminate the time arrow in the nondemolition condition amounts to the redundancy of the quantum consideration as all such (possible and actual) observables can be simultaneously represented as the functions of $(\sigma, \tau)$ as in the classical case.

4.1.2. *Transition from possible to actual.* The analysis above shows that as soon as dynamics is taken into consideration even in the form of just a single unitary transformation, the measurement process needs to specify the arrow of time, what is the predictable future and what is the reduced past, what is possible and what is actual with respect to this measurement. As soon as a measured observable $Y$ is specified, i.e. is taken as a *be*able, all other operators which do not commute with $Y$ become entirely redundant and are not among possible future *be*ables. The algebra $\mathcal{A}$ of all such potential future observables (not the state which stays invariant in the Schrödinger picture unless the selection due to an inference has taken place!) reduces to the subalgebra commuting with $Y$, and *this reduction doesn't change the reality* (the wave function remains the same and induces the same, now mixed, state on the smaller, reduced algebra!). Possible observables in an individual system are only those which are compatible with the actual observable/beables. This is another formulation of Bohr's complementarity which specifies *mathematically* which natural processes have the special status of 'measurements', and which was unknown to Bell (compare with "There is nothing in the mathematics to tell what is 'system' and what is 'apparatus', ", in [1], p.174). More specifically this can be rephrased in the form of a dynamical postulate of quantum causality called the Nondemolition Principle [77] which we first formulate for a single instant of time $t$ in quite an obvious form:

> In the appropriate representation of a quantum system by an algebra $\mathcal{A}$ of (necessarily not all) operators on a Hilbert space of the system plus measurement apparatus, causal, or nondemolition observables are represented only by those operators $Y$, which are compatible with $\mathcal{A}$:
>
> $$[X, Y] := XY - YX = 0, \quad \forall X \in \mathcal{A}$$
>
> (this is usually written as $Y \in \mathcal{A}'$, where $\mathcal{A}'$, called the commutant of $\mathcal{A}$, in this formulation is not necessarily contained in $\mathcal{A}$). Each measurement process of the history for a quantum system $\mathcal{A}$ can be represented as nondemolition by the causal observables in the appropriate representation of $\mathcal{A}$.

Note that the space of representation plays here the crucial role: the reduced operators

$$X_0 = (I \otimes \varphi_0)^* X (I \otimes \varphi_0), \ Y_0 = (I \otimes \varphi_0)^* Y (I \otimes \varphi_0)$$

for commuting $X$ and $Y$ might not commute on the smaller space $\mathfrak{h}_0 \subset \mathfrak{h} \otimes \mathfrak{g}$ of the initial states $\psi \otimes \varphi_0$ with a fixed $\varphi_0 \in \mathfrak{g}$. Even if the nondemolition observables $Y$ is faithfully represented by $Y_0$ on initial space $\mathfrak{h}_0$, as it is in the case $Y = G$ of the Schrödinger's cat with $\varphi_0(\tau) = \delta(\tau)$, where $Y_0$ is the multiplication operator $G_0 = \hat{g}$ for $\psi$:

$$G(\psi \otimes \delta)(\sigma, \tau) = g(\sigma + \tau)\psi(\sigma)\delta(\tau) = g(\sigma)\psi(\sigma)\delta(\tau) = (G_0\psi \otimes \delta)(\sigma, \tau),$$

there is usually no room in $\mathfrak{h}_0$ to represent all Heisenberg operators $X \in \mathcal{A}$ commuting with $Y$ on $\mathfrak{h} \otimes \mathfrak{g}$. The induced operators $Y_0$ do not commute with all operators $F$ of the system initially represented on $\mathfrak{h}_0$, and this is why the measurement of $Y_0$ is thought to cause demolition on $\mathfrak{h}_0$. However in all such cases the future operators $X$ reduced to $X_0$ on $\mathfrak{h}_0$, commute with $Y_0$ as they are decomposable with respect to $Y_0$ (however the reduction $X \mapsto X_0$ is not the Heisenberg one-to-one but instead

an irreversible dynamical map). This can be seen explicitly for the atom described by the Heisenberg operators $X = U^* (F \otimes I) U$ in the interaction representation with the cat:

$$X_0 = \sum_\tau E(\tau) F E(\tau), \quad Y_0 = \sum g(\tau) E(\tau).$$

The nondemolition principle can be considered not only as a restriction on the possible observations for a given dynamics but also as a condition for the causal dynamics to be compatible with the given observations (beables). As was proved in [86], the causality condition is necessary and sufficient for the existence of a conditional expectation for any state on the total algebra $\mathcal{A} \vee \mathcal{B}$ with respect to a commutative subalgebra $\mathcal{B}$ of nondemolition observables $Y$. Thus the nondemolition causality condition amounts exactly to the existence of the conditional states, i.e. to the predictability of the states on the algebra $\mathcal{A}$ upon the measurement results of the observables in $\mathcal{B}$. Then the transition from a prior $\rho$ to a posterior state $\rho_y = P_{V(y)\psi}$ is simply the result of gaining knowledge of $y$ defining the actual state in the decoherent mixture

$$\rho = \int V(y) P_\psi V(y)^* \, \mathrm{d}\mu = \int P_{V(y)\psi} f(y) \, \mathrm{d}\mu$$

of all possible states, where $f(y) = \|V(y)\psi\|^2$ is the probability density of $y$ defining the output measure $\mathrm{d}\nu = f\mathrm{d}\mu$. As Heisenberg always emphasized, "quantum jump" is contained in the transitions from possible to actual.

If an algebra $\mathcal{B}$ of beables is specified at a time $t$, there must be a causal representation $\mathcal{B}_t$ of $\mathcal{B}$ with respect to the present $\mathcal{A}_t$ and all future possible representations $\mathcal{A}_s$, $s > t$ of the quantum system on the same Hilbert space (they might not coincide with $\mathcal{A}_t$ if the system is open [90]). The past representations $\mathcal{A}_r$, $r < t$ which are incompatible with a $G \in \mathcal{B}_t$ are meaningless as noncausal for the observation at the time $t$, they should be replaced by the causal histories $\mathcal{B}_r$, $r < t$ of the beables which must be consistent in the sense of compatibility of all $\mathcal{B}_t$. Thus the dynamical formulation of the nondemolition principle of quantum causality and the consistency of histories reads as

$$\mathcal{A}_s \subseteq \mathcal{B}_r', \quad \mathcal{B}_s \subseteq \mathcal{B}_r', \quad \forall r \leq s.$$

These are the only possible conditions when the posterior states always exist as results of inference (filtering and prediction) of future quantum states upon the measurement results of the classical (i.e. commutative) past of a process of observation. The act of measurement transforms quantum propensities into classical realities. As Lawrence Bragg, another Nobel prize winner, once said, everything in the future is a wave, everything in the past is a particle.

4.1.3. *The true Heisenberg principle.* The time continuous solution of the quantum measurement problem was motivated by analogy with the classical stochastic filtering problem which obtains the prediction of future for an unobservable dynamical process $x(t)$ by time-continuous measuring of another, observable process $y(t)$. Such a problem was first considered by Wiener and Kolmogorov who found its solution in the form of causal spectral filter but only for the stationary Gaussian case. The differential solution in the form of a stochastic filtering equation was then obtained by Stratonovich [91] in 1958 for an arbitrary Markovian pair $(x, y)$. This was really a break through in the statistics of stochastic processes which soon

found many applications, in particular for solving the problems of stochastic control under incomplete information (it is possible that this was one of the reasons why the Russians were so successful in launching the rockets to the Moon and other planets of the Solar system in 60s).

If $X(t)$ is the unobservable process, a Heisenberg coordinate process of a quantum particle, say, and $Y(t)$ is an observable quantum process, describing the trajectories $y(t)$ of the particle in a cloud chamber, say, why don't we find a filtering equation for the a posteriori expectation $q(t)$ of $X(t)$ or any other function of $X(t)$ in the same way as we do it in the classical case if we know a history, i.e. a particular trajectory $y(r)$ up to the time $t$? This problem was first considered and solved for the case of quantum Markovian Gaussian pair $(X, Y)$ corresponding to a quantum open linear system with linear output channel, in particular for a quantum oscillator matched to a quantum transmission line [2, 34]. By studying this example, the nondemolition condition

$$[X(s), Y(r)] = 0, \qquad [Y(s), Y(r)] = 0 \quad \forall r \le s$$

was first found, and this allowed the solution in the form of the causal equation for $q(t) = \langle X(t) \rangle_y$.

Let us describe this exact dynamical model of the causal nondemolition measurement first in terms of quantum white noise analysis for a one-dimensional quantum nonrelativistic particle of mass $m$ which is conservative if not observed, in a potential field $\phi$. But we shall assume that the particle is under indirect observation by measuring of its Heisenberg position operator $X(t)$ with an additive random error $e(t)$ :

$$Y(t) = X(t) + e(t).$$

We take the simplest statistical model for the error process $e(t)$, the white noise model (the worst, completely chaotic error), assuming that it is a classical (i.e. commutative) Gaussian white noise given by the first momenta

$$\langle e(t) \rangle = 0, \quad \langle e(s) e(r) \rangle = \sigma^2 \delta(s - r).$$

The measurement process $Y(t)$ should be commutative, satisfying the causal nondemolition condition with respect to the noncommutative process $X(t)$ (and any other Heisenberg operator-process of the particle), this can be achieved by perturbing the particle Newton-Erenfest equation:

$$m \frac{\mathrm{d}^2}{\mathrm{d}t^2} X(t) + \nabla \phi(X(t)) = f(t).$$

Here $f(t)$ is the Langevin force perturbing the dynamics due to the measurement, which is assumed to be another classical (commutative) white noise.

$$\langle f(t) \rangle = 0, \quad \langle f(s) f(r) \rangle = \tau^2 \delta(s - r).$$

In classical measurement and filtering theory the white noises $e(t), f(t)$ are usually considered independent, and the intensities $\sigma^2$ and $\tau^2$ can be arbitrary, even zeros, corresponding to the ideal case of the direct unperturbing observation of the particle trajectory $X(t)$. However in quantum theory corresponding to the standard commutation relations

$$X(0) = \hat{x}, \quad \frac{\mathrm{d}}{\mathrm{d}t} X(0) = \frac{1}{m} \hat{p}, \quad [\hat{x}, \hat{p}] = i\hbar \hat{1}$$

the particle trajectories do not exist, and it was always understood that the measurement error $e(t)$ and perturbation force $f(t)$ should satisfy a sort of uncertainty relation. This "true Heisenberg principle" had never been mathematically formulated and proved before the discovery [2] of quantum causality and nondemolition condition in the above form of commutativity of $X(s)$ and $Y(r)$ for $r \leq s$. As we showed first in the linear case [2, 34], and later even in the most general case [89], these conditions are fulfilled if and only if $e(t)$ and $f(t)$ satisfy the canonical commutation relations

$$[e(r), e(s)] = 0, \ [e(r), f(s)] = \frac{\hbar}{i}\delta(r-s), \ [f(r), f(s)] = 0.$$

This proves that the pair $(e, f)$ must satisfy the uncertainty relation $\sigma\tau \geq \hbar/2$, i.e.

$$\Delta e_t \Delta f_t \geq \hbar t/2,$$

in terms of the standard deviations of the integrated processes

$$e_t = \int_0^t e(r)\,\mathrm{d}r, \quad f_t = \int_0^t f(s)\,\mathrm{d}s.$$

This inequality constitutes the precise formulation of the true Heisenberg principle for the square roots $\sigma$ and $\tau$ of the intensities of error $e$ and perturbation $f$: they are inversely proportional with the same coefficient of proportionality, $\hbar/2$, as for the pair $(\hat{x}, \hat{p})$. The canonical pair $(e, f)$ called quantum white noise cannot be considered classically despite of the possibility of the classical realizations of each process $e$ and $f$ separately due to the self-commutativity of the families $e$ and $f$. Thus, a generalized matrix mechanics for the treatment of quantum open systems under continuous nondemolition observation and the true Heisenberg principle was invented exactly 20 years ago in [2]. The nondemolition commutativity of $Y(t)$ with respect to the Heisenberg operators of the open quantum system was later rediscovered for the output of quantum stochastic fields in [81].

4.2. **Consistent Histories and Filtering.** Schrödinger believed that all quantum problems including the interpretation of measurement should be formulated in continuous time in the form of differential equations. He thought that the measurement problem would have been resolved if quantum mechanics had been made consistent with relativity theory and the time had been treated appropriately. However Einstein and Heisenberg did not believe this, each for his own reasons. While Einstein thought that the probabilistic interpretation of quantum mechanics was wrong, Heisenberg simply stated:-

> *Quantum mechanics itself, whatever its interpretation, does not account*
> *for the transition from 'possible to the actual'*

Perhaps the closest to the truth was Bohr when he said that it 'must be possible so to describe the extraphysical process of the subjective perception as if it were in reality in the physical world', extending the reality beyond the closed quantum mechanical form by including a subjective observer into a semiclassical world. He regarded the measurement apparatus, or meter, as a semiclassical object which interacts with the world in a quantum mechanical way but has only commuting observables - pointers. Thus Bohr accepted that *not all the world is quantum mechanical, there is a classical part of the physical world, and we belong partly to this classical world.*

In realizing this program I will follow the line suggested by John Bell [1] along which the "development towards greater physical precision would be to have the 'jump' in the equations and not just the talk – so that it would come about as a dynamical process in dynamically defined conditions."

4.2.1. *Stochastic decoherence equation.* The generalized wave mechanics which enables us to treat the quantum processes of time continuous observation, or in other words, quantum mechanics with trajectories $\omega = (y_t)$, was discovered only quite recently, in [36, 38, 92]. The basic idea of the theory is to replace the deterministic unitary Schrödinger propagation $\psi \mapsto \psi(t)$ by a linear causal stochastic one $\psi \mapsto \chi(t, \omega)$ which is not necessarily unitary for each history $\omega$, but unitary in the mean square sense with respect to a standard probability measure $\mu(\mathrm{d}\omega)$ for $\mathrm{d}\omega$. Due to this the positive measures

$$\mathrm{P}(t, \mathrm{d}\omega) = \|\chi(t, \omega)\|^2 \mu(\mathrm{d}\omega), \quad \tilde{\mu}(\mathrm{d}\omega) = \lim_{t \to \infty} \mathrm{P}(t, \mathrm{d}\omega)$$

are normalized (if $\|\psi\| = 1$) for each $t$, and are interpreted as the probability measure for the histories $\omega_t = (y_r)_{r < t}$.of the output stochastic process $y_t$ with respect to the measure $\tilde{\mu}$. In the same way as the abstract Schrödinger equation can be derived from only unitarity of propagation, the abstract decoherence wave equation can be derived from the mean square unitarity in the form of a linear stochastic differential equation. The reason that Bohr and Schrödinger didn't derive such an equation despite their firm belief that the measurement process can be described 'as if it were in reality in the physical world' is that the appropriate (stochastic) differential calculus had not been yet developed early in that century. As Newton had to invent the differential calculus in order to formulate the equations of classical dynamics, we had to develop the quantum stochastic calculus for nondemolition processes [36, 86] presented in the Appendix in order to derive the generalized wave equation for quantum dynamics with continual observation.

For the notational simplicity we shall consider here the one dimensional case, $d = 1$, the multi-dimensional case is discussed in the Appendix 2 and can be found elsewhere (e.g. in [36, 86]). The abstract stochastic wave equation can be written in this case as

$$\mathrm{d}\chi(t) + K\chi(t)\,\mathrm{d}t = L\chi(t)\,\mathrm{d}y_t, \quad \chi(0) = \psi.$$

Here $y_t(\omega)$ is assumed to be a martingale (e.g. the independent increment process with zero expectation, see the Appendix) representing a measurement noise with respect to the input probability measure $\mu(\mathrm{d}\omega) = \mathrm{P}(0, \mathrm{d}\omega)$ (but not with respect to the output probability measure $\tilde{\mu} = \mathrm{P}(\infty, \mathrm{d}\omega)$ for which $y_t(\omega)$ is an output process with dependent increments). If the stochastic process $\chi(t, \omega)$ is normalized in the mean square sense for each $t$, it represents a probability amplitude $\chi(t)$ in an extended Hilbert space describing the process of continual decoherence of the initial pure state $\rho(0) = P_\psi$ into the mixture

$$\rho(t) = \int P_{\psi_\omega(t)} \mathrm{P}(t, \mathrm{d}\omega) = \mathrm{M}\left[\chi(t)\chi(t)^\dagger\right]$$

of the posterior states corresponding to $\psi_\omega\left(t\right) = \chi\left(t, \omega\right) / \left\|\chi\left(t, \omega\right)\right\|$, where M denotes mean with respect to the measure $\mu$. Assuming that the conditional expectation $\left\langle \mathrm{d}y_t \mathrm{d}y_t \right\rangle_t$ in

$$\left\langle \mathrm{d}\left(\chi^\dagger \chi\right) \right\rangle_t = \left\langle \mathrm{d}\chi^\dagger \mathrm{d}\chi + \chi^\dagger \mathrm{d}\chi + \mathrm{d}\chi^\dagger \chi \right\rangle_t$$
$$= \chi^\dagger \left(L^* \left\langle \mathrm{d}y_t \mathrm{d}y_t \right\rangle_t L - \left(K + K^*\right) \mathrm{d}t\right) \chi$$

is $\mathrm{d}t$ (e.g. $\left(\mathrm{d}y_t\right)^2 = \mathrm{d}t + \varepsilon \mathrm{d}y_t$), the mean square normalization in its differential form $\left\langle \mathrm{d}\left(\chi^\dagger \chi\right) \right\rangle_t = 0$ can be expressed [38, 40] as $K + K^* = L^*L$, i.e.

$$K = \frac{1}{2}L^*L + \frac{i}{\hbar}H,$$

where $H = H^*$ is the Schrödinger Hamiltonian such that this is the Schrödinger equation if $L = 0$. One can also derive the corresponding Master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho\left(t\right) + K\rho\left(t\right) + \rho\left(t\right)K^* = L\rho\left(t\right)L^*$$

for mixing decoherence of the initially pure state $\rho\left(0\right) = \psi\psi^\dagger$, as well as a stochastic nonlinear wave equation for the dynamical prediction of the posterior state vector $\psi_\omega\left(t\right)$, the normalization of $\chi\left(t, \omega\right)$ at each $\omega$.

4.2.2. *Quantum jumps and diffusions.* Actually, there are two basic standard forms [92, 93] of such stochastic wave equations, corresponding to two basic types of stochastic integrators with independent increments: the Brownian standard type, $\varepsilon = 0$, $y_t \simeq w_t$, and the Poisson standard type $\varepsilon = 1$, $y_t \simeq n_t - t$ with respect to the basic measure $\mu$, see the Appendix. To get these we shall assume that $y_t$ is standard with respect to the input measure $\mu$, given by the multiplication table

$$\left(\mathrm{d}y\right)^2 = \mathrm{d}t + \nu^{-1/2}\mathrm{d}y, \quad \mathrm{d}y\mathrm{d}t = 0 = \mathrm{d}t\mathrm{d}y,$$

where $\nu > 0$ is the intensity of the Poisson process $n_t = \nu^{1/2}y_t + \nu t$, and

$$L = \nu^{1/2}(C - I), \quad H = E + i\frac{\nu}{2}\left(C - C^*\right),$$

with $C$ and $E$ called collapse and energy operators respectively. This corresponds to the stochastic decoherence equation of the form

$$\mathrm{d}\chi\left(t\right) + \left(\frac{\nu}{2}\left(C^*C - I\right) + \frac{i}{\hbar}E\right)\chi\left(t\right)\mathrm{d}t = \left(C - I\right)\chi\left(t\right)\mathrm{d}n_t,$$

which was derived for quantum jumps caused by the counting observation in [38, 94]. It correspond to the linear stochastic decoherence Master-equation

$$\mathrm{d}\varrho\left(t\right) + \left[G\varrho\left(t\right) + \varrho\left(t\right)G^* - \nu\varrho\left(t\right)\right]\mathrm{d}t = \left[C\varrho\left(t\right)C^* - \varrho\left(t\right)\right]\mathrm{d}n_t, \quad \varrho\left(0\right) = \rho,$$

for the not normalized (but normalized in the mean) density matrix $\varrho\left(t, \omega\right)$, where $G = \frac{\nu}{2}C^*C + \frac{i}{\hbar}E$ (it has the form $\chi\left(t, \omega\right)\chi\left(t, \omega\right)^\dagger$ in the case of a pure initial state $\rho = \psi\psi^\dagger$).

The nonlinear filtering equation for $\psi_\omega\left(t\right)$ in this case has the form [93]

$$\mathrm{d}\psi_\omega + \left(\frac{\nu}{2}\left(C^*C - \left\|C\psi_\omega\right\|^2\right) + \frac{i}{\hbar}E\right)\psi_\omega\mathrm{d}t = \left(C/\left\|C\psi_\omega\right\| - I\right)\psi_\omega\mathrm{d}n_\omega^\rho,$$

(see also [88] for the infinite-dimensional case). It corresponds to the nonlinear stochastic Master-equation

$$\mathrm{d}\rho_\omega + \left[G\rho_\omega + \rho_\omega G^* - \nu\rho_\omega \mathrm{Tr}C\rho_\omega C^*\right]\mathrm{d}t = \left[C\rho_\omega C^*/\mathrm{Tr}C\rho_\omega C^* - \rho\right]\mathrm{d}n_\omega^\rho$$

for the posterior density matrix $\rho_\omega(t)$ which is the projector $\psi_\omega(t)\psi_\omega(t)^\dagger$ for the pure initial state $\rho_\omega(0) = P_\psi$. Here $n^\rho(t) = n_t^{\rho(t)}$ is the output counting process which is described by the history probability measure

$$\mathrm{P}(t, \mathrm{d}\omega) = \pi(t, \omega)\,\mu(\mathrm{d}\omega), \quad \pi(t, \omega) = \mathrm{Tr}\varrho(t, \omega)$$

with the increment $\mathrm{d}n^\rho(t)$ independent of $n_t^\rho$ under the condition $\rho_\omega(t) = \rho$, with the conditional expectation

$$\mathrm{M}\left[\mathrm{d}n^\rho(t)\,|\rho_\omega(t) = \rho\right] = \nu\mathrm{Tr}C\rho C^*\mathrm{d}t$$

(or $\nu\|C\psi\|^2\,\mathrm{d}t$ for $\rho = P_\psi$). The derivation and solution of this equation was also considered in [87], and its solution was applied in quantum optics in [45, 49].

This nonlinear quantum jump equation can be written also in the quasi-linear form [92, 93]

$$\mathrm{d}\psi_\omega(t) + \tilde{K}(t)\psi_\omega(t)\,\mathrm{d}t = \tilde{L}(t)\psi_\omega(t)\,\mathrm{d}y_\omega^\rho(t),$$

where $y_\omega^\rho(t)$ is the innovating martingale with respect to the output measure which is described by the differential

$$\mathrm{d}y_\omega^\rho(t) = \nu^{-1/2}\|C\psi_\omega(t)\|^{-1}\,\mathrm{d}n_\omega^\rho(t) - \nu^{1/2}\|C\psi_\omega(t)\|\,\mathrm{d}t$$

with $\rho = P_\psi$ and the initial $y_\omega^\rho(0) = 0$, the operator $\tilde{K}(t)$ similar to $K$ has the form

$$\tilde{K}(t) = \frac{1}{2}\tilde{L}(t)^*\tilde{L}(t) + \frac{i}{\hbar}\tilde{H}(t),$$

and $\tilde{H}(t), \tilde{L}(t)$ depend on $t$ (and $\omega$) through the dependence on $\psi = \psi_\omega(t)$:

$$\tilde{L} = \nu^{1/2}\left(C - \|C\psi\|\right), \quad \tilde{H} = E + i\frac{\nu}{2}\left(C - C^*\right)\|C\psi\|.$$

The latter form of the nonlinear filtering equation admits the central limit $\nu \to \infty$ corresponding to the standard Wiener case $\varepsilon = 0$ when $y_t = w_t$ with respect to the input Wiener measure $\mu$. If $L$ and $H$ do not depend on $\nu$, i.e. $C$ and $E$ depend on $\nu$ as

$$C = I + \nu^{-1/2}L, \; E = H + \frac{\nu^{1/2}}{2i}\left(L - L^*\right),$$

then $y_\omega^\rho(t) \to \tilde{y}_t$ as $\varepsilon^2 = \nu^{-1} \to 0$, where the innovating diffusion process $\tilde{y}_t$ defined as

$$\mathrm{d}\tilde{y}_t(\omega) = \mathrm{d}y_t(\omega) - 2\,\mathrm{Re}\,\langle\psi_\omega(t)\,|L\psi_\omega(t)\rangle\,\mathrm{d}t$$

can be identified with another standard Wiener process $\tilde{w}_t$ respectively to the output probability measure such that $\tilde{\mu}(\mathrm{d}\omega) = \mu(\mathrm{d}\tilde{\omega})$. If $\|\psi_\omega(t)\| = 1$ (which follows from the initial condition $\|\psi\| = 1$), the stochastic operator-functions $\tilde{L}(t), \tilde{H}(t)$ defining the nonlinear filtering equation have the limits

$$\tilde{L} = L - \mathrm{Re}\,\langle\psi|L\psi\rangle, \quad \tilde{H} = H + \frac{i}{2}\left(L - L^*\right)\mathrm{Re}\,\langle\psi|L\psi\rangle.$$

The corresponding nonlinear stochastic diffusion equation

$$\mathrm{d}\psi_\omega(t) + \tilde{K}(t)\psi_\omega(t)\,\mathrm{d}t = \tilde{L}(t)\psi_\omega(t)\,\mathrm{d}\tilde{w}_t$$

was first derived in the general multi-dimensional density-matrix form

$$\mathrm{d}\rho_\omega + \left[K\rho_\omega + \rho_\omega K - L\rho_\omega L^*\right]\mathrm{d}t = \left[L\rho_\omega + \rho_\omega L^* - \rho_\omega\mathrm{Tr}\left(L + L^*\right)\rho_\omega\right]\mathrm{d}\tilde{w}_t$$

in [36, 42] from the microscopic reversible quantum stochastic evolution models by the quantum filtering method. It has been recently applied in quantum optics [46, 47, 48, 50] for the description of counting, homodyne and heterodyne time-continuous measurements introduced in [94]. A particular case of this filtering equation for the quantum particle in a potential field $\phi$ was also derived phenomenologically by Diosi [37]. It was solved for the case of linear and quadratic potential $\phi$ and the Gaussian initial wave function in [36, 96]. The general microscopic derivation for the case of multi-dimensional complete and incomplete measurements and solution in the linear-Gaussian case is given in [89]. As in the classical case this solution coincides with the optimal quantum linear filtering (quantum Kalman filter) earlier obtained in [2, 34] for the complex amplitude of the quantum open oscillator continuously observed in the complex noise $y_t$ with $(\mathrm{d}y)^2 = 0, |\mathrm{d}y|^2 = \mathrm{d}t$. The applications of this quantum stochastic model to homodyne and heterodyne continuous measurements in quantum optics can be found in [44, 48, 50]. The explicit solution of this stochastic wave equation for the free particle and the Gaussian initial wave packet is given in [97]. One can show [98, 99] that the nondemolition observation of such a particle is described by filtering of the quantum noise which results in the continual collapse of any initial wave packet to the Gaussian stationary one localized at the position posterior expectation.

The connection between the above diffusive nonlinear filtering equation and our linear decoherence Master-equation

$$\mathrm{d}\varrho\,(t) + \left[K\varrho\,(t) + \varrho\,(t)\,K^* - L\varrho\,(t)\,L^*\right]\mathrm{d}t = \left[L\varrho\,(t) + \varrho\,(t)\,L^*\right]\mathrm{d}w_t, \quad \varrho\,(0) = \rho,$$

for the stochastic density operator $\varrho\,(t, \omega)$, defining the output probability density $\mathrm{Tr}\varrho\,(t, \omega)$, was well understood and presented in [41, 50, 51]. However it has also found an incorrect mathematical treatment in recent Quantum State Diffusion theory [95] based on the case $\varepsilon = 0$ of our filtering equation (this particular nonlinear filtering equation is empirically postulated as the 'primary quantum state diffusion', and its more fundamental linear version $\mathrm{d}\chi + K\chi\mathrm{d}t = L\chi\mathrm{d}w$ is 'derived' in [95] simply by dropping the non-linear terms without appropriate change of the probability measures for the processes $\tilde{y}_t = \tilde{w}_t$ and $y_t = w_t$). The most general stochastic decoherence Master equation is given in the Appendix 2.

4.2.3. *Q-bit trajectories and localizations.* Let us describe the exact Markovian model of an open quantum bit in a white noise and nondemolition measurement. It was introduced in [40, 86] even in the case of multidimensional quantum noise, but we shall consider here just one dimension.

We assume that in an interaction representation picture the q-bit operators $S = \sigma\,(\mathbf{s})$ (spins $1/2$, see the notations of the Appendix 1) evolve as $S\,(t) = U\,(t)^*\,SU\,(t)$, where $U\,(t)$ is stochastic unitary transformation in the Hilbert space $\mathfrak{h} = \mathbb{C}^2$ satisfying the Itô-Schrödinger equation

$$\mathrm{d}U\,(t) + \left(\frac{i}{\hbar}H + \frac{1}{2}L^*L\right)U\,(t)\,\mathrm{d}t = \frac{i}{\hbar}LU\,(t)\,\mathrm{d}f_t, \quad U\,(0) = I.$$

Here $H = \sigma\,(\mathbf{h})$ is a Hamiltonian and $L = \sigma\,(\mathbf{l})$ is a spin-operator given by real 3-vectors $\mathbf{h}$ and $\mathbf{l}$ (we assume for simplicity that $L^* = L$), and $f_t = i\hbar\,(\Lambda_- - \Lambda^+)_t$ is the integral of the Langevin force which is defined as the input field momentum process in the notations of the Appendix 2. The operators $S\,(t)$ satisfy the

perturbed Heisenberg equation in the Itô form

$$\mathrm{d}S\left(t\right) + \left(\frac{i}{\hbar}\left[S\left(t\right), H\left(t\right)\right] + \frac{1}{2}\left[\left[S\left(t\right), L\left(t\right)\right], L\left(t\right)\right]\right)\mathrm{d}t = \frac{i}{\hbar}\left[S\left(t\right), L\left(t\right)\right]\mathrm{d}f_t$$

and it can be written in the form of a vector stochastic equation

$$\mathrm{d}\mathbf{s}\left(t\right) + \left(\mathbf{k}\left(t\right)\times\mathbf{s}\left(t\right) + 2\mathbf{l}\left(t\right)^2\mathbf{s}\left(t\right) - 2\left(\mathbf{l}\left(t\right)\cdot\mathbf{s}\left(t\right)\right)\mathbf{l}\left(t\right)\right)\mathrm{d}t = \frac{i}{\hbar}\left(\mathbf{l}\left(t\right)\times\mathbf{s}\left(t\right)\right)\mathrm{d}f_t,$$

where $\mathbf{k} = 2\hbar^{-1}\mathbf{h}$, and all time dependence of all coefficients is given in the interaction representation of the corresponding operators.

Now we assume that indirect observation of this bit is the counting of the output photon numbers $N_t = U\left(t\right)^*\left(I\otimes n_t\right)U\left(t\right)$, where

$$n_t = \nu t + \nu^{1/2}\left(\Lambda_- + \Lambda^+\right)_t + \Lambda_t$$

is the Poisson process represented in Fock space as the quantum number process of intensity $\nu$. One can easily prove by quantum Itô formula that this process, given by

$$\mathrm{d}N_t = \mathrm{d}\Lambda + \nu^{1/2}\left(C\left(t\right)^*\mathrm{d}\Lambda_- + C\left(t\right)\mathrm{d}\Lambda^+\right) + \nu C\left(t\right)^* C\left(t\right)\mathrm{d}t,$$

where $C = I + \nu^{-1/2}\sigma\left(\mathbf{l}\right) = C^*$, is nondemolition so that the state of the q-bit can be predicted continuously while it is measured in time. The optimal prediction in the mean square sense of the q-bit operators $S = \sigma\left(\mathbf{s}\right)$ is given by the posterior expectations

$$\left\langle S\right\rangle_\omega\left(t\right) = \mathrm{Tr}S\rho_\omega\left(t\right) = \mathbf{s}\cdot\mathbf{r}_\omega\left(t\right),$$

where $\mathbf{r}_\omega\left(t\right)$ is the posterior polarization defining the posterior q-bit state

$$\rho_\omega\left(t\right) = \frac{1}{2}\left(\sigma\left(\mathbf{r}_\omega\left(t\right)\right) + I\right).$$

It satisfies the nonlinear filtering equation

$$\mathrm{d}\mathbf{r}_\omega\left(t\right) + \left(2\nu^{1/2}\mathbf{l} + \mathbf{r}_\omega\left(t\right)\times\mathbf{k} - 2\nu^{1/2}\left(\mathbf{r}_\omega\left(t\right)\cdot\mathbf{l}\right)\mathbf{r}_\omega\left(t\right)\right)\mathrm{d}t$$

$$= 2\left(\frac{\nu^{1/2}\left(\mathbf{l} - \left(\mathbf{r}_\omega\left(t\right)\cdot\mathbf{l}\right)\mathbf{r}_\omega\left(t\right)\right) + \left(\mathbf{l}\cdot\mathbf{r}_\omega\left(t\right)\right)\mathbf{l} - \left(\mathbf{l}\cdot\mathbf{l}\right)\mathbf{r}_\omega\left(t\right)}{\nu + 2\nu^{1/2}\mathbf{l}\cdot\mathbf{r}_\omega\left(t\right) + \mathbf{l}\cdot\mathbf{l}}\right)\mathrm{d}n_\omega^\rho\left(t\right)$$

with respect to the counting process $n_\omega^\rho\left(t\right)$ of the conditional intensity

$$\nu\mathrm{Tr}C\rho_\omega\left(t\right)C^* = \nu + 2\nu^{1/2}\mathbf{l}\cdot\mathbf{r}_\omega\left(t\right) + \mathbf{l}\cdot\mathbf{l}.$$

The solution of this quantum filtering equation can be obtained in the form

$$\mathbf{r}_\omega\left(t\right) = \mathbf{p}\left(t, \omega\right)/\pi\left(t, \omega\right),$$

where $\pi\left(t\right)$ is the probability density of the output counting process $n_\omega^\rho\left(t\right)$ with respect to the input probability measure for the Poisson process $n_t\left(\omega\right)$, and $\mathbf{p}\left(t\right)$ is the polarization for the stochastic density matrix $\varrho\left(t, \omega\right)$:

$$\pi\left(t\right) = \mathrm{Tr}\varrho\left(t\right), \quad \varrho\left(t\right) = \frac{1}{2}\left(\sigma\left(\mathbf{p}\left(t\right)\right) + \pi\left(t\right)I\right).$$

The 4-vector $p = \left(\pi, \mathbf{p}\right)$ is the solution of the linear osteoclastic system

$$\mathrm{d}\pi\left(t\right) = \left(\nu^{-1/2}\left(\mathbf{l}\cdot\mathbf{l}\right)\pi\left(t\right) + 2\mathbf{l}\cdot\mathbf{p}\left(t\right)\right)\mathrm{d}y_t, \quad \pi\left(0\right) = 1,$$

$$\mathrm{d}\mathbf{p}\left(t\right) + \left(\mathbf{p}\left(t\right) \times \mathbf{k} + 2\left(\mathbf{l}\cdot\mathbf{l}\right)\mathbf{p}\left(t\right) - 2\left(\mathbf{l}\cdot\mathbf{p}\left(t\right)\right)\mathbf{l}\right)\mathrm{d}t$$
$$= \left(2\mathbf{l}\pi\left(t\right) + 2\nu^{-1/2}\left(\mathbf{l}\cdot\mathbf{p}\left(t\right)\right)\mathbf{l} - \nu^{-1/2}\left(\mathbf{l}\cdot\mathbf{l}\right)\mathbf{p}\left(t\right)\right)\mathrm{d}y_t, \quad \mathbf{p}\left(0\right) = \mathbf{r},$$

where $y_t = \nu^{-1/2}n_t - \nu^{1/2}t$ is given by the stationary Poisson process $n_t$ of the intensity $\nu$. The expectation

$$\mathbf{r}\left(t\right) = \int \mathbf{r}_\omega\left(t\right)\mathrm{P}\left(t, \mathrm{d}\omega\right) = \mathrm{M}\left[\mathbf{p}\left(t\right)\right]$$

gives the solution to the Bloch Master-equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}\left(t\right) + \mathbf{r}\left(t\right) \times \mathbf{k} + 2\left(\mathbf{l}\cdot\mathbf{l}\right)\mathbf{r}\left(t\right) = 2\left(\mathbf{l}\cdot\mathbf{r}\left(t\right)\right)\mathbf{l}, \quad \mathbf{r}\left(0\right) = \mathbf{r}$$

for the polarization of the averaged density matrix $\rho\left(t\right) = \left(\sigma\left(\mathbf{r}\left(t\right) + I\right)\right)/2$.

Passing to the limit $\nu \to \infty$ of infinite intensity of the counting nondemolition process we obtain the system of diffusive equations

$$\mathrm{d}\pi\left(t\right) = 2\mathbf{l}\cdot\mathbf{p}\left(t\right)\mathrm{d}w_t, \quad \pi\left(0\right) = 1,$$

$$\mathrm{d}\mathbf{p}\left(t\right) + \left(\mathbf{p}\left(t\right) \times \mathbf{k} + 2\left(\mathbf{l}\cdot\mathbf{l}\right)\mathbf{p}\left(t\right) - 2\left(\mathbf{l}\cdot\mathbf{p}\left(t\right)\right)\mathbf{l}\right)\mathrm{d}t$$
$$= 2\mathbf{l}\pi\left(t\right)\mathrm{d}w_t, \quad \mathbf{p}\left(0\right) = \mathbf{r},$$

where it is taken into account that the limit of the process $y_t$ is the standard Wiener process $w_t$. This linear system, which was derived in [40], represents the stochastic decoherence Master-equation for the quantum bit under the continuous observation of the nondemolition process $Y_t = U\left(t\right)^*\left(I \otimes w_t\right)U\left(t\right)$ given by

$$\mathrm{d}Y_t = \left(L\left(t\right) + L\left(t\right)^*\right)\mathrm{d}t + \mathrm{d}\Lambda_- + \mathrm{d}\Lambda^+ = 2L\left(t\right)\mathrm{d}t + \mathrm{d}w_t, \quad Y_0 = 0,$$

Here $w_t = \left(\Lambda_- + \Lambda^+\right)_t$ is the standard Wiener process, defined as the input field coordinate process in Fock space in the Appendix 2. It is the central limit of the Poisson process $n_t$ in the Fock space, that is the limit of

$$y_t = \nu^{-1/2}\Lambda_t + \Lambda_- + \Lambda^+ = \nu^{-1/2}n_t - \nu^{1/2}t$$

at $\nu \to \infty$. This limit quantum diffusion model for the open quantum bit under the continuous observation coincides with the signal + noise model $Y\left(t\right) = |\mathbf{l}|\, X\left(t\right) + e\left(t\right)$ considered for derivation of the generalized Heisenberg principle. Here $|\mathbf{l}| = \left(\mathbf{l}\cdot\mathbf{l}\right)^{1/2}$, $X = \sigma\left(\mathbf{e}\right)$, $\mathbf{e} = \mathbf{l}/\left|\mathbf{l}\right|$ and $e\left(t\right)$ is the one half of the standard white noise, the generalized derivative $w\left(t\right) = \mathrm{d}w_t/\mathrm{d}t$. It was proved in [36], see also [89] for the infinite dimensional case, that $Y_t$ is a commutative nondemolition process with respect to the Heisenberg processes due to the canonical commutation relations of $e\left(t\right) = w\left(t\right)/2$ and the Langevin force $f\left(t\right) = \mathrm{d}f_t/\mathrm{d}t$. Note that the quantum error process $w_t = 2e_t$ does not commute with the perturbing quantum process $f_t$ in Fock space due to the multiplication table

$$\mathrm{d}f\mathrm{d}w = i\hbar\mathrm{d}t, \quad \mathrm{d}w\mathrm{d}f = -i\hbar\mathrm{d}t.$$

This corresponds to the canonical commutation relations for the normalized derivatives $e\left(t\right)$ and $f\left(t\right)$ such that the true Heisenberg principle is fulfilled at the boundary $\sigma\tau = \hbar/2$ of the standard deviation $\sigma = 1/2$ for $e$ and $\tau = \hbar$ for $f$. Thus our quantum stochastic model of nondemolition observation is the minimal perturbation model of the continual indirect measurement of the quantum bit position $X\left(t\right) = \sigma\left(\mathbf{e}\left(t\right)\right)$.

The solution of this linear diffusive system gives the solution $\mathbf{r}(t) = \mathbf{p}(t) / \pi(t)$ to the nonlinear filtering equation

$$\mathrm{d}\mathbf{r}_\omega(t) + (\mathbf{r}_\omega(t) \times \mathbf{k} + 2(\mathbf{l} \cdot \mathbf{l})\,\mathbf{r}_\omega(t) - 2(\mathbf{l} \cdot \mathbf{r}_\omega(t))\,\mathbf{l})\,\mathrm{d}t$$
$$= 2(\mathbf{l} - (\mathbf{r}_\omega(t) \cdot \mathbf{l})\,\mathbf{r}_\omega(t))\,\mathrm{d}\tilde{w}_t, \quad \mathbf{r}_\omega(0) = \mathbf{r},$$

for the posterior diffusion of the quantum bit state under continuous observation, derived in [86]. Here $\tilde{w}_t$ is an innovating diffusive process given by the equation

$$\mathrm{d}\tilde{w}_t = \mathrm{d}w_t - 2\mathbf{l} \cdot \mathbf{r}_\omega(t)\,\mathrm{d}t, \quad \tilde{w}_0 = 0.$$

It can be seen as another standard Wiener process, however not with respect the input but the output probability measure corresponding to the histories density $\pi(t, \omega)$. It is central limit of $n_\omega^\rho(t)$ at $\nu \to \infty$, that is the limit of the innovating counting martingale $y_\omega^\rho(t)$ given by the equation

$$\mathrm{d}y_\omega^\rho(t) = \nu^{-1/2}\mathrm{d}n_\omega^\rho(t) - \left(\nu^{1/2} + 2\mathbf{l} \cdot \mathbf{r}_\omega(t) + \nu^{-1/2}\mathbf{l} \cdot \mathbf{l}\right)\mathrm{d}t.$$

This solution can be easily obtained under the condition that $\mathbf{k} = k\mathbf{e}_z$ is colinear to $\mathbf{l}$, i.e. $\mathbf{e} = \mathbf{e}_z$, when this system splits into two independent systems

$$\mathrm{d}\pi(t) = 2|\mathbf{l}|\,p_z(t)\,\mathrm{d}w_t, \ \mathrm{d}p_z(t) = 2|\mathbf{l}|\,\pi(t)\,\mathrm{d}w_t,$$

$$\mathrm{d}\mathbf{p}_\perp(t) + \left(2|\mathbf{l}|^2\,\mathbf{p}_\perp(t) + k\mathbf{p}_\perp(t) \times \mathbf{e}_z\right)\mathrm{d}t = 0,$$

where $p_z = \mathbf{e}_z \cdot \mathbf{p}$, $\mathbf{p}_\perp = \mathbf{p} - p_z\mathbf{e}_z$. The first stochastic system, diagonalized for $\pi_\pm = (\pi \pm p_z)/2$ as

$$\mathrm{d}\pi_\pm(t) = \pm 2|\mathbf{l}|\,\pi_\pm(t)\,\mathrm{d}w_t, \quad \pi_\pm(0) = \frac{1}{2}(1 \pm z),$$

has apparent solution $\pi = \pi_- + \pi_+$, $p_l = \pi_+ - \pi_-$ where

$$\pi_\pm(t, \omega) = \frac{1}{2}(1 \pm z)\exp\left(\pm 2|\mathbf{l}|\,w_t - 2|\mathbf{l}|^2\,t\right)$$

are the joint propensity densities of the spin-projection $L = \sigma(\mathbf{l})$ to be $\pm|\mathbf{l}|$ and the trajectory of $Y$ to be $w$ up to the time $t$ with respect to the standard Wiener probability measure $\mu$. This gives

$$\pi(t, \omega) = (\cosh 2|\mathbf{l}|\,w_t + z\sinh 2|\mathbf{l}|\,w_t)\exp\left(-2|\mathbf{l}|^2\,t\right),$$

$$p_z(t, \omega) = (\sinh 2|\mathbf{l}|\,w_t + z\cosh 2|\mathbf{l}|\,w_t)\exp\left(-2|\mathbf{l}|^2\,t\right).$$

The orthogonal component $\mathbf{p}_\perp$ has the spiral nonstochastic rotation

$$p_x(t) = r_x^0(t)\exp\left(-2|\mathbf{l}|^2\,t\right), \ r_x^0(t) = x\cos kt - y\sin kt,$$

$$p_y(t) = r_y^0(t)\exp\left(-2|\mathbf{l}|^2\,t\right), \ r_y^0(t) = y\cos kt + x\sin kt,$$

which clearly suggests that $\mathbf{p}_\perp(t) \to 0$ if $t \to \infty$ or $|\mathbf{l}| \to \infty$ for each $t > 0$.

Thus the stochastic components

$$\mathbf{r}_\omega^\perp(t) = \frac{\mathbf{p}_\perp(t)}{\pi(t, \omega)} = x_\omega(t)\,\mathbf{e}_x + y_\omega(t)\,\mathbf{e}_y, \quad z_\omega(t) = \frac{p_z(t, \omega)}{\pi(t, \omega)}$$

of the posterior polarization $\mathbf{r}_\omega(t)$ are found as

$$\mathbf{r}_\omega^\perp(t) = \left(\cosh 2\,|\mathbf{l}|\,w_t + z \sinh 2\,|\mathbf{l}|\,w_t\right)^{-1}\mathbf{r}_\perp^0(t)\ ,$$

$$z_\omega(t) = \left(1 + z \tanh 2\,|\mathbf{l}|\,w_t\right)^{-1}\left(\tanh 2\,|\mathbf{l}|\,w_t + z\right).$$

Note that in order to express this as the solution to the nonlinear filtering equation one has to make the substitution

$$\mathrm{d}w_t = 2\mathbf{l}\cdot\mathbf{r}_\omega(t)\,\mathrm{d}t + \mathrm{d}\tilde{w}_t, \quad w_0 = 0.$$

At the limit $|\mathbf{l}| \to \infty$ of the infinite accuracy of the nondemolition measurement $\mathbf{r}_\omega^\perp(t) \to 0$, $z_\omega(t) \to \mathrm{sign}w_t = \pm 1$ for each finite $t$ and $w_t \neq 0$ independent of the values $\mathbf{r}_\perp = x\mathbf{e}_x + y\mathbf{e}_y$ and $z$ for the components of the initial polarization $\mathbf{r}$. One can find also the solution

$$\mathbf{r}(t) = \int \mathbf{p}(t,\omega)\,\mu(\mathrm{d}\omega) = \mathbf{p}_\perp(t) + z\mathbf{e}_z$$

to the Bloch Master-equation in the case of colinear $\mathbf{k}$ and $\mathbf{l}$.as the expectation of $\mathbf{r}_\omega(t)$ with respect to the output measure $\tilde{\mu}$. This apparently has the limit $\mathbf{r}(t) \to z\mathbf{e}_z$ at $t \to \infty$.

Thus the stochastic decoherence, continuous trajectories and spontaneous localizations of an open q-bit are derived as the result of continuous nondemolition measurement of a spin-projection $\sigma(\mathbf{l})$ plus white noise $e(t)$ from the unitary evolution perturbed by another white noise $f(t)$.

## 5. CONCLUSION: A QUANTUM MESSAGE FROM THE FUTURE

Although Schrödinger didn't derive the stochastic filtering equation for the continuously decohering wave function $\chi(t)$, describing the state of the semiclassical system including the observable nondemolition process $y_t$ in continuous time in the same way as we did it for his cat just in one step, he did envisage a possibility of how to get it 'if one introduces two symmetric systems of waves, which are traveling in opposite directions; one of them presumably has something to do with the known (or supposed to be known) state of the system at a later point in time' [100]. This desire coincides with the "transactional" attempt of interpretation of quantum mechanics suggested in [101] on the basis that the relativistic wave equation yields in the nonrelativistic limit two Schrödinger type equations, one of which is the time reversed version of the usual equation: 'The state vector $\psi$ of the quantum mechanical formalism is a real physical wave with spatial extension and it is identical with the initial "offer wave" of the transaction. The particle (photon, electron, etc.) and the collapsed state vector are identical with the completed transaction.' There was no mathematical proof of this statement in [101], and it is obviously not true for the deterministic state vector $\psi(t)$ satisfying the conventional Schrödinger equation, but we are going to show that this interpretation is true for the stochastic wave $\chi(t)$ satisfying our decoherence equation.

First let us note that the stochastic equation for the offer wave $\chi(t)$ and the standard input probability measure $\mu$ can be represented in Fock space as

$$\mathrm{d}\chi(t) + K\chi(t)\,\mathrm{d}t = L\mathrm{d}y_t\chi(t), \quad \chi(0) = \psi \otimes \delta_0,$$

where $y_t = \Lambda^+ + \Lambda_- + \varepsilon\Lambda$ in the notation explained in the Appendix. It coincides on the noise vacuum state $\delta_0$ with the quantum stochastic Schrödinger equation

$$\mathrm{d}\varphi(t) + K\varphi(t)\,\mathrm{d}t = \left(L\mathrm{d}\Lambda^+ - L^*\mathrm{d}\Lambda_-\right)\varphi(t), \quad \varphi(0) = \psi \otimes \delta_0$$

corresponding to the generalized Heisenberg equation with the Langevin force, $i f_t = \hbar (\Lambda^+ - \Lambda_-)_t$, if $L^* = L$. Indeed, as it was noted in [89], due to adaptedness both $L dy$ and $L d\Lambda^+ - L^* d\Lambda_-$ act on the tensor product states with future vacuum $\delta_0$ on which they have the same action since $\Lambda_- \delta_0 = 0$, $\Lambda \delta_0 = 0$ (the annihilation process $\Lambda_-$ is zero on the vacuum $\delta_0$, as well as the number process $\Lambda$). Thus when extended from $\delta_0$ to any initial Fock vector $\varphi_0$, quantum stochastic evolution is the HP unitary propagation [80] which is a unitary cocycle on Fock space over $L^2 (\mathbb{R}_+)$ with respect to the free time-shift evolution $\varphi (t, s) = \varphi (0, s + t)$ in the Fock space. This free plain wave evolution in the half space $s > 0$ in the extra dimension is the input, or offer wave evolution for our three dimensional (or more?) world located at the boundary of $\mathbb{R}_+$. The single offer waves do not interact in the Fock space until they reach the boundary $s = 0$ where they produce the quantum jumps described by the stochastic differential equation.

As has been recently shown in [102, 103], by doubling the Fock space it is possible to extend the cocycle to a unitary group evolution which will also include the free propagation of the output waves in the opposite direction. The conservative boundary condition corresponding to the interaction with our world at the boundary, includes the creation, annihilation and exchange of the input-output waves. The corresponding "Schrödinger" boundary value problem is the second quantization of the Dirac wave equation on the half line, with a boundary condition in Fock space which is responsible for the stochastic interaction of quantum noise with our world in the course of the transaction of the input-output waves. These nondemolition continual observations are represented in this picture by measurement at the boundary of the arrival times and positions of the particles corresponding to the quantized waves in Fock space with respect to an "offer state", the input vacuum, dressed into the output wave. The continual reduction process for our world wave function is then simply represented as the decohering input wave function in the extended space, which is filtered from the corresponding mixture of pure states by the process of innovation of the initial knowledge during the continual measurement. The result of this filtering gives the best possible prediction of future states which is allowed by the quantum causality. As was shown on the example of a free quantum particle under observation, the filtering appears as a dissipation, oscillation and gravitation as a result of nondemolition observation.

My friend Robin Hudson wrote in his Lecture Notes on Quantum Theory:

> *Quantum theory is a beautiful mathematical theory. If only it didn't have to mean something, to be interpreted.*

Obviously here he used beautiful in the sense of simple: Everything that is simple is indeed beautiful. However Nature is beautiful but not simple: we live at the edge of two worlds, one is quantum, the other one is classical, everything in the future is quantized waves, everything in the past is trajectories of recorded particles.

Certainly all great founders of quantum theory are followers of those about whom Aristotle wrote in his Metaphysica:

> *they fancied that the principles of mathematics are the principles of all things', and '...these are the greatest forms of beauty.*

APPENDIX 1: **On Bell's "Proof" that von Neumann's Proof was in Error.**

To "disprove" the von Neumann's theorem on the nonexistence of hidden variables in quantum mechanics Bell [66] argued that the dispersion-free states specified

by a hidden parameter $\lambda$ should be additive only for commuting pairs from the space $\mathcal{L} = \mathcal{L}(\mathfrak{h})$ of all Hermitian operators on the system Hilbert space $\mathfrak{h}$. One can assume even less, that the corresponding probability function $E \mapsto \langle E \rangle_\lambda$ should be additive with respect to only orthogonal decompositions in the subset $\mathcal{E} = \mathcal{E}(\mathfrak{h})$ of all Hermitian projectors $E$, as only orthogonal events are simultaneously verifiable by measuring an observable $L \in \mathcal{L}$. In the case of finite-dimensional Hilbert space $\mathfrak{h}$ it is equivalent to the Bell's assumption, but we shall reformulate his only counterexample it terms of the propositions, or events $E \in \mathcal{E}$ in order to dismiss his argument that he 'is not dealing with logical propositions, but with measurements involving, for example, differently oriented magnets' (p.6 in [1]).

Bell constructed an example of hidden dispersion-free states for the quantum-mechanical states, described by one-dimensional projectors

$$\rho = \frac{1}{2}\left(I + \sigma\left(\mathbf{r}\right)\right) \equiv P\left(\mathbf{r}\right), \quad \sigma\left(\mathbf{r}\right) = x\sigma_x + y\sigma_y + z\sigma_z$$

in two-dimensional space $\mathfrak{h} = \mathbb{C}^2$, given by the points $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$ on the unit sphere $\mathbf{S} \subset \mathbb{R}^3$ and Pauli matrices $\sigma$. He assigned to spin operators $\sigma\left(\mathbf{e}\right)$ describing the spin projections in the directions $\mathbf{e} \in \mathbf{S}$ the simultaneously definite values

$$s_\lambda\left(\mathbf{e}\right) = \pm 1 \equiv \langle \sigma\left(\mathbf{e}\right) \rangle_\lambda, \quad \mathbf{e} \in \mathbf{S}_\lambda^\pm\left(\mathbf{r}\right),$$

which can be taken as their dispersion-free expectations $\langle \sigma\left(\mathbf{e}\right) \rangle_\lambda$ due to

$$\sigma\left(-\mathbf{e}\right) = -\sigma\left(\mathbf{e}\right), \quad \sigma\left(\mathbf{e}\right)^2 = I$$

and $\langle I \rangle_\lambda = 1$ if $s_\lambda\left(-\mathbf{e}\right) = -s_\lambda\left(\mathbf{e}\right)$. This is specified by a reflection-symmetric partition

$$\mathbf{S}_\lambda^- = -\mathbf{S}_\lambda^+, \quad \mathbf{S}_\lambda^- \cup \mathbf{S}_\lambda^+ = \mathbf{S}, \quad \mathbf{S}_\lambda^- \cap \mathbf{S}_\lambda^+ = \emptyset$$

of the unit sphere $\mathbf{S}$. Obviously there are plenty of such partitions, but Bell took a special family

$$\mathbf{S}_\lambda^\pm\left(\mathbf{r}\right) = \left[\mathbf{S}^\pm\left(\mathbf{r}\right) \backslash \mathbf{S}_\lambda\left(\pm\mathbf{r}\right)\right] \cup \left[\mathbf{S}^\mp\left(\mathbf{r}\right) \backslash \mathbf{S}_{-\lambda}\left(\pm\mathbf{r}\right)\right],$$

where $\mathbf{S}^\pm$ are south and north hemispheres of the standard reflection-symmetric partition with $\mathbf{r}$ pointing north, and

$$\mathbf{S}_\lambda\left(\mathbf{r}\right) = \{\mathbf{e} \in \mathbf{S} : \mathbf{e} \cdot \mathbf{r} < 2\lambda\}$$

is parametrized by $\lambda \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ in such a way that

$$\int_{-1/2}^{1/2} s_\lambda\left(\mathbf{e}\right) \mathrm{d}\lambda = \Pr\left\{\lambda : \mathbf{S}_\lambda^+\left(\mathbf{r}\right) \ni \mathbf{e}\right\} - \Pr\left\{\lambda : \mathbf{S}_\lambda^-\left(\mathbf{r}\right) \ni \mathbf{e}\right\} = \mathbf{e} \cdot \mathbf{r}.$$

In his formula $\mathbf{r} = \mathbf{e}_z$, but it can be extended to the case $|\mathbf{r}| \leq 1$ of not completely polarized quantum states $\rho$ defining the quantum-mechanical expectations $\langle \sigma\left(\mathbf{e}\right) \rangle$ and quantum probabilities $\Pr\left\{P\left(\mathbf{e}\right) = 1\right\}$ of the propositions $E = P\left(\mathbf{e}\right)$ as the linear and affine forms in the unit ball of such $\mathbf{r}$:

$$\mathrm{Tr}\sigma\left(\mathbf{e}\right)\rho = \mathbf{e} \cdot \mathbf{r}, \quad \mathrm{Tr}P\left(\mathbf{e}\right)\rho = \frac{1}{2}\left(1 + \mathbf{e} \cdot \mathbf{r}\right).$$

Each $\lambda$ assigns the zero-one probabilities $\langle P\left(\pm\mathbf{e}\right) \rangle_\lambda = \chi_\lambda^\pm\left(\mathbf{e}\right)$ given by the characteristic functions $\chi_\lambda^\pm$ of $\mathbf{S}_\lambda^\pm$ simultaneously for all quantum events $P\left(\pm\mathbf{e}\right)$, the

eigen-projectors of $\sigma\left(\mathbf{e}\right)$ corresponding to the eigenvalues $\pm 1$:

$$P\left(\pm\mathbf{e}\right) = \frac{1}{2}\left(I \pm \sigma\left(\mathbf{e}\right)\right) \mapsto \chi_\lambda^\pm\left(\mathbf{e}\right) = \frac{1}{2}\left(1 \pm s_\lambda\left(\mathbf{e}\right)\right).$$

The additivity of the probability function $E \mapsto \langle E\rangle_\lambda$ in $\mathcal{E} = \{O, P\left(\mathbf{S}\right), I\}$ at each $\lambda$ follows from $\langle O\rangle_\lambda = 0$:

$$\langle O\rangle_\lambda + \langle I\rangle_\lambda = 1 = \langle O + I\rangle_\lambda,$$

as $O + I = I$, and from $\chi_\lambda^+\left(-\mathbf{e}\right) = \chi_\lambda^-\left(\mathbf{e}\right)$:

$$\langle P\left(\mathbf{e}\right)\rangle_\lambda + \langle P\left(-\mathbf{e}\right)\rangle_\lambda = 1 = \langle P\left(\mathbf{e}\right) + P\left(-\mathbf{e}\right)\rangle_\lambda,$$

as $P\left(\mathbf{e}\right) + P\left(-\mathbf{e}\right) = I$.

Thus a classical hidden variable theory reproducing the affine quantum probabilities $\mathrm{P}\left(\mathbf{e}\right) = \langle P\left(\mathbf{e}\right)\rangle$ as the uniform mean value

$$\mathrm{M}\langle P\left(\mathbf{e}\right)\rangle. = \int_{-1/2}^{1/2} \frac{1}{2}\left(1 + s_\lambda\left(\mathbf{e}\right)\right)\mathrm{d}\lambda = \frac{1}{2}\left(1 + \mathbf{e}\cdot\mathbf{r}\right) = \mathrm{Tr}P\left(\mathbf{e}\right)\rho$$

of the classical yes-no observables $\chi^+\left(\mathbf{e}\right) = \langle P\left(\mathbf{e}\right)\rangle$ was constructed by Bell. However it does not contradict to the von Neumann theorem even if the latter is strengthened by the restriction of the additivity only to the orthogonal projectors $E \in \mathcal{E}$.

Indeed, apart from partial additivity (the sums are defined in $\mathcal{E}$ only for the orthogonal pairs from $\mathcal{E}$), the von Neumann theorem restricted to $\mathcal{E} \subset \mathcal{L}$ should also inherit the *physical continuity*, induced by ultra-strong topology in $\mathcal{L}$. In the finite dimensional case it is just ordinary continuity in the projective topology $\mathfrak{h}$, and in the case $\dim\mathfrak{h} = 2$ it is the continuity on the projective space $\mathbf{S}$ of all one-dimensional projectors $P\left(\mathbf{e}\right)$, $\mathbf{e} \in \mathbf{S}$. It is obvious that the zero-one probability function $E \mapsto \langle E\rangle_\lambda$ constructed by Bell is not *physically continuous* on the restricted set: the characteristic function $\chi_\lambda^+\left(\mathbf{e}\right) = \langle P\left(\mathbf{e}\right)\rangle_\lambda$ of the half-sphere $\mathbf{S}_\lambda^+\left(\mathbf{r}\right)$ is discontinuous in $\mathbf{e}$ on the whole sphere $\mathbf{S}$ for any $\lambda$ and $\mathbf{r}$. Measurements of the spin projections in the physically close directions $\mathbf{e}_n \to \mathbf{e}$ should be described by close probabilities $\langle P\left(\mathbf{e}_n\right)\rangle_\lambda \to \langle P\left(\mathbf{e}\right)\rangle_\lambda$ in any physical state specified by $\lambda$, otherwise the state cannot have physical meaning!

Moreover, the mean $\mathrm{M}$ over $\lambda$ cannot be considered as the conditional averaging of a classical partially hidden world with respect to the quantum observable part because it gives nonlinear expectations with respect to the states $\rho$ even if it is restricted to the smallest commutative algebra generated by the characteristic functions $\{\chi^+\left(\mathbf{e}\right) : \mathbf{e} \in \mathbf{S}\}$ of $\lambda : \mathbf{S}_\lambda^+\left(\mathbf{r}\right) \ni \mathbf{e}$. One can see this by the uniform averaging of the commutative products $\chi_\lambda^+\left(\mathbf{e}\right)\chi_\lambda^+\left(\mathbf{f}\right)$: such mean values ( i.e. the second order moments) are affine with respect to $\mathbf{r}$ only for colinear $\mathbf{e}$ and $\mathbf{f} \in \mathbf{S}$.

The continuity argument might be considered to be as purely mathematical, but in fact it is not: even in classical probability theory with a discrete phase space the pure states defined by Dirac $\delta$-measure, are uniformly continuous, as any positive probability measure is on the space of classical observables defined by bounded measurable functions on any continuous phase space. In quantum theory an expectation defined as a linear positive functional on $\mathcal{L}$ is also uniformly continuous, hence the von Neumann assumption of physical (ultra-weak) continuity is only a restriction in the infinite-dimensional case. Even if the state is defined only on $\mathcal{E} \subset \mathcal{L}$ as a probability function which is additive only on the orthogonal projectors, the uniform continuity follows from its positivity in the case of $\dim\mathfrak{h} \geq 3$

[67]. In fact, Gleason obtained more than this: He proved that the case $\dim \mathfrak{h} = 2$ is the only exceptional one when a probability function on $\mathcal{E}(\mathfrak{h})$ (which should be countably additive in the case $\dim \mathfrak{h} = \infty$) may not be induced by a density operator $\rho$, and thus cannot be extended to a linear expectation on the operator space $\mathcal{L}(\mathfrak{h})$. Such irregular states cannot be extended by linearity on the algebra of all (not just Hermitian) operators in $\mathfrak{h} = \mathbb{C}^2$ even if it is continuous.

To rule out even this exceptional case we note that an irregular states $E \mapsto \langle E \rangle$ on $\mathcal{E}(\mathbb{C}^2)$ cannot be composed with any state of an additional quantum system even if the latter is given by a regular probability function $\langle F \rangle = \mathrm{Tr} F \sigma$ on a set $\mathcal{E}(\mathfrak{g})$ of ortho-projectors of another Hilbert space. There is no additive probability function on the set $\mathcal{E}(\mathbb{C}^2 \otimes \mathfrak{g})$ of all verifiable events for the compound quantum system described by a nontrivial Hilbert space $\mathfrak{g}$ such that

$$\langle E \rangle = \langle E \otimes I \rangle, \quad \langle I \otimes F \rangle = \mathrm{Tr} F \sigma,$$

where $\sigma = P_\varphi$ is the density operator of wave function $\varphi \in \mathfrak{g}$. Indeed, if it could be possible for some $\mathfrak{g}$ with $\dim \mathfrak{g} > 1$, it would be possible for $\mathfrak{g} = \mathbb{C}^2$. By virtue of Gleason's theorem any probability function which is additive for orthogonal projectors on $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$ is regular on $\mathcal{E}(\mathbb{C}^4)$, given by a density operator $\hat{\varrho}$. Hence

$$\langle E \rangle = \mathrm{Tr}\,(I \otimes E)\,\hat{\varrho} = \mathrm{Tr} E \rho$$

i.e. the state on $\mathcal{E}(\mathbb{C}^2)$ is also regular, with the density operator in $\mathfrak{h} = \mathbb{C}^2$ given by the partial trace

$$\rho = \mathrm{Tr}\,[\hat{\varrho}|\mathfrak{h}] = \mathrm{Tr}_\mathfrak{g}\,\hat{\varrho}.$$

In order to obtain an additive product-state on $\mathcal{E}(\mathbb{C}^2 \otimes \mathfrak{g})$ satisfying

$$\langle E \otimes F \rangle = \langle E \rangle \,\mathrm{Tr} F P_\varphi, \quad E \in \mathcal{E}(\mathbb{C}^2)\,, F \in \mathcal{E}(\mathfrak{g})$$

for a finite-dimensional $\mathfrak{g} = \mathbb{C}^n$ with $n > 1$ it necessary to define the state as an expectation on the whole unit ball $\mathcal{B}_1$ of the algebra $\mathcal{B}$ of all (not just Hermitian) operators in $\mathbb{C}^2$. Indeed, any one-dimensional Hermitian projector in $\mathbb{C}^2 \otimes \mathbb{C}^n = \mathbb{C}^{2n}$ can be described as an $n \times n$-matrix $\mathbf{E} = [A_j A_i^*]$ with $2 \times 2$-entries $A_j \in \mathcal{B}_1(\mathbb{C}^2)$, $j = 1, \ldots, n$ satisfying the normalization condition

$$\sum_{j=1}^n A_j^* A_j = P(\mathbf{e}) = \frac{1}{2}(I + \sigma(\mathbf{e}))$$

for some $\mathbf{e} \in \mathbf{S}$. These entries have the form

$$A = \alpha P(\mathbf{e}) + a Q(\mathbf{e}_\perp), \quad Q(\mathbf{e}_\perp) = \frac{1}{2}\sigma(\mathbf{e}_\perp),$$

where $\mathbf{e}_\perp$ is an orthogonal complex vector such that

$$i\mathbf{e}_\perp \times \mathbf{e} = \mathbf{e}_\perp \quad \mathbf{e}_\perp^* \cdot \mathbf{e}_\perp = 2, \quad i\mathbf{e}_\perp^* \times \mathbf{e}_\perp = 2\mathbf{e},$$

and $\sum \left(\left|\alpha_j^2\right| + \left|a_j^2\right|\right) = 1$ corresponding to $\mathrm{Tr}\mathbf{E} = 1$. The matrix elements

$$A_j A_i^* = \alpha_j \alpha_i^* P(\mathbf{e}) + a_j a_i^* P(-\mathbf{e}) + \alpha_j a_i^* Q(\mathbf{e}_\perp^*) + a_j \alpha_i^* Q(\mathbf{e}_\perp)$$

for these orthoprojectors in $\mathbb{C}^{2n}$ are any matrices from the unit ball $\mathcal{B}_1(\mathbb{C}^2)$, not just Hermitian orthoprojectors. By virtue of Gleason's theorem the product-state

of such events $\mathbf{E}$ must be defined by the additive probability

$$\langle \mathbf{E} \rangle = \sum_{i,j=1}^{n} \varphi^j \left( A_j A_\iota^* \right) \bar{\varphi}^i = \varrho \left( B \right),$$

where $B = A \left( \varphi \right) A \left( \varphi \right)^* = \beta I + \sigma \left( \mathbf{b} \right)$ is given by $\alpha \left( \varphi \right) = \varphi^j \alpha_j$, $a \left( \varphi \right) = \varphi^j a_j$ for $\varphi \in \mathbb{C}^n$ with the components $\varphi^j = \bar{\varphi}_j$, and $\varrho \left( B \right) = \mathrm{Tr} B \rho$ is the linear expectation

$$\varrho \left( B \right) = \frac{1}{2} \left( \beta_+ \left( 1 + r_1 \right) + \beta_- \left( 1 - r_1 \right) + b_\perp \bar{r}_\perp + \bar{b}_\perp r_\perp \right) = \beta + \mathbf{b} \cdot \mathbf{r}$$

with $r_1 = \mathbf{e} \cdot \mathbf{r}$, $\mathbf{r}_\perp = \mathbf{e}_\perp \cdot \mathbf{r}$, $\beta_+ = \left| \alpha \left( \varphi \right) \right|^2$, $\beta_- = \left| a \left( \varphi \right) \right|^2$, $b_\perp = \alpha \left( \varphi \right) \overline{a \left( \varphi \right)}$. It these terms we can formulate the definition of a regular state without assuming a priori the linearity and even continuity conditions also for the case $\mathfrak{h} = \mathbb{C}^2$.

A complex-valued map $B \mapsto \varrho \left( B \right)$ on the unit ball $\mathcal{B}_1 \left( \mathfrak{h} \right)$ normalized as $\varrho \left( I \right) = 1$ is called *state* for a quantum system described by the Hilbert space $\mathfrak{h}$ (including the case $\dim \mathfrak{h} = 2$) if it is positive on all Hermitian projective matrices $\mathbf{E} = [A_j A_k^*]$ with entries $A_j \in \mathcal{B}_1 \left( \mathfrak{h} \right)$ in the sense

$$\sum_j A_j^* A_j = P \in \mathcal{E} \left( \mathfrak{h} \right) \Rightarrow \varrho \left( \mathbf{E} \right) = \left[ \varrho \left( A_j A_k^* \right) \right] \geq 0,$$

of positive-definiteness of the matrices $\varrho \left( \mathbf{E} \right)$ with the complex entries $\left[ \varrho \left( A_j A_k^* \right) \right]$. It is called a *regular state* if

$$\varrho \left( E \otimes P_\varphi \right) = \varrho \left( E \right) P_\varphi$$

for any one-dimensional projector $P_\varphi = \left[ \varphi_j \varphi_i^* \right]$, and if it is countably-additive with respect to the orthogonal decompositions $\mathbf{E} = \sum \mathbf{E} \left( k \right)$:

$$\sum_j A_j \left( i \right)^* A_j \left( k \right) = 0, \forall i \neq k \Rightarrow \varrho \left( \sum_k A_j \left( k \right) A_j \left( k \right)^* \right) = \sum_k \varrho \left( A_j \left( k \right) A_i \left( k \right)^* \right).$$

It obvious that the state thus defined can be uniquely extended to a regular product-state on $\mathcal{E} \left( \mathfrak{h} \otimes \mathbb{C}^n \right)$ by

$$\sum_{j,k} \bar{\varphi}_j \varrho \left( A_j A_i^* \right) \varphi_i \geq 0, \quad \forall \varphi_j \in \mathbb{C}, \quad \sum \left| \varphi_j \right| = 1,$$

which proves that it is continuous and is given by a density operator: $\varrho \left( B \right) = \mathrm{Tr} B \rho$. Thus the composition principle rules out the existence of the hidden variable representation for the quantum bits corresponding to the case $\mathfrak{h} = \mathbb{C}^2$.

APPENDIX 2: **Symbolic Calculus for Quantum Noise.**

In order to formulate the differential nondemolition causality condition and to derive a filtering equation for the posterior states in the time-continuous case we need quantum stochastic calculus.

The classical differential calculus for the infinitesimal increments

$$\mathrm{d}x = x \left( t + \mathrm{d}t \right) - x \left( t \right)$$

became generally accepted only after Newton gave a simple algebraic rule $\left( \mathrm{d}t \right)^2 = 0$ for the formal computations of the differentials $\mathrm{d}x$ for smooth trajectories $t \mapsto x \left( t \right)$.

In the complex plane $\mathbb{C}$ of phase space it can be represented by a one-dimensional algebra $\mathfrak{a} = \mathbb{C}\eth$ of the elements $a = \alpha\eth$ with involution $a^\star = \bar{\alpha}\eth$. Here

$$\eth = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}\left(\sigma_1 + i\sigma_2\right)$$

for $\mathrm{d}t$ is the nilpotent matrix, which can be regarded as Hermitian $\eth^\star = \eth$ with respect to the Minkowski metrics $(\mathbf{z}|\mathbf{z}) = 2\,\mathrm{Re}\,z_-\bar{z}_+$ in $\mathbb{C}^2$.

This formal rule was generalized to non-smooth paths early in the last century in order to include the calculus of forward differentials $\mathrm{d}w \simeq (\mathrm{d}t)^{1/2}$ for continuous diffusions $w_t$ which have no derivative at any $t$, and the forward differentials $\mathrm{d}n \in \{0,1\}$ for left continuous counting trajectories $n_t$ which have zero derivative for almost all $t$ (except the points of discontinuity where $\mathrm{d}n = 1$). The first is usually done by adding the rules

$$(\mathrm{d}w)^2 = \mathrm{d}t, \quad \mathrm{d}w\mathrm{d}t = 0 = \mathrm{d}t\mathrm{d}w$$

in formal computations of continuous trajectories having the first order forward differentials $\mathrm{d}x = \alpha\mathrm{d}t + \beta\mathrm{d}w$ with the diffusive part given by the increments of standard Brownian paths $w$. The second can be done by adding the rules

$$(\mathrm{d}n)^2 = \mathrm{d}n, \quad \mathrm{d}n\mathrm{d}t = 0 = \mathrm{d}t\mathrm{d}n$$

in formal computations of left continuous and smooth for almost all $t$ trajectories having the forward differentials $\mathrm{d}x = \alpha\mathrm{d}t + \gamma\mathrm{d}m$ with jumping part given by the increments of standard compensated Poisson paths $m_t = n_t - t$. These rules were developed by Itô [104] into the form of a stochastic calculus.

The linear span of $\mathrm{d}t$ and $\mathrm{d}w$ forms the Wiener-Itô algebra $\mathfrak{b} = \mathbb{C}\eth + \mathbb{C}\mathrm{d}_w$, while the linear span of $\mathrm{d}t$ and $\mathrm{d}n$ forms the Poisson-Itô algebra $\mathfrak{c} = \mathbb{C}\eth + \mathbb{C}\mathrm{d}_m$, with the second order nilpotent $\mathrm{d}_w = \mathrm{d}_w^\star$ and the idempotent $\mathrm{d}_m = \mathrm{d}_m^\star$. They are represented together with $\eth$ by the triangular Hermitian matrices

$$\eth = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathrm{d}_w = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathrm{d}_m = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

on the Minkowski space $\mathbb{C}^3$ with respect to the inner Minkowski product $(\mathbf{z}|\mathbf{z}) = z_-z^- + z_\circ z^\circ + z_+z^+$, where $z^\mu = \bar{z}_{-\mu}$, $-(-,\circ,+) = (+,\circ,-)$.

Although both algebras $\mathfrak{b}$ and $\mathfrak{c}$ are commutative, the matrix algebra $\mathfrak{a}$ generated by $\mathfrak{b}$ and $\mathfrak{c}$ on $\mathbb{C}^3$ is not:

$$\mathrm{d}_w\mathrm{d}_m = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \mathrm{d}_m\mathrm{d}_w.$$

The four-dimensional $\star$-algebra $\mathfrak{a} = \mathbb{C}\eth + \mathbb{C}\mathrm{d}_- + \mathbb{C}\mathrm{d}^+ + \mathbb{C}\mathrm{d}$ of triangular matrices with the canonical basis

$$\mathrm{d}_- = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathrm{d}^+ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \mathrm{d} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

given by the algebraic combinations

$$\mathrm{d}_- = \mathrm{d}_w\mathrm{d}_m - \eth, \ \mathrm{d}^+ = \mathrm{d}_m\mathrm{d}_w - \eth, \ \mathrm{d} = \mathrm{d}_m - \mathrm{d}_w$$

is the canonical representation of the differential $\star$-algebra for one-dimensional vacuum noise in the unified quantum stochastic calculus [82, 86]. It realizes the HP (Hudson-Parthasarathy) table [80]

$$d\Lambda_- d\Lambda^+ = dt, \quad d\Lambda_- d\Lambda = d\Lambda_-, \quad d\Lambda d\Lambda^+ = d\Lambda^+, \quad (d\Lambda)^2 = d\Lambda,$$

with zero products for all other pairs, for the multiplication of the canonical counting $d\Lambda = \lambda(\mathrm{d})$, creation $d\Lambda^+ = \lambda(\mathrm{d}^+)$, annihilation $d\Lambda_- = \lambda(\mathrm{d}_-)$, and preservation $dt = \lambda(\mathrm{d})$ quantum stochastic integrators in Fock space over $L^2(\mathbb{R}_+)$. As was proved recently in [105], any generalized Itô algebra describing a quantum noise can be represented in the canonical way as a $\star$-subalgebra of a quantum vacuum algebra

$$d\Lambda_\mu^\iota d\Lambda_\kappa^\nu = \delta_\kappa^\iota d\Lambda_\mu^\nu, \quad \kappa, \mu \in \{-, 1, \ldots, d\}; \; \iota, \nu \in \{1, \ldots, d, +\},$$

in the Fock space with several degrees of freedom $d$, where $d\Lambda_-^+ = dt$ and $d$ is restricted by the doubled dimensionality of quantum noise (could be infinite), similar to the representation of every semi-classical system with a given state as a subsystem of quantum system with a pure state. Note that in this quantum Itô product formula $\delta_\kappa^\iota = 0$ if $\iota = +$ or $\kappa = -$ as $\delta_\kappa^\iota \neq 0$ only when $\iota = \kappa$.

The quantum Itô product gives an explicit form

$$d\chi\chi^\dagger + \chi d\chi^\dagger + d\chi d\chi^\dagger = \left(\alpha_\nu^\mu \chi^\dagger + \chi\alpha_\nu^{\star\mu} + \alpha_k^\mu a_\nu^{\star k}\right)_\nu^\mu d\Lambda_\mu^\nu$$

of the term $d\chi d\chi^\dagger$ for the adjoint quantum stochastic differentials

$$d\chi = \alpha_\nu^\mu d\Lambda_\mu^\nu, \quad d\chi^\dagger = \alpha_\nu^{\star\mu} d\Lambda_\mu^\nu,$$

for evaluation of the product differential

$$d\left(\chi\chi^\dagger\right) = (\chi + d\chi)(\chi + d\chi)^\dagger - \chi\chi^\dagger.$$

Here $\alpha_{-\nu}^{\star\mu} = \alpha_{-\mu}^{\nu\dagger}$ is the quantum Itô involution with respect to the switch $-(-,+) = (+,-)$, $-(1, \ldots, d) = (1, \ldots, d)$, introduced in [82], and the Einstein summation is always understood over $\nu = 1, \ldots, d, +; \mu = -, 1, \ldots, d$ and $k = 1, \ldots, d$. This is the universal Itô product formula which lies in the heart of the general quantum stochastic calculus [82, 86] unifying the Itô classical stochastic calculi with respect to the Wiener and Poisson noises and the quantum differential calculi [80, 81] based on the particular types of quantum Itô algebras for the vacuum or finite temperature noises. It was also extended to the form of quantum functional Itô formula and even for the quantum nonadapted case in [106, 107].

In particular, any real-valued process $y_t$ with zero mean value $\langle y_t \rangle = 0$ and independent increments generating a two-dimensional Itô algebra has the differential $dy$ in the form of a commutative combination of $d\Lambda, d\Lambda_-, d\Lambda^+$. The Itô formula for the process $y_t$ can be obtained from the HP product

$$d\chi d\chi^\dagger = \alpha\alpha^\dagger d\Lambda + \alpha^- \alpha^\dagger d\Lambda_- + \alpha\alpha^{-\dagger} d\Lambda^+ + \alpha^- \alpha^{-\dagger} dt$$

for the quantum stochastic differential

$$d\chi = \alpha d\Lambda + \alpha^- d\Lambda_- + \alpha_+ d\Lambda^+ d\chi + \alpha_+^- dt.$$

The noise $y_t$ is called standard if it has stationary increments with the standard variance $\langle y_t^2 \rangle = t$. In this case

$$y_t = \left(\Lambda^+ + \Lambda_- + \varepsilon\Lambda\right)_t = \varepsilon m_t + (1 - \varepsilon) w_t,$$

where $\varepsilon \geq 0$ is defined by the equation $(\mathrm{d}y)^2 - \mathrm{d}t = \varepsilon \mathrm{d}y$. Such, and indeed higher dimensional, quantum noises for continual measurements in quantum optics were considered in [108, 109].

The general form of a quantum stochastic decoherence equation, based on the canonical representation of the arbitrary Itô algebra for a quantum noise in the vacuum of $d$ degrees of freedom, can be written as

$$\mathrm{d}\chi\left(t\right) = \left(L_\nu^\mu - \delta_\nu^\mu\right)\chi\left(t\right)\mathrm{d}\Lambda_\mu^\nu, \quad \chi\left(0\right) = \psi.$$

Here $L_\nu^\mu$ are the operators in the system Hilbert space $\mathfrak{h} \ni \psi$ with $L_\kappa^{\star -} L_+^\kappa = 0$ for the mean square normalization

$$\left\langle \chi\left(t\right)^\dagger \chi\left(t\right) \right\rangle = \mathrm{M}\chi\left(t\right)^\dagger \chi\left(t\right) = \psi^\dagger \psi$$

with respect to the vacuum of Fock space of the quantum noise, where the Einstein summation is understood over all $\kappa = -, 1, \ldots, d, +$ with the agreement

$$L_-^- = I = L_+^+, \quad L_-^k = 0 = L_k^+, \quad k = 1, \ldots, d$$

and $\delta_\nu^\mu = 1$ for all coinciding $\mu, \nu \in \{-, 1, \ldots, d, +\}$ such that $L_\nu^\mu - \delta_\nu^\mu = 0$ whenever $\mu = +$ or $\nu = -$. In the notations $L_+^i = L^i$, $L_+^- = -K$, $L_j^- = -K_j$, $i, j = 1, \ldots, d$ the decoherence wave equation takes the standard form [110, 111]

$$\mathrm{d}\chi\left(t\right) + \left(K\mathrm{d}t + K_j\mathrm{d}\Lambda_-^j\right)\chi\left(t\right) = \left(L^i\mathrm{d}\Lambda_i^+ + \left(L_j^i - \delta_j^i\right)\mathrm{d}\Lambda_i^j\right)\chi\left(t\right),$$

where $\Lambda_i^+\left(t\right), \Lambda_-^j\left(t\right), \Lambda_i^j\left(t\right)$ are the canonical creation, annihilation and exchange processes respectively in Fock space, and the normalization condition is written as $L_k L^k = K + K^*$ with $L_k^* = L^k$ (the Einstein summation is over $i, j, k = 1, \ldots, d$).

Using the quantum Itô formula one can obtain the corresponding equation for the quantum stochastic density operator $\hat{\varrho} = \chi\chi^\dagger$ which is the particular case $\kappa = -, 1, \ldots, d, +$ of the quantum stochastic Master equation

$$\mathrm{d}\hat{\varrho}\left(t\right) = \left(L_\nu^\kappa \hat{\varrho}\left(t\right) L_\kappa^{\star \mu} - \hat{\varrho}\left(t\right)\delta_\nu^\mu\right)\mathrm{d}\Lambda_\mu^\nu, \quad \hat{\varrho}\left(0\right) = \rho,$$

where the summation over $\kappa = -, k, +$ should be in general extended to infinity, $k = 1, 2, \ldots$. This general form of the decoherence equation with $L_\kappa^{\star -} L_+^\kappa = 0$ corresponding to the normalization condition $\left\langle \hat{\varrho}\left(t\right) \right\rangle = \mathrm{Tr}\rho$ in the vacuum mean, was recently derived in terms of quantum stochastic completely positive maps in [110, 111]. Denoting $L_\nu^- = -K_\nu$, $L_+^{\star \mu} = -K^\mu$ such that $K_\mu^* = K^\mu$, this can be written as

$$\mathrm{d}\hat{\varrho}\left(t\right) + K_\nu \hat{\varrho}\left(t\right)\mathrm{d}\Lambda_-^\nu + \hat{\varrho}\left(t\right)K^\mu\mathrm{d}\Lambda_\mu^+ = \left(L_\nu^k \hat{\varrho}\left(t\right) L_k^{\star \mu} - \hat{\varrho}\left(t\right)\delta_\nu^\mu\right)\mathrm{d}\Lambda_\mu^\nu,$$

or in the notation above, $K_+ = K, K^- = K^*, L_+^k = L^k, L_k^{\star -} = L_k, L_k^{\star i} = L_i^{k*}$ as

$$\mathrm{d}\hat{\varrho}\left(t\right) + \left(K\hat{\varrho}\left(t\right) + \hat{\varrho}\left(t\right)K^* - L^k \hat{\varrho}\left(t\right) L_k\right)\mathrm{d}t = \left(L_j^k \hat{\varrho}\left(t\right) L_k^{*i} - \hat{\varrho}\left(t\right)\delta_j^i\right)\mathrm{d}\Lambda_i^j$$

$$+ \left(L_j^k \hat{\varrho}\left(t\right) L_k - K_j \hat{\varrho}\left(t\right)\right)\mathrm{d}\Lambda_-^j + \left(L^k \hat{\varrho}\left(t\right) L_k^{*i} - \hat{\varrho}\left(t\right)K^i\right)\mathrm{d}\Lambda_i^+,$$

with $K + K^* = L_k L^k$, $L^k = L_k^*$, $L_k^{*i} = L_i^{k*}$ for any number of $k$'s, and arbitrary $K^i = K_i^*$, $L_j^k$, $i, j = 1, \ldots, d$. This is the quantum stochastic generalization of the general form [112] for the non-stochastic (Lindblad) Master equation corresponding to the case $d = 0$. In the case $d > 0$ with pseudo-unitary block-matrix $\mathbf{L} = [L_\nu^\mu]_{\nu=-,\circ,+}^{\mu=-,\circ,+}$ in the sense $\mathbf{L}^\star = \mathbf{L}^{-1}$, it gives the general form of quantum stochastic Langevin equation corresponding to the HP unitary evolution for $\chi\left(t\right)$ [80].

The nonlinear form of this decoherence equation for the exactly normalized density operator $\hat{\rho}(t) = \hat{\varrho}(t) / \mathrm{Tr}_\mathfrak{h} \hat{\varrho}(t)$ was obtained for different commutative Itô algebras in [42, 94, 86].

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